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MORAVA K-THEORY RINGS FOR THE DIHEDRAL, SEMIDIHEDRAL AND GENERALIZED QUATERNION GROUPS IN CHERN CLASSES

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Abstract. Morava K-theory rings of classifying spaces of the dihedral, semidihedral and generalized quaternion groups are presented in terms of Chern classes.

1. Introduction and statements

The study of Morava K-theory of groups has attracted attention in the literature, for the generalized braid groups (also called Brieskorn groups or Artin groups); see for example [4].

The classifying spaces of p-groups with a cyclic maximal subgroup have been considered in [5, 10, 11, 12]. In [11, 12] it was shown that $K(s)^*$ of these spaces is generated as a $K(s)^*(pt)$-module by Chern classes of complex vector bundles. However, the multiplicative structure in terms of Chern classes has been determined in [11] only modulo certain indeterminacy. As for [5, 10], there the multiplicative structure is given completely but in terms of artificial generators not equal to Chern classes. Our aim here is to determine the aforementioned multiplicative structure completely in terms of Chern classes by applying the formula for transfer of the first Chern class along double coverings [2], [3].

In this paper we will consider the dihedral, semidihedral and generalized quaternion 2-groups. The modular and quasidihedral groups will be considered in [1].

Let

$$G = \langle a, b \mid a^{2^{m+1}} = 1,\ b^2 = a^e, bab^{-1} = a^r \rangle, \quad m \geq 1,$$

and either $e = 0$, $r = -1$ (the dihedral group $D_{2^{m+2}}$ of order 2^{m+2}), $e = 2^m$, $r = -1$ (the generalized quaternion group $Q_{2^{m+2}}$) or $m \geq 2$, $e = 0$, $r = 2^m - 1$ (the semidihedral group $SD_{2^{m+2}}$).

Consider the following Chern classes c, x, c_1, c_2 of dimensions $|c| = |x| = |c_1| = 2$, $|c_2| = 4$:

$$c = c_1(\eta_1),\ \eta_1 : G/\langle a \rangle \cong \mathbf{Z}/2 \to \mathbb{C}^*,\ b \mapsto -1,$$
$$x = c_1(\eta_2),\ \eta_2 : G/\left\langle a^2, b \right\rangle \cong \mathbf{Z}/2 \to \mathbb{C}^*,\ a \mapsto -1,$$

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and $c_i = c_i(\xi_{\pi_1})$, where

$$\xi_{\pi_1} \to B\langle a, b\rangle$$

is the plane bundle transferred from the canonical line bundle $\xi \to B\langle a\rangle$, for the double covering

$$\pi_1 : B\langle a\rangle \to B\langle a, b\rangle$$

corresponding to η_1.

Theorem 1.1. i) *$K(s)^*(BG) = K(s)^*[c, x, c_2]/R$ and the relations R are determined by*

(a)

$$c^{2^s} = x^{2^s} = 0,$$

(b)

$$v_s c c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} c^{2^s - 2^i + 1} c_2^{2^{i-1}} + \begin{cases} 0 & \text{if } G \text{ is dihedral,} \\ c^2 & \text{if } G \text{ is quaternion,} \\ cx & \text{if } G \text{ is semidihedral,} \end{cases}$$

(c)

$$v_s^2 c_2^{2^s} = \begin{cases} cx + x^2 & \text{if } G = D_8, \\ c^2 + cx + x^2 & \text{if } G = Q_8 \end{cases}$$

and for $m > 1$,

$$v_s^{2\kappa(m)} c_2^{2^{ms}} = cx + x^2$$

for G of all three types,

(d)

$$v_s x c_2^{2^{s-1}} = v_s \sum_{i=1}^{s-1} x^{2^s - 2^i + 1} c_2^{2^{i-1}} + \begin{cases} cx + x^2 & \text{if } G = D_8, \\ x^2 & \text{if } G = Q_8; \end{cases}$$

for $m > 1$,

$$v_s x c_2^{2^{s-1}} = v_s x \sum_{i=1}^{s-1} c^{2^s - 2^i} c_2^{2^{i-1}} + \sum_{i=1}^{ms} v_s^{1+\kappa(m)+2^{ms}-2^i} c_2^{(2^{ms}+1)2^{s-1} - (2^s-1)2^{i-1}}$$
$$+ \begin{cases} 0 & \text{if } G \text{ is dihedral,} \\ cx & \text{if } G \text{ is quaternion or semidihedral,} \end{cases}$$

where $\kappa(m) = \frac{2^{ms}-1}{2^s-1}$.

ii) $c^2 x = cx^2$, $c_1^{2^{ms}+1} = 0$, $c_2^{(2^{ms}+1)2^{s-1}} = 0$.

2. Preliminaries

Together with the covering π_1 we can consider the covering

$$\pi_2 : B\left\langle a^2, b\right\rangle \to B\langle a, b\rangle,$$

corresponding to η_2. Then let

$$\eta_{\pi_2} \to BG$$

be the transferred line bundle associated with the double covering $\left\langle a^4, b\right\rangle \to \left\langle a^2, b\right\rangle$. The bundles ξ_{π_1}, η_{π_2} coincide if $m = 1$, but if $m > 1$, then

$$\eta_{\pi_2} = (\xi^{\otimes 2^{m-1}})_{\pi_1}. \tag{1}$$

The following bundle relations hold.

Lemma 2.1. i) $\eta_i^{\otimes 2} = \mathbb{C}$, $\eta_1 \otimes \xi_{\pi_1} = \xi_{\pi_1}$;

ii) $\eta_i \otimes \eta_{\pi_2} = \eta_{\pi_2}$;

iii) $\eta_{\pi_2}^{\otimes 2} = \mathbb{C} \oplus \eta_1 \oplus \eta_2 \oplus \eta_1 \otimes \eta_2$;

iv) $\det \xi_{\pi_1}$ *is* η_1 *if* G *is dihedral, the trivial bundle* $\mathbb{C}$ *if* G *is quaternion and* $\eta_1 \otimes \eta_2$ *if* G *is semidihedral, and for* $m > 1$ *one has* $\det \eta_{\pi_2} = \eta_1$ *in all three cases;*

v) $((\xi^{\otimes 2^i})_{\pi_1})^{\otimes 2} = (\xi^{\otimes 2^{i+1}})_{\pi_1} \oplus \mathbb{C} \oplus \eta_1$, *for* $1 \le i < m-1$. *The bundle* $\xi_{\pi_1} \otimes \xi_{\pi_1} = (\xi^{\otimes 2})_{\pi_1} \oplus (\xi \otimes \xi^{\otimes r})_{\pi_1}$ *is* $(\xi^{\otimes 2})_{\pi_1} \oplus \mathbb{C} \oplus \eta_1$ *if* G *is dihedral or quaternion and is* $(\xi^{\otimes 2})_{\pi_1} \oplus \eta_1 \otimes \eta_2 \oplus \eta_2$ *if* G *is semidihedral.*

Proof. These relations are the consequences of the Frobenius reciprocity of the transfer in complex K-theory. For example,

$$\begin{aligned}\eta_{\pi_2}^{\otimes 2} &= (\xi^{2^{m-1}})_{\pi_1}^{\otimes 2} = (\xi^{2^{m-1}} \otimes \pi_1^*((\xi^{2^{m-1}})_{\pi_1}))_{\pi_1} = (\xi^{2^m} \oplus \mathbb{C})_{\pi_1} \\ &= \eta_2 \otimes (\mathbb{C})_{\pi_1} \oplus \mathbb{C} \oplus \eta_1 = \eta_2 \otimes (\mathbb{C} \oplus \eta_1) \oplus \mathbb{C} \oplus \eta_1. \qquad \square\end{aligned}$$

We recall the transfer formula from [3] (see also [2]).

Let $X \to X/\pi$ be a regular double covering defined by a free involution on X, let $\xi \to X$ be a complex line bundle, let ξ_π be the transferred bundle and let

$$\mathrm{Tr}_\pi^* : K(s)^*(X) \to K(s)^*(X/\pi)$$

be the associated transfer homomorphism [8], [6]. Then

$$c_1(\xi_\pi) = c_1(\psi) + v_s \sum_{i=1}^{s-1} c_1(\psi)^{2^s - 2^i} c_2(\xi_\pi)^{2^{i-1}} + \mathrm{Tr}_\pi^*(c_1(\xi)), \tag{2}$$

where $\psi \to X/\pi$ is the complex line bundle associated to the covering $X \to X/\pi$.

The following lemma is an easy consequence of the recursive formula for the FGL given in 4.3.9 of [9]. See [2], Lemma 5.3.

Lemma 2.2. i) *For the Honda formal group law at* $p = 2$, $s > 1$, *one has* $F(y,z) = y + z + v_s(yz)^{2^{s-1}}$ *modulo* $y^{2^{2(s-1)}}$ *(or modulo* $z^{2^{2(s-1)}}$*).*

ii) $F(y,z) = y + z + v_s \Phi(v_s, y, z)^{2^{s-1}}$, *where* $\Phi(v_s, y, z) = yz + v_s(yz)^{2^{s-1}}(y+z)$ *modulo* $(yz)^{2^{s-1}}(y+z)^{2^{s-1}}$.

For two line bundles with the Chern classes y and z, respectively, $\Phi(v_s, y, z)$ can be regarded as the $K(s)^*$ orientation class of their sum.

Lemma 2.3. *Let* $m > 1$ *and either* $r = -1$ *or* $r = 2^m - 1$. *Then one has in* $K(s)^*[u]/(u^{2^{(m+1)s}})$,

$$u^{2^{ms}} = \sum_{i=1}^{ms} v_s^{2^{ms} - 2^i} (u[r](u))^{2^{(m+1)s-1} - (2^s-1)2^{i-1}} + [r](u)(u + [r](u))^{2^{ms}-1}.$$

Proof. The obvious decomposition in $\mathbb{F}_2[y,z]$,

$$y^{2^k} = \sum_{i=1}^{k} (y+z)^{2^k - 2^i} (yz)^{2^{i-1}} + y(y+z)^{2^k - 1},$$

implies for $y = u$, $z = [r](u)$, $k = ms$ that

$$u^{2^{ms}} = \sum_{i=1}^{ms} (u + [r](u))^{2^{ms} - 2^i} (u[r](u))^{2^{i-1}} + u(u + [r](u))^{2^{ms}-1}.$$

We want to equate the monomials

$$(u+[r](u))^{2^{ms}-2^i} = (u+[r](u))^{2^i+\ldots+2^{ms-1}}$$

to the monomials

$$v_s^{2^{ms}-2^i}(u[r](u))^{(2^{ms}-2^i)2^{s-1}} = v_s^{2^i+\ldots+2^{ms-1}}(u[r](u))^{(2^i+\ldots+2^{ms-1})2^{s-1}}$$

by the equation $(u+[r](u))^2 = v_s^2(u[r](u))^{2^s}$ modulo some irrelevant factor as follows.

The nilpotence degree for u is $2^{(m+1)s}$, hence is $2^{(m+1)s-1}$ for $u[r](u)$. Then as it is 2^s for $F(u,[2^m-1])$ (whereas $F(u,[-1](u))=0$), the nilpotence degree for $u+[r](u)$ is 2^{ms} by Lemma 2.2 ii).

Thereupon it suffices to show

$$(u+[r](u))^2 = v_s^2(u[r](u))^{2^s}\ mod\ (u+[r](u))^4.$$

Lemma 2.2 ii) implies

$$(u+[r](u))^2 = v_s^2(u[r](u))^{2^s} + F(u,[r](u))^2\, mod\ (u+[r](u))^{2^s}$$

and the dihedral and quaternion cases follow.

For the semidihedral group one has $F(u,[2^m-1](u)) = v_s^{\kappa(m)}u^{2^{ms}}$. Also, $u^{2^{ms+1}} = (u[r]u)^{2^{ms}}$ as $u^{2^{ms}} = ([r](u))^{2^{ms}}$. Therefore, one obtains modulo $(u+[r](u))^{2^s}$ ignoring powers of v_s,

$$(u[r](u))^{2^s} = (u+[r](u))^2 + (u[r](u))^{2^{ms}},$$

$F(u,[r](u))^2 = (u[r](u))^{2^{ms}} = ((u+[r](u))^2+(u[r](u))^{2^{ms}})^{2^{ms-s}} = 0$ as $ms-s+1 > s$. The result follows. □

3. Proofs

As mentioned in the introduction, it was proved in [11] that as a $K(s)^*(pt)$-module, $K(s)^*$ of the spaces we consider is generated by the Chern classes c, x, c_2 defined above. Let $\tilde{c_1}$, $\tilde{c_2}$ be the Chern classes of the bundle η_{π_2}.

Lemma 2.1 i) implies $c^{2^s}=0$ and $x^{2^s}=0$ as $[2](c)=v_sc^{2^s}=0$ and similarly for x.

Let

$$c_1^* = c_1 + c + v_s\sum_{i=1}^{s-1} c^{2^s-2^i}c_2^{2^{i-1}},\ \ c_1^{**} = \tilde{c}_1 + x + v_s\sum_{i=1}^{s-1} x^{2^s-2^i}\tilde{c}_2^{2^{i-1}}. \tag{3}$$

By (2), $c_1^* \in \operatorname{Im}\operatorname{Tr}^*_{\pi_1}$, $c_1^{**} \in \operatorname{Im}\operatorname{Tr}^*_{\pi_2}$, hence $c_1^{2^s} \in \operatorname{Im}\operatorname{Tr}^*_{\pi_1}$, $\tilde{c}_1^{2^s} \in \operatorname{Im}\operatorname{Tr}^*_{\pi_2}$ as $c^{2^s}=x^{2^s}=0$.

By the Frobenius reciprocity $cc_1^*=0$, hence by (2) $c^{2^{s-1}}c_1^{2^{s-1}}=0$ and $x^{2^{s-1}}c_1^{2^{s-1}} = c^{2^{s-1}}x^{2^{s-1}}$ modulo $Tr^*_{\pi_1}(u)$, $u=c_1(\xi)$. From (2) one obtains $c_1^{2^{s-1}} = c^{2^{s-1}}$ modulo $Tr^*_{\pi_1}(u)$. Hence Lemma 2.2 ii) implies modulo $Tr^*_{\pi_1}(u)$,

$$c_1(\det\xi_{\pi_1}) = c_1 + v_sc_2^{2^{s-1}}. \tag{4}$$

Then note $F(c,x) = c+x+v_sc^{2^{s-1}}x^{2^{s-1}}$, hence combining (3) and (4) we get modulo $Tr^*_{\pi_1}(u)$ and $c^{2^{s-1}}x^{2^{s-1}}$,

$$v_sc_2^{2^{s-1}} + v_s\sum_{i=1}^{s-1} c^{2^s-2^i}c_2^{2^{i-1}} = \begin{cases} 0 & \text{if } G \text{ is dihedral,}\\ c & \text{if } G \text{ is quaternion,}\\ x & \text{if } G \text{ is semidihedral.}\end{cases} \tag{5}$$

Also, one has

$$c_1(det\ \eta_{\pi_2}) = \tilde{c}_1 + v_s\tilde{c}_2^{2^{s-1}} + v_s\tilde{c}_1^{2^s}. \tag{6}$$

To see (6) we need the relations ii) of Theorem 1.1. These are consequences of the relations (7)–(10).

Lemma 2.1 v) and (1) imply that modulo c, x the Chern classes $\tilde{c}_1$, $\tilde{c}_2$ coincide with the first and second Chern classes of $(\xi_{\pi_1})^{\otimes 2^{m-1}}$, respectively,

$$\tilde{c}_1 = v_s^{\frac{2^{(m-1)s}-1}{2^s-1}} c_1^{2^{(m-1)s}}; \quad \tilde{c}_2 = v_s^{\frac{2(2^{(m-1)s}-1)}{2^s-1}} c_2^{2^{(m-1)s}}. \tag{7}$$

On the other hand, consecutively equating Chern classes of both sides of the equation in Lemma 2.1 iii) gives, respectively,

$$\tilde{c}_1^{2^s} = c^{2^{s-1}} x^{2^{s-1}}; \tag{8}$$

for $m = 1$,

$$v_2^2 c_2^{2^s} = \begin{cases} cx + x^2 & \text{if } G = D_8, \\ c^2 + cx + x^2 & \text{if } G = Q_8; \end{cases} \tag{9}$$

for $m > 1$,

$$v_2^2 \tilde{c}_2^{2^s} = cx + x^2$$

in all cases, and

$$c^2x + cx^2 = 0, \tag{10}$$

and we get (c) and relations ii). Here we use the splitting principle and write formally $\eta_{\pi_2} = \lambda_1 \oplus \lambda_2$, $\eta_{\pi_2}^{\otimes 2} = \lambda_1^{\otimes 2} \oplus \lambda_2^{\otimes 2} \oplus 2\lambda_1 \otimes \lambda_2$. Also, we take into account that the determinant $\lambda_1 \otimes \lambda_2$ is known by Lemma 2.1 iv). Let $m > 1$, $\lambda_1 \otimes \lambda_2 = \eta_1$. Then by the first equation for the Chern classes $v_s\tilde{c}_1^{2^s} = c+x+c+x+v_s c^{2^{s-1}} x^{2^{s-1}} \Rightarrow$ (8). By (7) and (a) $\tilde{c}_1^{2^s} \in ImTr^*_{\pi_1}$. Also, as before, $\tilde{c}_1^{2^s} \in ImTr^*_{\pi_2}$. Hence $c\tilde{c}_1^{2^s} = x\tilde{c}_1^{2^s} = 0$ and by (8) $c^i x^j = 0$ for $i+j > 2^s$. Multiplying (3) by $c^{2^{s-1}} x^{2^{s-1}}$ we get $c_1^{2^{ms}+1} = 0$; therefore, Lemma 2.2 implies $c_2^{(2^{ms}+1)2^{s-1}} = 0$. Then the second equation gives $v_s^2\tilde{c}_2^{2^s} + c^2 = cx + (c+x)(c + x + v_s c^{2^{s-1}} x^{2^{s-1}}) \Rightarrow$ (9). The third equation gives $0 = v_s\tilde{c}_1^{2^s} c^2 = cx(c + x + v_s c^{2^{s-1}} x^{2^{s-1}}) = cx(c+x)$ and (10) follows. Similarly, for $m = 1$, but $\tilde{c}_i = c_i$ and for $G = Q_8$ the determinant $\lambda_1 \otimes \lambda_2$ is trivial.

To prove (b) for $m > 1$ we raise (4) to the power $2^{ms-s} > 2^s$. One obtains

$$c_2^{2^{ms-1}} = 0 \ mod\ Tr^*_{\pi_1}(u). \tag{11}$$

By the Frobenius reciprocity of the transfer, (11) implies

$$cc_2^{2^{ms-1}} = 0. \tag{12}$$

Then as above $c^i x^j = 0$ for $i + j \geq 2^s + 1$. Multiplying (5) by c we get (b).

Now let $m = 1$. Then $\tilde{c}_i = c_i$ and $cc_1^* = 0$ by (2). Hence multiplying (6) by c we get (b).

Proof of (d). Let $m > 1$. By the above definitions one has $\pi_1^*(\xi_{\pi_1}) = \xi \oplus \xi^{\otimes r}$ and $\pi^*(\eta_2) = \xi^{\otimes 2^m}$. Then Lemma 2.3 implies

$$\begin{aligned} v_s^{-\kappa(m)}\pi_1^*(x) &= u^{2^{ms}} \\ &= \pi_1^*\Big(\sum_{i=1}^{ms} v_s^{2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}\Big) + [r](u)\pi_1^*(c_1^{2^{ms}-1}). \end{aligned} \tag{13}$$

We want to apply transfer to (13) after multiplying by u. By (2) $c_1^{2^{ms}-1} \in ImTr^*_{\pi_1}$ is in the annihilator of c, hence

$$Tr^*_{\pi_1}(u\ [r](u))c_1^{2^{ms}-1} = Tr^*_{\pi_1}(1)c_2c_1^{2^{ms}-1} = v_s c^{2^s-1}c_2c_1^{2^{ms}-1} = 0$$

and we get

$$Tr^*_{\pi_1}(u)(x + \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}) = 0. \tag{14}$$

Now multiplying (5) by $x + \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}$ and using (12) and (14), the dihedral and quaternion cases follow. For the semidihedral group it remains to show that

$$x(x + \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}) = cx. \tag{15}$$

Let us denote for ease of reading, $\Sigma = \sum_{i=1}^{ms} v_s^{\kappa(m)+2^{ms}-2^i} c_2^{2^{(m+1)s-1}-(2^s-1)2^{i-1}}$.

Then $\Sigma(x+\Sigma) = 0$ as (11) implies $\Sigma = 0$ modulo $Tr^*_{\pi_1}(u)$ and $\Sigma^2 = v_s^{2\kappa(m)}c_2^{2^{ms}}$ as the nilpotence degree of c_2 is $2^{(m+1)s-1} + 2^{s-1}$. Thus $x\Sigma = v_s^{2\kappa(m)}c_2^{2^{ms}}$ and (15) follows from (c).

Now let $m = 1$. Then $\tilde{c}_i = c_i$, $cc_1^* = 0$, $xc_1^{**} = 0$ by (2); and by (1) det η_{π_2} is η_1 (for $G = D_8$) or a trivial bundle (for $G = Q_8$). Hence multiplying (6) by x we get (d).

There remains to show that the given relations give a ring of correct rank, which is

$$2^{(m+1)s-1} + 2^{2s} - 2^{s-1}$$

according to the generalized character theory [7]. This follows by counting the obvious explicit bases of these rings according to Theorem 1.1: for $G = D_8$ or Q_8,

$$\{c^ic_2^j, x^ic_2^j, c^ixc_2^j, c_2^k | 1 \le i < 2^s, 0 \le j < 2^{s-1}, 0 \le k < 2^s\};$$

and for $m > 1$ and all three cases,

$$\{c^ic_2^j, c^ixc_2^j, c_2^k | 1 \le i < 2^s, 0 \le j < 2^{s-1}, 0 \le k < (2^{ms}+1)2^{s-1}\}.$$

Of course there are alternative bases. For example, if one considers cx as the decomposable in Theorem 1.1, then for $m > 1$ the $K(s)^*$ base for $K(s)^*(BG)$ is: for $G = D_{2^{m+2}}$,

$$\{c^ic_2^j, x^ic_2^j, c_2^k | 1 \le i < 2^s - j, 0 \le j < 2^s - 1, 0 \le k < (2^{ms}+1)2^{s-1}\};$$

for $G = Q_{2^{m+2}}$, $\{cc_2^i, xc_2^i, c_2^j | 0 \le i < (2^s-1)2^{s-1}, 0 \le j < (2^{ms}+1)2^{s-1}\}$; and for $G = SD_{2^{m+2}}$, $\{c^ic_2^j, xc_2^k, c_2^l | 1 \le i < 2^s - j, 0 \le j < 2^s - 1, 0 \le k < (2^s-1)2^{s-1}$, $0 \le l < (2^{ms}+1)2^{s-1}\}$.

A natural question arises about the relationship between our calculations and those of [10] and [12], in terms of an alternative generating set.

The authors are grateful to Mamuka Jibladze for computer calculations of the following example, and to Björn Schuster for helpful discussions clarifying this relationship.

Example ($K(2)^*(BD_8)$)**.** This example shows that the ring structures given in [10] have to be corrected. For D_8 they are correct modulo the minimal (one-dimensional) ideal, lying in the kernel of the restriction maps corresponding to all proper subgroups.

Let A be the version of $K(2)^*(BD_8)$ of Theorem 1.1 and B its [10] version. Then

$$\begin{aligned} A &= \mathbb{F}_2[v_2^{\pm 1}][c, x, c_2]/ \\ &\quad (c^4, x^4, c^3c_2 + cc_2^2, v_2x^3c_2 + v_2xc_2^2 + cx + x^2, v_2^2c_2^4 + cx + x^2), \\ B &= \mathbb{F}_2[v_2^{\pm 1}][y_1, y_2, \hat{c}_2]/ \\ &\quad (y_1^4, y_2^4, v_2^2\hat{c}_2^4 + v_2y_1\hat{c}_2^2, v_2y_1\hat{c}_2^2 + v_2y_2\hat{c}_2^2, v_2y_2\hat{c}_2^2 + y_1y_2). \end{aligned}$$

Choose the following basis in A over $\mathbb{F}_2[v_2^{\pm 1}]$,

$$\begin{aligned} \langle 1, c, x, c^2, cx, x^2, c_2, c^3, x^3, cc_2, xc_2, cx^2, cx^2c_2, c^2c_2, \\ x^2c_2, cxc_2, c_2^2, cc_2^2, xc_2^2, cxc_2^2, c_2^3, xc_2^3 \rangle \end{aligned}$$

and suppose there is a graded isomorphism $f : B \to A$. Then by dimension considerations,

$$\begin{aligned} f(y_1) &= \epsilon_{11}c + \epsilon_{12}x + \epsilon_{13}v_2c_2^2 + \epsilon_{14}v_2x^2c_2 + \epsilon_{15}v_2cxc_2 + \epsilon_{16}v_2c^2c_2 + \epsilon_{17}v_2^2xc_2^3, \\ f(y_2) &= \epsilon_{21}c + \epsilon_{22}x + \epsilon_{23}v_2c_2^2 + \epsilon_{24}v_2x^2c_2 + \epsilon_{25}v_2cxc_2 + \epsilon_{26}v_2c^2c_2 + \epsilon_{27}v_2^2xc_2^3, \\ f(\hat{c}_2) &= \alpha_1c_2 + \alpha_2c^2 + \alpha_3cx + \alpha_4x^2 + \alpha_5v_2xc_2^2 + \alpha_6v_2cc_2^2 + \alpha_7v_2cx^2c_2, \end{aligned}$$

where ϵ_{ij}, $\alpha_k \in \mathbb{F}_2$. Then, $y_1^4 = 0$ implies $(\epsilon_{11}c + \epsilon_{12}x + \epsilon_{13}v_2c_2^2 + \epsilon_{14}v_2x^2c_2 + \epsilon_{15}v_2cxc_2 + \epsilon_{16}v_2c^2c_2 + \epsilon_{17}v_2^2xc_2^3)^4 = \epsilon_{13}c_2^8 = \epsilon_{13}x^2c_2 + \epsilon_{13}cxc_2 + \epsilon_{13}v_2xc_2^3 = 0$, hence $\epsilon_{13} = 0$.

Similarly, $y_2^4 = 0$ implies $\epsilon_{23} = 0$.

Next, $f((y_1 - y_2)\hat{c}_2^2)c_2^5 = 0$ implies $(\epsilon_{12} + \epsilon_{22})\alpha_1cxc_2^2 = 0$. Necessarily, $\alpha_1 \neq 0$, since otherwise c_2 would not be in the image of f. Thus we have $\epsilon_{12} = \epsilon_{22}$. Moreover, these are not zero since otherwise x would not be in the image of f. Thus we have

$$\begin{aligned} f(y_1) &= \epsilon_{11}c + x + \epsilon_{14}v_2x^2c_2 + \epsilon_{15}v_2cxc_2 + \epsilon_{16}v_2c^2c_2 + \epsilon_{17}v_2^2xc_2^3, \\ f(y_2) &= \epsilon_{21}c + x + \epsilon_{24}v_2x^2c_2 + \epsilon_{25}v_2cxc_2 + \epsilon_{26}v_2c^2c_2 + \epsilon_{27}v_2^2xc_2^3, \\ f(\hat{c}_2) &= c_2 + \alpha_2c^2 + \alpha_3cx + \alpha_4x^2 + \alpha_5v_2xc_2^2 + \alpha_6v_2cc_2^2 + \alpha_7v_2cx^2c_2. \end{aligned}$$

Taking this into account, $f(y_1\hat{c}_2^2 - y_1y_2)c_2^2 = 0$ implies $\epsilon_{11} + \epsilon_{21} + \epsilon_{24} + \epsilon_{27} = 0$ and $f(y_2\hat{c}_2^2 - y_1y_2)c_2^2 = 0$ implies $\epsilon_{11} + \epsilon_{21} + \epsilon_{14} + \epsilon_{17} = 0$, whereas $f(y_1y_2 - \hat{c}_2^4)c_2^2 = 0$ implies $\epsilon_{11} + \epsilon_{21} + \epsilon_{14} + \epsilon_{24} + \epsilon_{17} + \epsilon_{27} = 1$. Hence $\epsilon_{11} + \epsilon_{21} = \epsilon_{14} + \epsilon_{17} = \epsilon_{24} + \epsilon_{27} = 1$.

But these relations imply that $(f(y_1)f(\hat{c}_2)^2 - f(y_2)f(\hat{c}_2)^2)x = cxc_2^2$, which should be actually zero as $(y_1 - y_2)\hat{c}_2^2 = 0$.

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