

## UNIVERSAL LOCALIZATION OF TRIANGULAR MATRIX RINGS

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ABSTRACT. If  $R$  is a triangular  $2 \times 2$  matrix ring, the columns  $P$  and  $Q$  are f.g. projective  $R$ -modules. We describe the universal localization of  $R$  which makes invertible an  $R$ -module morphism  $\sigma : P \rightarrow Q$ , generalizing a theorem of A. Schofield. We also describe the universal localization of  $R$ -modules.

### 1. INTRODUCTION

Suppose  $R$  is an associative ring (with 1) and  $\sigma : P \rightarrow Q$  is a morphism between finitely generated projective  $R$ -modules. There is a universal way to localize  $R$  in such a way that  $\sigma$  becomes an isomorphism. More precisely, there is a ring morphism  $R \rightarrow \sigma^{-1}R$  which is universal for the property that

$$\sigma^{-1}R \otimes_R P \xrightarrow{1 \otimes \sigma} \sigma^{-1}R \otimes_R Q$$

is an isomorphism (Cohn [7, 9, 8, 6], Bergman [4, 5], Schofield [17]). Although it is often difficult to understand universal localizations when  $R$  is non-commutative<sup>1</sup> there are examples where elegant descriptions of  $\sigma^{-1}R$  have been possible (e.g. Cohn and Dicks [10], Dicks and Sontag [11, Thm. 24], Farber and Vogel [12] Ara, González-Barroso, Goodearl and Pardo [1, Example 2.5]). The purpose of this paper is to describe and to generalize some particularly interesting examples due to A. Schofield [17, Thm. 13.1] which have application in topology (e.g., Ranicki [16, Part 2]).

We consider a triangular matrix ring  $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$  where  $A$  and  $B$  are associative rings (with 1) and  $M$  is an  $(A, B)$ -bimodule. Multiplication in  $R$  is given by

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}$$

for all  $a, a' \in A$ ,  $m, m' \in M$  and  $b, b' \in B$ . The columns  $P = \begin{pmatrix} A \\ 0 \end{pmatrix}$  and  $Q = \begin{pmatrix} M \\ B \end{pmatrix}$  are f.g. projective left  $R$ -modules with

$$P \oplus Q \cong R.$$

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<sup>1</sup>If  $R$  is commutative one obtains a ring of fractions; see Bergman [4, p. 68].

For the general theory of triangular matrix rings see Haghany and Varadarajan [13, 14].

We shall describe in Theorem 2.4 the universal localization  $R \rightarrow \sigma^{-1}R$  which makes invertible a morphism  $\sigma : P \rightarrow Q$ . Such a morphism can be written  $\sigma = \begin{pmatrix} j \\ 0 \end{pmatrix}$  where  $j : A \rightarrow M$  is a morphism of left  $A$ -modules. Examples follow, in which restrictions are placed on  $A$ ,  $B$ ,  $M$  and  $\sigma$ . In particular, Example 2.8 recovers Theorem 13.1 of Schofield [17]. We proceed to describe the universal localization  $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$  of an arbitrary left module  $N$  for the triangular matrix ring  $R$  (see Theorem 2).

The structure of this paper is as follows: definitions, statements of results and examples are given in Section 2 and the proofs are collected in Section 3.

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## 2. STATEMENTS AND EXAMPLES

Let us first make more explicit the universal property of localization:

**Definition 2.1.** A ring morphism  $R \rightarrow R'$  is called  $\sigma$ -invertible if

$$\text{id} \otimes \sigma : R' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow R' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

is an isomorphism. The universal localization  $i_\sigma : R \rightarrow \sigma^{-1}R$  is the initial object in the category of  $\sigma$ -invertible ring morphisms  $R \rightarrow R'$ . In other words, every  $\sigma$ -invertible ring morphism  $R \rightarrow R'$  factors uniquely as a composite  $R \rightarrow \sigma^{-1}R \rightarrow R'$ .

**Definition 2.2.** An  $(A, M, B)$ -ring  $(S, f_A, f_M, f_B)$  is a ring  $S$  together with ring morphisms  $f_A : A \rightarrow S$  and  $f_B : B \rightarrow S$  and an  $(A, B)$ -bimodule morphism  $f_M : M \rightarrow S$ .

$$\begin{array}{ccc} A & \xrightarrow{f_A} & S & \xleftarrow{f_B} & B \\ & & f_M \uparrow & & \\ & & M & & \end{array}$$

It is understood that the  $(A, B)$ -bimodule structure on  $S$  is induced by  $f_A$  and  $f_B$ , so that  $f_A(a)f_M(m) = f_M(am)$  and  $f_M(m)f_B(b) = f_M(mb)$  for all  $a \in A$ ,  $b \in B$  and  $m \in M$ .

A morphism  $(S, f_A, f_M, f_B) \rightarrow (S', f'_A, f'_M, f'_B)$  of  $(A, M, B)$ -rings is a ring morphism  $\theta : S \rightarrow S'$  such that i)  $\theta f_A = f'_A$ , ii)  $\theta f_M = f'_M$  and iii)  $\theta f_B = f'_B$ .

**Definition 2.3.** Suppose  $p \in M$ . Let  $(T(M, p), \rho_A, \rho_M, \rho_B)$  denote the initial object in the subcategory of  $(A, M, B)$ -rings with the property  $\rho_M(p) = 1$ . For brevity we often write  $T = T(M, p)$ .

The ring  $T$  can be explicitly described in terms of generators and relations as follows. We have one generator  $x_m$  for each element  $m \in M$  and relations:

- (+)  $x_m + x_{m'} = x_{m+m'}$ ,
- (a)  $x_{ap}x_m = x_{am}$ ,
- (b)  $x_mx_{pb} = x_{mb}$ ,
- (id)  $x_p = 1$ ,

for all  $m, m' \in M$ ,  $a \in A$  and  $b \in B$ . The morphisms  $\rho_A, \rho_M, \rho_B$  are

$$\begin{aligned} \rho_A &: A \rightarrow T; a \mapsto x_{ap}, \\ \rho_B &: B \rightarrow T; b \mapsto x_{pb}, \\ \rho_M &: M \rightarrow T; m \mapsto x_m. \end{aligned}$$

Suppose  $\sigma : \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} M \\ B \end{pmatrix}$  is a morphism of left  $R$ -modules. We may write  $\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$  for some  $p \in M$ . Let  $T = T(M, p)$ .

**Theorem 2.4.** *The universal localization  $R \rightarrow \sigma^{-1}R$  is (isomorphic to)*

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \xrightarrow{\begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}} \begin{pmatrix} T & T \\ T & T \end{pmatrix}.$$

- Example 2.5.** (1) Suppose  $A = B = M$  and multiplication in  $A$  defines the  $(A, A)$ -bimodule structure on  $M$ . If  $p = 1$ , then  $T = A$  and  $\rho_A = \rho_M = \rho_B = \text{id}_A$ .  
 (2) Suppose  $A = B$  and  $M = A \oplus A$  with the obvious bimodule structure. If  $p = (1, 0)$ , then  $T$  is the polynomial ring  $A[x]$  in a central indeterminate  $x$ . The map  $\rho_A = \rho_B$  is the inclusion of  $A$  in  $A[x]$  while  $\rho_M(1, 0) = 1$  and  $\rho_M(0, 1) = x$ .

The universal localizations corresponding to Example 2.5 are

- (1)  $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \rightarrow \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ ;  
 (2)  $\begin{pmatrix} A & A \oplus A \\ 0 & A \end{pmatrix} \rightarrow \begin{pmatrix} A[x] & A[x] \\ A[x] & A[x] \end{pmatrix}$ .

*Remark 2.6.* One can regard the triangular matrix rings in these examples as path algebras over  $A$  for the quivers



The universal localizations  $R \rightarrow \sigma^{-1}R$  are obtained by introducing an inverse to the arrow in (1) and by introducing an inverse to one of the arrows in (2). See for example Benson [2, p. 99] for an introduction to quivers.

The following examples subsume these.

- Example 2.7.** (1) (Amalgamated free product; Schofield [17, Thm. 4.10])  
 Suppose  $i_A : C \rightarrow A$  and  $i_B : C \rightarrow B$  are ring morphisms and  $M = A \otimes_C B$ . If  $p = 1 \otimes 1$ , then  $T$  is the amalgamated free product  $A \sqcup_C B$  and appears in the pushout square

$$\begin{array}{ccc} C & \xrightarrow{i_A} & A \\ i_B \downarrow & & \downarrow \rho_A \\ B & \xrightarrow{\rho_B} & T \end{array}$$

The map  $\rho_M$  is given by  $\rho_M(a \otimes b) = \rho_A(a)\rho_B(b)$  for all  $a \in A$  and  $b \in B$ . We recover part (1) of Example 2.5 by setting  $A = B = C$  and  $i_A = i_B = \text{id}$ .

- (2) (HNN extension) Suppose  $A = B$  and  $i_1, i_2 : C \rightarrow A$  are ring morphisms. Let  $A \otimes_C A$  denote the tensor product with  $C$  acting via  $i_1$  on the first copy of  $A$  and by  $i_2$  on the second copy. Let  $M = A \oplus (A \otimes_C A)$  and  $p = (1, 0 \otimes 0)$ . Now  $T = A *_C \mathbb{Z}[x]$  is generated by the elements in  $A$  together with an indeterminate  $x$  and has the relations in  $A$  together with  $i_1(c)x = xi_2(c)$  for each  $c \in C$ . We have  $\rho_A(a) = \rho_B(a) = a$  for all  $a \in A$  while  $\rho_M(1, 0 \otimes 0) = 1$  and  $\rho_M(0, a_1 \otimes a_2) = a_1xa_2$ . If  $C = A$  and  $i_1 = i_2 = \text{id}_A$ , we recover part (2) of Example 2.5.

The following example is Theorem 13.1 of Schofield [17] and generalizes Example 2.7.

- Example 2.8.** (1) Suppose  $p$  generates  $M$  as a bimodule, i.e.,  $M = ApB$ . Now  $T$  is generated by the elements of  $A$  and the elements of  $B$  subject to the relation  $\sum_{i=1}^n a_i b_i = 0$  if  $\sum_{i=1}^n a_i p b_i = 0$  (with  $a_i \in A$  and  $b_i \in B$ ). This ring  $T$  is denoted  $A \sqcup_{(M,p)} B$  in [17, Ch. 13]. The maps  $\rho_A$  and  $\rho_B$  are obvious and  $\rho_M$  sends  $\sum_i a_i p b_i$  to  $\sum_i a_i b_i$ .
- (2) Suppose  $M = ApB \oplus N$  for some  $(A, B)$ -bimodule  $N$ . Now  $T$  is the tensor ring over  $A \sqcup_{(M,p)} B$  of

$$(A \sqcup_{(M,p)} B) \otimes_A N \otimes_B (A \sqcup_{(M,p)} B).$$

We may vary the choice of  $p$  as the following example illustrates:

- Example 2.9.** Suppose  $A = B = M = \mathbb{Z}$  and  $p = 2$ . In this case  $T = \mathbb{Z}[\frac{1}{2}]$  and  $\rho_A = \rho_B$  is the inclusion of  $\mathbb{Z}$  in  $\mathbb{Z}[\frac{1}{2}]$  while  $\rho_M(n) = n/2$  for all  $n \in \mathbb{Z}$ .

Example 2.9 can be verified by direct calculation using Theorem 2.4 or deduced from part (1) of Example 2.5 by setting  $a_0 = b_0 = 2$  in the following more general proposition. Before stating it, let us remark that the universal property of  $T = T(M, p)$  implies that  $T(M, p)$  is functorial in  $(M, p)$ . An  $(A, B)$ -bimodule morphism  $\phi : M \rightarrow M'$  with  $\phi(p) = p'$  induces a ring morphism  $T(M, p) \rightarrow T(M', p')$ .

**Proposition 2.10.** *Suppose  $A$  and  $B$  are rings,  $M$  is an  $(A, B)$ -bimodule and  $p \in M$ . If  $a_0 \in A$  and  $b_0 \in B$  satisfy  $a_0 m = m b_0$  for all  $m \in M$ , then:*

- (1) *The element  $\rho_M(a_0 p) = x_{a_0 p} = x_{p b_0}$  is central in  $T(M, p)$ .*
- (2) *The ring morphism  $\phi : T(M, p) \rightarrow T(M, a_0 p) = T(M, p b_0)$  induced by the bimodule morphism  $\phi : M \rightarrow M; m \mapsto a_0 m = m b_0$  is the universal localization of  $T(M, p)$  making invertible the element  $x_{a_0 p}$ .*

Since  $x_{a_0 p}$  is central, each element in  $T(M, a_0 p)$  can be written as a fraction  $\alpha/\beta$  with numerator  $\alpha \in T(M, p)$  and denominator  $\beta = x_{a_0 p}^r$  for some nonnegative integer  $r$ .

Having described universal localization of the ring  $R$  in Theorem 2.4 we may also describe the universal localization  $\sigma^{-1}R \otimes_R N$  of a left  $R$ -module  $N$ .

For the convenience of the reader let us first recall the structure of modules over a triangular matrix ring.

**Lemma 2.11.** *Every left  $R$ -module  $N$  can be written canonically as a triple*

$$(N_A, N_B, f : M \otimes_B N_B \rightarrow N_A)$$

where  $N_A$  is a left  $A$ -module,  $N_B$  is a left  $B$ -module and  $f$  is a morphism of left  $A$ -modules.

A proof of this lemma is included in Section 3 below. Localization of modules can be expressed as follows.

**Theorem 2.12.**<sup>2</sup> For any left  $R$ -module  $N = (N_A, N_B, f)$  the localization left  $\sigma^{-1}R$ -module  $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$  is isomorphic to  $\begin{pmatrix} L \\ L \end{pmatrix}$  with  $\sigma^{-1}R = M_2(T)$ ,  $T = T(M, p)$ ,  $L$  the left  $T$ -module defined by

$$\begin{aligned} L &= (T \ T) \otimes_R N \\ &= \operatorname{coker} \left( \begin{pmatrix} 1 \otimes f \\ g \otimes 1 \end{pmatrix} : T \otimes_A M \otimes_B N_B \rightarrow (T \otimes_A N_A) \oplus (T \otimes_B N_B) \right) \end{aligned}$$

with  $g$  the  $(T, B)$ -bimodule morphism

$$g : T \otimes_A M \rightarrow T ; t \otimes m \mapsto -tx_m,$$

and  $M_2(T)$  acting on the left of  $\begin{pmatrix} L \\ L \end{pmatrix}$  by matrix multiplication.

### 3. PROOFS

The remainder of this paper is devoted to the proofs of Theorem 2.4, Proposition 2.10 and Theorem 2.

**3.1. Localization as pushout.** Before proving Theorem 2.4 we show that there is a pushout diagram

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \\ R & \longrightarrow & \sigma^{-1}R \end{array}$$

where  $\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$ . Bergman observed [4, p. 71] that more generally, up to Morita equivalence every localization  $R \rightarrow \sigma^{-1}R$  appears in such a pushout diagram.

It suffices to check that the lower horizontal arrow in any pushout

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \theta \downarrow \\ R & \xrightarrow{i} & S \end{array}$$

is i)  $\sigma$ -inverting and ii) universal among  $\sigma$ -inverting ring morphisms. The universal property of a pushout will be shown to be the universal property of a universal localization, so that such a commutative diagram is a pushout if and only if  $S$  is a universal localization  $\sigma^{-1}R$ .

<sup>2</sup>This corrects Theorem 2.12 in the preprint version arXiv:math.RA/0407407.

i) The map  $\text{id} \otimes \sigma : S \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow S \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$  has inverse given by the composite

$$S \otimes_R \begin{pmatrix} M \\ B \end{pmatrix} \subset S \otimes_R R \cong S \xrightarrow{\gamma} S \cong S \otimes_R R \rightarrow S \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where  $\gamma$  multiplies on the right by  $\theta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ .

ii) If  $i' : R \rightarrow S'$  is a  $\sigma$ -inverting ring morphism, then there is an inverse  $\psi : S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix} \rightarrow S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix}$  to  $\text{id} \otimes \sigma$ . It is argued shortly below that there is a (unique) diagram

$$(1) \quad \begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \theta' \\ R & \xrightarrow{i} & S \\ & \searrow i' & \swarrow \psi \\ & & S' \end{array}$$

where  $\theta'$  sends  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  to  $\psi \left( 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \subset S'$ . Since  $S$  is a pushout, there is a unique morphism  $S \rightarrow S'$  to complete the diagram and so  $i'$  factors uniquely through  $i$ .

To show uniqueness of (1), note that in  $S'$  multiplication on the right by  $\theta' \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  must coincide with the morphism

$$\begin{pmatrix} 0 & 0 \\ \text{id} \otimes \sigma & 0 \end{pmatrix} : S' \otimes \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes \begin{pmatrix} M \\ B \end{pmatrix} \longrightarrow S' \otimes \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes \begin{pmatrix} M \\ B \end{pmatrix},$$

so multiplication on the right by  $\theta' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  coincides with  $\begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix}$ . Now  $1 \in S'$  may be written

$$\left( 1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

so  $\theta' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \psi \left( 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ . The reader may verify that this formula demonstrates the existence of a commutative diagram (1).

**3.2. Identifying  $\sigma^{-1}R$ .**

*Proof of Theorem 2.4.* It suffices to show that the diagram of ring morphisms

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \\ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow{\rho} & \begin{pmatrix} T & T \\ T & T \end{pmatrix} \end{array}$$

is a pushout, where  $T = T(M, p)$ ,  $\rho = \begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}$  and  $\alpha$  is defined as in Section 3.1. Given a diagram of ring morphisms

$$(2) \quad \begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ \alpha \downarrow & & \downarrow \\ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow{\rho} & \begin{pmatrix} T & T \\ T & T \end{pmatrix} \\ & \searrow \rho' & \nearrow \theta \\ & & S \end{array}$$

we must show that there is a unique morphism  $\gamma$  to complete the diagram. The map  $\theta$  induces a decomposition of  $S$  as a matrix ring  $M_2(S') = \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$  with  $S'$  the centralizer of  $\theta(M_2(\mathbb{Z})) \subset S$ . In particular,  $\theta(e_{ij}) = e_{ij}$  for  $i, j \in \{1, 2\}$ . Any morphism  $\gamma$  which makes the diagram commute must be of the form  $\gamma = M_2(\gamma')$  for some ring morphism  $\gamma' : T \rightarrow S'$  (e.g. Cohn [9, p. 1] or Lam [15, (17.7)]). Commutativity of the diagram implies that  $\rho'$  also respects the  $2 \times 2$  matrix structure and we may write

$$\rho' = \begin{pmatrix} \rho'_A & \rho'_M \\ 0 & \rho'_B \end{pmatrix} : \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \longrightarrow \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$$

with  $\rho'_M(p) = 1$  as one sees by considering the images of  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  in  $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$  under the maps in diagram (2) above. Since  $\rho'$  is a ring morphism, one finds

$$\begin{pmatrix} \rho'_A(aa') & \rho'_M(am' + mb') \\ 0 & \rho'_B(bb') \end{pmatrix} = \begin{pmatrix} \rho'_A(a)\rho'_A(a') & \rho'_A(a)\rho'_M(m') + \rho'_M(m)\rho'_B(b') \\ 0 & \rho'_B(b)\rho'_B(b') \end{pmatrix}$$

for all  $a, a' \in A, b, b' \in B$  and  $m, m' \in M$ . Hence the maps  $\rho'_A : A \rightarrow S'$  and  $\rho'_B : B \rightarrow S'$  are ring morphisms and  $\rho'_M$  is a morphism of  $(A, B)$ -bimodules. Thus  $S'$  is an  $(A, M, B)$ -ring with respect to the maps  $\rho'_A, \rho'_M, \rho'_B$  such that  $\rho'_M(p) = 1$ . By the universal property of  $T$  there exists a unique morphism  $\gamma' : T \rightarrow S'$  such that  $M_2(\gamma') : M_2(T) \rightarrow M_2(S') = S$  completes diagram (2) above.  $\square$

*Proof of Proposition 2.10.* 1. In  $T(M, p)$  we have  $x_{a_0p}x_m = x_{a_0m} = x_{mb_0} = x_mx_{pb_0} = x_mx_{a_0p}$  for all  $m \in M$ .

2. The map  $\phi : M \rightarrow M; m \mapsto a_0m$  induces

$$(3) \quad \begin{aligned} \phi : T(M, p) &\rightarrow T(M, a_0p), \\ x_m &\mapsto x_{a_0m}. \end{aligned}$$

In particular,  $\phi(x_{a_0p}) = x_{a_0^2p} \in T(M, a_0p)$  and we have

$$x_{a_0^2p}x_p = x_{a_0(a_0p)}x_p = x_{a_0p} = 1 = x_{pb_0} = x_px_{pb_0^2} = x_px_{a_0^2p},$$

so  $\phi(x_{a_0p})$  is invertible.

We must check that (3) is universal. If  $f : T(M, p) \rightarrow S$  is a ring morphism and  $f(x_{a_0p})$  is invertible, we claim that there exists a unique  $\tilde{f} : T(M, a_0p) \rightarrow S$  such that  $f\phi = \tilde{f}$ .

*Uniqueness:* Suppose  $\tilde{f}\phi = f$ . For each  $m \in M$  we have

$$\tilde{f}(x_{a_0m}) = \tilde{f}\phi(x_m) = f(x_m).$$

Now  $f(x_{a_0p})\tilde{f}(x_m) = \tilde{f}\phi(x_{a_0p})\tilde{f}(x_m) = \tilde{f}(x_{a_0(a_0p)x_m}) = \tilde{f}(x_{a_0m}) = f(x_m)$ , so

$$(4) \quad \tilde{f}(x_m) = (f(x_{a_0p}))^{-1}f(x_m).$$

*Existence:* It is straightforward to check that equation (4) provides a definition of  $\tilde{f}$  which respects the relations (+), (a), (b) and (id) in  $T(M, a_0p)$ . Relation (b), for example, is proved by the equations

$$\tilde{f}(x_m)\tilde{f}(x_{a_0pb}) = f(x_{a_0p})^{-1}f(x_m)f(x_{pb}) = f(x_{a_0p})^{-1}f(x_{mb}) = \tilde{f}(x_{mb})$$

and the other relations are left to the reader. □

**3.3. Module localization.** We turn finally to the universal localization  $\sigma^{-1}R \otimes_R N$  of an  $R$ -module  $N$ .

*Proof of Lemma 2.11.* If  $N$  is a left  $R$ -module, set  $N_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} N$  and set  $N_B = N/N_A$ . If  $m \in M$  and  $n_B \in N_B$  choose a lift  $x \in N$  and define the map  $f : M \otimes N_B \rightarrow N_A$  by  $f(m \otimes n_B) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} x$ . Conversely, given a triple  $(N_A, N_B, f)$  one recovers a left  $R$ -module  $\begin{pmatrix} N_A \\ N_B \end{pmatrix}$  with

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} = \begin{pmatrix} an_A + f(m \otimes n_B) \\ bn_B \end{pmatrix}$$

for all  $a \in A, b \in B, m \in M, n_A \in N_A, n_B \in N_B$ . □

*Proof of Theorem 2.* As in the statement, let  $T = T(M, p)$  and define the left  $T$ -module

$$L = \text{coker} \left( \begin{pmatrix} 1 & f \\ g & 1 \end{pmatrix} : T \otimes_A M \otimes_B N_B \rightarrow (T \otimes_A N_A) \oplus (T \otimes_B N_B) \right).$$

We shall establish an isomorphism of left  $T$ -modules

$$(5) \quad (T \quad T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \cong L$$

and leave to the reader the straightforward deduction that there is an isomorphism of  $\sigma^{-1}R$ -modules

$$\sigma^{-1}R \otimes_R N = \begin{pmatrix} T & T \\ T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \cong \begin{pmatrix} L \\ L \end{pmatrix}.$$

The left  $T$ -module morphism

$$\alpha : L \rightarrow (T \quad T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix};$$

$$(t \otimes n_A, t' \otimes n_B) \mapsto (t \quad 0) \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} + (0 \quad t') \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix}$$

is well defined, since

$$\begin{aligned}
 & \alpha(t \otimes_A f(m, n_B), g(t, m) \otimes_B n_B) \\
 &= \alpha(t \otimes_A f(m, n_B), -tx_m \otimes_B n_B) \\
 &= (t \ 0) \otimes_R \begin{pmatrix} f(m, n_B) \\ 0 \end{pmatrix} - (0 \ tx_m) \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} \\
 &= (t \ 0) \otimes_R \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ n_B \end{pmatrix} - (t \ 0) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} \\
 &= 0 \in (T \ T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix}.
 \end{aligned}$$

The left  $T$ -module morphism

$$\beta : (T \ T) \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \rightarrow L; (t \ t') \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} \mapsto (t \otimes n_A, t' \otimes n_B)$$

is well defined, since

$$\begin{aligned}
 & \beta((t \ t') \otimes_R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} - (t \ t') \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix}) \\
 &= (t \otimes (an_A + f(m, n_b)), t' \otimes bn_B) - (ta \otimes n_A, (tx_m + t'b) \otimes n_B) \\
 &= (t \otimes f(m, n_B), -tx_m \otimes n_B) \\
 &= (1 \otimes f, g \otimes 1)(t \otimes m \otimes n_B) = 0 \in L.
 \end{aligned}$$

It is immediate that  $\beta\alpha = \text{id}$ . To prove (5) we must check that  $\alpha\beta = \text{id}$  or in other words, that

$$(t \ t') \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} = (t \ 0) \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} + (0 \ t') \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix}.$$

This equation follows from the next two calculations:

$$\begin{aligned}
 (t \ 0) \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} &= (t \ 0) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} = 0; \\
 (0 \ t') \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} &= (0 \ t') \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} = 0.
 \end{aligned}$$

□

#### REFERENCES

- [1] P. Ara, M. A. González-Barroso, K. R. Goodearl, and E. Pardo. Fractional skew monoid rings. *Journal of Algebra*, 278(1):104–126, 2004. MR2068068 (2005f:16042)
- [2] D. J. Benson. *Representations and cohomology. I. Basic Representation Theory of finite groups and associative algebras*. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, 1995. MR1110581 (92m:20005)
- [3] G. M. Bergman. Modules over coproducts of rings. *Transactions of the American Mathematical Society*, 200:1–32, 1974. MR0357502 (50:9970)
- [4] G. M. Bergman. Coproducts and some universal ring constructions. *Transactions of the American Mathematical Society*, 200:33–88, 1974. MR0357503 (50:9971)
- [5] G. M. Bergman and W. Dicks. Universal derivations and universal ring constructions. *Pacific Journal of Mathematics*, 79(2):293–337, 1978. MR0531320 (81b:16024)

- [6] P. M. Cohn. Localization in general rings, a historical survey. Proceedings of the Conference on Noncommutative Localization in Algebra and Topology, ICMS, Edinburgh, 29-30 April, 2002, *London Mathematical Society Lecture Notes* 330, Cambridge University Press, 5–23, 2006.
- [7] P. M. Cohn. *Free Rings and their Relations*. London Mathematical Society Monographs, 2. Academic Press, London, 1971. MR0371938 (51:8155)
- [8] P. M. Cohn. Rings of fractions. *American Mathematical Monthly*, 78:596–615, 1971. MR0285561 (44:2779)
- [9] P. M. Cohn. *Free Rings and their Relations*. London Mathematical Society Monographs, 19. Academic Press, London, second edition, 1985. MR0800091 (87e:16006)
- [10] P.M. Cohn and W. Dicks. Localization in semifirs. II. *J. London Math.Soc. (2)*, 13(3):411–418, 1976. MR0424856 (54:12814)
- [11] W. Dicks and E. Sontag. Sylvester domains. *J. Pure Appl. Algebra*, 13(3):243–275, 1978. MR0509164 (80j:16014)
- [12] M. Farber and P. Vogel. The Cohn localization of the free group ring. *Mathematical Proceedings of the Cambridge Philosophical Society*, 111(3):433–443, 1992. MR1151322 (93b:16049)
- [13] A. Haghany and K. Varadarajan. Study of formal triangular matrix rings. *Communications in Algebra*, 27(11):5507–5525, 1999. MR1713049 (2000g:16003)
- [14] A. Haghany and K. Varadarajan. Study of modules over formal triangular matrix rings. *Journal of Pure and Applied Algebra*, 147(1):41–58, 2000. MR1744654 (2000k:16004)
- [15] T. Y. Lam. *Lectures on Modules and Rings*. Number 189 in Graduate Texts in Mathematics. Springer, New York, 1999. MR1653294 (99i:16001)
- [16] A. A. Ranicki. Noncommutative localization in topology. Proceedings of the Conference on Noncommutative Localization in Algebra and Topology, ICMS, Edinburgh, 29-30 April, 2002. arXiv:math.AT/0303046, *London Mathematical Society Lecture Notes* 330, Cambridge University Press, 81–102, 2006.
- [17] A. H. Schofield. *Representations of rings over skew fields*, Volume 92 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1985. MR0800853 (87c:16001)

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