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ON BURCH'S INEQUALITY AND A REDUCTION SYSTEM OF A FILTRATION

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Abstract. Let $\mathcal{F} = \{F_n\}$ be a multiplicative filtration of a local ring such that the Rees algebra $\mathrm{R}(\mathcal{F})$ is Noetherian. We recall Burch's inequality for $\mathcal{F}$ and give an upper bound of the a-invariant of the associated graded ring $\mathrm{a}(\mathrm{G}(\mathcal{F}))$ using a reduction system of $\mathcal{F}$. Applying those results, we study the symbolic Rees algebra of certain ideals of dimension 2.

1. Introduction

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let I be an ideal of A. The analytic spread of I is defined by $\ell(I) = \dim A/\mathfrak{m} \otimes_A \mathrm{G}(I)$, where $\mathrm{G}(I)$ is the associated graded ring of I. Then we have $\ell(I) \leq \dim\, A - \inf\{\text{depth } A/I^n \mid n > 0\}$. This is a well-known result due to Burch [3, Corollary]. In this paper we will recall that such an inequality remains true for a general filtration.

We call a family of ideals $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a filtration if (i) $F_n \supseteq F_{n+1}$ for any n, (ii) $F_0 = A$ and $F_1 \neq A$, and (iii) $F_m F_n \subseteq F_{m+n}$ for any m, n. Once a filtration $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is given, then we can define the following graded rings:

$$\mathrm{R}(\mathcal{F}) = \textstyle\sum_{n\geq 0} F_n t^n \subseteq A[t] \quad \text{and} \quad \mathrm{G}(\mathcal{F}) = \mathrm{R}(\mathcal{F})/\mathfrak{A} = \bigoplus_{n\geq 0} F_n/F_{n+1}\,,$$

where t is an indeterminate over A and $\mathfrak{A} = \sum_{n\geq 0} F_{n+1}t^n$. These algebras are called the Rees algebra and the associated graded ring of $\mathcal{F}$, respectively. We assume that $\mathrm{R}(\mathcal{F})$ is Noetherian and $\dim \mathrm{R}(\mathcal{F}) = d + 1$ (cf. [5, (2.2) of Part II]). Then $\mathrm{G}(\mathcal{F})$ is also Noetherian and $\dim \mathrm{G}(\mathcal{F}) = d$. We denote by $\ell(\mathcal{F})$ the Krull dimension of $A/\mathfrak{m} \otimes_A \mathrm{G}(\mathcal{F})$. Then $\mathrm{ht}_A F_1 \leq \ell(\mathcal{F}) \leq \dim A$ and the following assertion is implicitly known to hold.

Theorem 1.1. *We have $\ell(\mathcal{F}) \leq \dim\, A - \inf\{\text{depth } A/F_n \mid n > 0\}$, and the equality holds if $\mathrm{G}(\mathcal{F})$ is a Cohen-Macaulay ring.*

As is noted in [9, Section 2], the analytic spread of a filtration is related to a reduction system. Let $a_0, a_1, \ldots, a_\ell$ be elements of A such that $a_0 = 0$ and $a_1 \in F_{k_1}, \ldots, a_\ell \in F_{k_\ell}$ for some positive integers $k_1, \ldots, k_\ell$. We put $k_0 = 0$. We say that $\{a_i \in F_{k_i}\}_{0\leq i\leq \ell}$ is a reduction system of $\mathcal{F}$ if $F_n = \sum_{i=0}^{\ell} a_i F_{n-k_i}$ for $n \gg 0$.

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When this is the case, we have $\ell \geq \ell(\mathcal{F})$. Of course, there exist a reduction system consisting of $\ell(\mathcal{F})$ elements. By using a reduction system we provide an upper bound of the a-invariant of $\mathrm{G}(\mathcal{F})$ as follows.

Theorem 1.2. *Let $\{a_i \in F_{k_i}\}_{0 \leq i \leq \ell}$ be a reduction system of $\mathcal{F}$. We assume that the following two conditions are satisfied for a nonnegative integer r:*

(a) *$F_n = \sum_{i=0}^{\ell} a_i F_{n-k_i}$ for any $n > r$.*
(b) *If $\mathfrak{p} \in \mathrm{V}(F_1)$ and $\mathrm{ht}_A \mathfrak{p} = i < \ell$, then $F_n A_\mathfrak{p} = \sum_{j=0}^{i} a_j F_{n-k_j} A_\mathfrak{p}$ for any $n > \max\{0, r - \sum_{j=i+1}^{\ell} k_j\}$.*

Then we have $\mathrm{a}(\mathrm{G}(\mathcal{F})) \leq r - \sum_{i=0}^{\ell} k_i$.

Although the proofs of Theorems 1.1 and 1.2 can be done following the standard argument in the ideal case, it is quite useful to extend Burch's result and the estimation of the a-invariant as in those theorems. For example, applying Theorems 1.1 and 1.2, we can prove the following result, which was already proved by Conca [4] in the case where $A = K[x_1, \ldots, x_5]$ is a polynomial ring over a field K.

Theorem 1.3. *Let A be a Gorenstein local ring with $\dim A = 5$. Let I be the ideal of A generated by the maximal minors of the matrix*

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix},$$

where $x_1, x_2, \ldots, x_5$ is a system-of-parameters (sop) for A. Then the symbolic Rees algebra $\mathrm{R}_s(I) = \sum_{n \geq 0} I^{(n)} t^n$ is generated in degree 2 and is a Cohen-Macaulay ring, where $I^{(n)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}_A A/I} (I^n A_\mathfrak{p} \cap A)$.

In the proof of the assertion above, Theorem 1.1 plays a crucial role when we determine the symbolic powers. The importance of considering general filtration can be seen in such argument.

Throughout this paper $(A, \mathfrak{m})$ is a Noetherian local ring with $\dim A = d$ and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration of A. We denote by $\mathrm{V}(F_1)$ the set of prime ideals containing F_1. For $\mathfrak{p} \in \mathrm{V}(F_1)$, we set $\mathcal{F}_\mathfrak{p} = \{F_n A_\mathfrak{p}\}_{n \in \mathbb{Z}}$, which is a filtration of $A_\mathfrak{p}$. For a module M over a commutative ring R, $\mathrm{H}^i_{\mathfrak{B}}(B)$ stands for the i-th local cohomology module of M with respect to an ideal $\mathfrak{B}$ of R. When R is $\mathbb{Z}$-graded and M is a graded R-module, we express the n-th homogeneous component of M by M_n or $[M]_n$. Moreover, for an integer k, $M(k)$ means the graded R-module such that $[M(k)]_n = M_{n+k}$ for any $n \in \mathbb{Z}$. If R is a nonnegatively graded Noetherian local ring such that R_0 is local and $\dim R = m$, we set $\mathrm{a}(R) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^m_N(R)]_n \neq 0\}$, where N is the graded maximal ideal of R. This is called the a-invariant of R (cf. [7]).

2. Burch's inequality for a filtration

Although 1.1 may be well known, let us recall its proof in this section. We put $G = \mathrm{G}(\mathcal{F})$.

Lemma 2.1 ([2, (9.23)])**.** grade $\mathfrak{m}G = \inf\{\text{depth } A/F_n \mid n > 0\}$.

Proof. Put $g = \mathrm{grade}\, \mathfrak{m}G$. We prove by induction on g.

Let $g = 0$. Then there exists $P \in \mathrm{Ass}\, G$ such that $\mathfrak{m}G \subseteq P$. We can choose a homogeneous element $f \in G$ so that $P = 0 :_G f$. We set $n = \deg f$. Since $\mathfrak{m}f = 0$, we have $\mathfrak{m} \in \mathrm{Ass}_A G_n \subseteq \mathrm{Ass}_A A/F_{n+1}$, and so depth $A/F_{n+1} = 0$.

On the other hand, if $g > 0$, then $\inf\{\operatorname{depth} A/F_n \mid n > 0\} > 0$. Indeed, given an $a \in \mathfrak{m}$ such that its image in G is G-regular, then it is easy to see that a is a nonzerodivisor on A/F_n for any $n > 0$. We set $B = A/aA$, $\mathfrak{n} = \mathfrak{m}B$ and $\mathcal{H} = \{F_nB\}_{n\in\mathbb{Z}}$. Then $G/aG \cong \mathrm{G}(\mathcal{H})$ (cf. [9, 2.7]) and grade $\mathfrak{n}\cdot \mathrm{G}(\mathcal{H}) = g-1$. By the induction hypothesis, we have $\inf\{\operatorname{depth} B/F_nB \mid n > 0\} = g-1$. Since $(A/F_n)/a(A/F_n) \cong B/F_nB$, we have $\operatorname{depth} A/F_n = \operatorname{depth} B/F_nB + 1$ for any $n > 0$. Thus we get $\inf\{\operatorname{depth} A/F_n \mid n > 0\} = g$.

Proof of Theorem 1.1. We get the first assertion since

$$\begin{aligned}\ell(\mathcal{F}) &= \dim G/\mathfrak{m}G \\ &\le \dim G - \operatorname{ht}_G \mathfrak{m}G \\ &\le \dim G - \operatorname{grade} \mathfrak{m}G \\ &= d - \inf\{\operatorname{depth} A/F_n \mid n > 0\} \quad \text{(by Lemma 2.1)}.\end{aligned}$$

Both of inequalities above become equalities when G is a Cohen-Macaulay ring. Hence we get the second assertion. □

3. An upper bound of $\mathrm{a}(\mathrm{G}(\mathcal{F}))$

In this section, we aim to prove Theorem 1.2. However, we consider a more general situation. Let $R = \bigoplus_{n\ge0} R_n$ be a nonnegatively graded Noetherian ring such that $(R_0, \mathfrak{n})$ is local and $\dim R = m$. We put $R_+ = \bigoplus_{n>0} R_n$ and $N = \mathfrak{n}R + R_+$. For $p \in \operatorname{Spec} R_0$, we denote by R_p the graded ring $(R_0)_p \otimes_{R_0} R = \bigoplus_{n\ge0}(R_n)_p$. Let $f_0, f_1, \ldots, f_\ell$ be homogeneous elements of R such that $f_0 = 0$ and $\deg f_i = k_i > 0$ for $1 \le i \le \ell$. We put $k_0 = 0$ and $\mathfrak{B}_i = (f_0, f_1, \ldots, f_i)R$ for $0 \le i \le \ell$. Moreover, we set

$$\begin{aligned}\mathrm{r}(f_0, f_1, \ldots, f_\ell; R) &= \sup\{n \ge 0 \mid R_n \ne \textstyle\sum_{i=0}^{\ell} f_i R_{n-k_i}\} \quad (R_j = 0 \text{ if } j < 0), \\ \mathrm{a}(f_0, f_1, \ldots, f_\ell; R) &= \max\{n \in \mathbb{Z} \mid [\mathrm{H}^{\ell}_{(f_0,f_1,\ldots,f_\ell)}(R)]_n \ne 0\}.\end{aligned}$$

Of course, these numbers may be infinite. However, by the same argument as [1, 7.2] we can prove the following.

Lemma 3.1. *Suppose that $\ell = m$ and $f_1, \ldots, f_m$ form an R_+-filter regular sequence, that is, f_i does not belong to any $P \in \operatorname{Ass}_R R/(f_0, f_1, \cdots, f_{i-1})R$ for $1 \le i \le m$, with $R_+ \not\subseteq P$. Then $\mathrm{r}(f_0, f_1, \ldots, f_m; R) < \infty$.*

Proof. Let us take any minimal prime ideal Q of $(f_0, f_1, \ldots, f_m)R$. Suppose $R_+ \not\subseteq Q$. Then $f_1, \ldots, f_m$ is an R_Q-regular sequence, and so $\dim R_Q \ge m = \dim R$. This means $Q = N$, which contradicts the assumption $R_+ \not\subseteq Q$. Thus we get $R_+ \subseteq \sqrt{(f_0, f_1, \ldots, f_m)R}$, which implies $\mathrm{r}(f_0, f_1, \ldots, f_m; R) < \infty$. □

Proposition 3.2. *Let $\ell = \dim R/\mathfrak{n}R$. We can choose $f_0, f_1, \ldots, f_\ell$ so that they satisfy the following conditions*:

(i) $\mathrm{r}(f_0, f_1, \ldots, f_\ell; R) < \infty$.
(ii) *If $p \in \operatorname{Spec} R_0$ and $\dim R_p = i \le \ell$, then $\mathrm{r}(f_0, f_1, \ldots, f_i; R_p) < \infty$.*

Proof. We may assume $\ell > 0$. Then we can choose homogeneous elements $f_1, \ldots, f_\ell \in R_+$ so that they form an sop for $R/\mathfrak{n}R$ and an R_+-filter regular sequence. Because, for any $p \in \operatorname{Spec} R_0$, the images of $f_1, \ldots, f_\ell$ in R_p still form an $(R_p)_+$-filter regular sequence, we get the assertion by Lemma 3.1. □

Corollary 3.3. *There exists a reduction system* $\{a_i \in F_{k_i}\}_{0\leq i\leq \ell}$ *of* $\mathcal{F}$ *such that* $\{a_j \in F_{k_j}A_{\mathfrak{p}}\}_{0\leq j\leq i}$ *is a reduction system of* $\mathcal{F}_{\mathfrak{p}}$ *for any* $\mathfrak{p} \in \mathrm{V}(F_1)$ *with* $\mathrm{ht}_A\, \mathfrak{p} = i \leq \ell$.

Proof. We take $\mathrm{G}(\mathcal{F})$ as R and choose $f_0, f_1, \ldots, f_\ell$ as in Proposition 3.2. Let us take $a_i \in F_{k_i}$ so that the class of $a_i t^{k_i}$ in $\mathrm{G}(\mathcal{F})$ is equal to f_i for $0 \leq i \leq \ell$. Then $\{a_i \in F_{k_i}\}_{0\leq i\leq \ell}$ satisfies the required condition. □

Lemma 3.4. $\mathrm{a}(f_0, f_1, \ldots, f_\ell; R) \leq \mathrm{r}(f_0, f_1, \ldots, f_\ell; R) - \sum_{i=0}^{\ell} k_i$.

Proof. It is enough to show in the case where $\mathrm{r}(f_0, f_1, \ldots, f_\ell; R) < \infty$. Let us prove by induction on ℓ. The assertion is obvious when $\ell = 0$. Let $\ell > 0$. We put $S = R/f_\ell R$ and $L = 0 :_R f_\ell$. We note that $\mathrm{H}^i_{\mathfrak{B}_\ell}(S) = \mathrm{H}^i_{\mathfrak{B}_{\ell-1}}(S)$ and $\mathrm{H}^i_{\mathfrak{B}_\ell}(L) = \mathrm{H}^i_{\mathfrak{B}_{\ell-1}}(L)$ for any $i \in \mathbb{Z}$. The short exact sequence $0 \to f_\ell R \to R \to S \to 0$ induces an exact sequence $\mathrm{H}^{\ell-1}_{\mathfrak{B}_{\ell-1}}(S) \to \mathrm{H}^\ell_{\mathfrak{B}_\ell}(f_\ell R) \to \mathrm{H}^\ell_{\mathfrak{B}_\ell}(R) \to 0$ since $\mathrm{H}^\ell_{\mathfrak{B}_{\ell-1}}(S) = 0$. On the other hand, from the exact sequence $0 \to L(-k_\ell) \to R(-k_\ell) \xrightarrow{f_\ell} f_\ell R \to 0$ we get $\mathrm{H}^\ell_{\mathfrak{B}_\ell}(R)(-k_\ell) \cong \mathrm{H}^\ell_{\mathfrak{B}_\ell}(f_\ell R)$ since $\mathrm{H}^i_{\mathfrak{B}_{\ell-1}}(L) = 0$ for $i \geq \ell$. Thus we get an exact sequence $[\mathrm{H}^{\ell-1}_{\mathfrak{B}_{\ell-1}}(S)]_{n+k_\ell} \to [\mathrm{H}^\ell_{\mathfrak{B}_\ell}(R)]_n \to [\mathrm{H}^\ell_{\mathfrak{B}_\ell}(R)]_{n+k_\ell} \to 0$ for any $n \in \mathbb{Z}$.

Let us denote by g_i the class of f_i in S. Then we have $\mathrm{a}(g_0, g_1, \ldots, g_{\ell-1}; S) \leq \mathrm{r}(g_0, g_1, \ldots, g_{\ell-1}; S) - \sum_{i=0}^{\ell-1} k_i$ by the induction hypothesis. Furthermore, we obviously have $\mathrm{r}(g_0, g_1, \ldots, g_{\ell-1}; S) = \mathrm{r}(f_0, f_1, \ldots, f_\ell; R)$. Now we take any $n > \mathrm{r}(f_0, f_1, \ldots, f_\ell; R) - \sum_{i=0}^{\ell} k_i$. Then $n + k_\ell > \mathrm{a}(g_0, g_1, \ldots, g_{\ell-1}; S)$, and so $[\mathrm{H}^{\ell-1}_{\mathfrak{B}_{\ell-1}}(S)]_{n+k_\ell} = 0$. Hence $[\mathrm{H}^\ell_{\mathfrak{B}_\ell}(R)]_n$ can be embedded in $[\mathrm{H}^\ell_{\mathfrak{B}_\ell}(R)]_{n+k_\ell}$. As a consequence we get $[\mathrm{H}^\ell_{\mathfrak{B}_\ell}(R)]_n = 0$ for any $n > \mathrm{r}(f_0, f_1, \ldots, f_\ell; R) - \sum_{i=0}^{\ell} k_i$, which means the required inequality. □

Lemma 3.5 ([8, (2.1)])**.** *Let* $n \in \mathbb{Z}$. *If* $[\mathrm{H}^{\dim R_p}_{(R_p)_+}(R_p)]_n = 0$ *for any* $p \in \mathrm{Spec}\, R_0$, *then we have* $[\mathrm{H}^m_N(R)]_n = 0$.

Proposition 3.6. *Let* $r \geq 0$. *If* $\mathrm{r}(f_0, f_1, \ldots, f_\ell; R) < r$ *and* $\mathrm{r}(f_0, f_1, \ldots, f_i; R_p) \leq r - \sum_{j=i+1}^{\ell} k_j$ *for any* $p \in \mathrm{Spec}\, R_0$ *with* $\dim R_p = i < \ell$, *then we have* $\mathrm{a}(R) \leq r - \sum_{j=0}^{\ell} k_j$.

Proof. Let us take any $n > r - \sum_{j=0}^{\ell} k_j$. By Lemma 3.5 it is enough to show that $[\mathrm{H}^{\dim R_p}_{(R_p)_+}(R_p)]_n = 0$ for any $p \in \mathrm{Spec}\, R_0$. This is obvious when $\dim R_p > \ell$ since $\mathrm{H}^{\dim R_p}_{(R_p)_+}(R_p) = \mathrm{H}^{\dim R_p}_{\mathfrak{B}_\ell}(R_p) = 0$. If $\dim R_p = i \leq \ell$, then

$$\begin{aligned}\mathrm{a}(f_0, f_1, \ldots, f_i; R_p) &\leq \mathrm{r}(f_0, f_1, \ldots, f_i; R_p) - \sum_{j=0}^{i} k_j \quad \text{(by Lemma 3.4)}\\ &\leq r - \sum_{j=0}^{\ell} k_j\,,\end{aligned}$$

and so $[\mathrm{H}^{\dim R_p}_{(R_p)_+}(R_p)]_n = [\mathrm{H}^i_{\mathfrak{B}_i}(R_p)]_n = 0$. □

Proof of Theorem 1.2. Let $R = \mathrm{G}(\mathcal{F})$ and f_i be the class of $a_i t^{k_i}$ in $\mathrm{G}(\mathcal{F})$ for $0 \leq i \leq \ell$. Then the assumption of Proposition 3.6 is satisfied, and we get the required assertion. □

4. Application

Let A be a 5-dimensional Gorenstein local ring and I the ideal described in Theorem 1.3. It is well known that A/I is a Cohen-Macaulay ring with $\dim A/I = 2$. As an application of Theorems 1.1 and 1.2, we will prove that the symbolic Rees algebra $\mathrm{R}_s(I) = \sum_{n\geq 0} I^{(n)}t^n$ is a Cohen-Macaulay ring.

For $1 \leq i < j \leq 4$, we set

$$\Delta_{ij} = \det \begin{pmatrix} x_i & x_j \\ x_{i+1} & x_{j+1} \end{pmatrix}.$$

Then we have the following relations:

$$(1) \qquad x_2\Delta_{34} - x_3\Delta_{24} + x_4\Delta_{23} = 0\,,$$
$$(2) \qquad x_2\Delta_{23} - x_3\Delta_{13} + x_4\Delta_{12} = 0\,,$$
$$(3) \qquad x_1\Delta_{24} - x_2\Delta_{14} + x_4\Delta_{12} = 0\,,$$
$$(4) \qquad x_1\Delta_{23} - x_2\Delta_{13} + x_3\Delta_{12} = 0\,,$$
$$(5) \qquad x_2\Delta_{34} - x_4\Delta_{14} + x_5\Delta_{13} = 0\,,$$
$$(6) \qquad x_3\Delta_{34} - x_4\Delta_{24} + x_5\Delta_{23} = 0\,.$$

By (1) and (2) we get

$$\Delta_{12}(x_3\Delta_{24} - x_2\Delta_{34}) = \Delta_{12}\cdot x_4\Delta_{23} = x_4\Delta_{12}\cdot\Delta_{23} = (x_3\Delta_{13} - x_2\Delta_{23})\Delta_{23}\,,$$

and so

$$x_2(\Delta_{12}\Delta_{34} - \Delta_{23}^2) = x_3(\Delta_{12}\Delta_{24} - \Delta_{23}\Delta_{13})\,.$$

Hence there exists $\delta \in A$ such that

$$(7) \qquad x_2\delta = \Delta_{12}\Delta_{24} - \Delta_{23}\Delta_{13}\,,$$
$$(8) \qquad x_3\delta = \Delta_{12}\Delta_{34} - \Delta_{23}^2\,.$$

These equalities imply $\delta \in I^{(2)}$ since any $\mathfrak{p} \in \mathrm{Min}_A\, A/I$ does not contain x_2 nor x_3. Next, we have

$$\begin{aligned} x_4\cdot x_2\delta &= x_4(\Delta_{12}\Delta_{24} - \Delta_{23}\Delta_{13}) \quad \text{(by (7))} \\ &= x_4\Delta_{12}\cdot\Delta_{24} - x_4\Delta_{23}\cdot\Delta_{13} \\ &= (x_3\Delta_{13} - x_2\Delta_{23})\Delta_{24} - (x_3\Delta_{24} - x_2\Delta_{34})\Delta_{13} \quad \text{(by (2) and (1))} \\ &= x_2\Delta_{13}\Delta_{34} - x_2\Delta_{23}\Delta_{24}\,, \end{aligned}$$

which yields

$$(9) \qquad x_4\delta = \Delta_{13}\Delta_{34} - \Delta_{23}\Delta_{24}\,.$$

Furthermore, we get

$$\begin{aligned} x_1\cdot x_2\delta &= x_1(\Delta_{12}\Delta_{24} - \Delta_{23}\Delta_{13}) \quad \text{(by (7))} \\ &= \Delta_{12}\cdot x_1\Delta_{24} - x_1\Delta_{23}\cdot\Delta_{13} \\ &= \Delta_{12}(x_2\Delta_{14} - x_4\Delta_{12}) - (x_2\Delta_{13} - x_3\Delta_{12})\Delta_{13} \quad \text{(by (3) and (4))} \\ &= x_2\Delta_{12}\Delta_{14} - x_2\Delta_{13}^2 + \Delta_{12}(x_3\Delta_{13} - x_4\Delta_{12}) \\ &= x_2\Delta_{12}\Delta_{14} - x_2\Delta_{13}^2 + \Delta_{12}\cdot x_2\Delta_{23} \quad \text{(by (2))}\,, \end{aligned}$$

and so

$$(10) \qquad x_1\delta = \Delta_{12}\Delta_{14} - \Delta_{13}^2 + \Delta_{12}\Delta_{23}\,.$$

Similarly, we have

$$\begin{aligned} x_5 \cdot x_4\delta &= x_5(\Delta_{13}\Delta_{34} - \Delta_{23}\Delta_{24}) \quad \text{(by (9))} \\ &= x_5\Delta_{13} \cdot \Delta_{34} - x_5\Delta_{23} \cdot \Delta_{24} \\ &= (x_4\Delta_{14} - x_2\Delta_{34})\Delta_{34} - (x_4\Delta_{24} - x_3\Delta_{34})\Delta_{24} \quad \text{(by (5) and (6))} \\ &= x_4\Delta_{14}\Delta_{34} - x_4\Delta_{24}^2 + \Delta_{34}(x_3\Delta_{24} - x_2\Delta_{34}) \\ &= x_4\Delta_{14}\Delta_{34} - x_4\Delta_{24}^2 + \Delta_{34} \cdot x_4\Delta_{23} \quad \text{(by (1))}, \end{aligned}$$

which induces

$$x_5\delta = \Delta_{14}\Delta_{34} - \Delta_{24}^2 + \Delta_{23}\Delta_{34}\,. \tag{11}$$

Now we set

$$F_n = \sum_{i+2j=n} \delta^j I^i \quad (i \text{ and } j \text{ run nonnegative integers})$$

for any $n > 0$ and $F_n = A$ for any $n \leq 0$. Then $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is a filtration of A. Obviously, $\mathrm{R}(\mathcal{F}) = A[It, \delta t^2]$.

Lemma 4.1. *$F_n = \delta F_{n-2} + (\Delta_{12}, \Delta_{34}, \Delta_{14})F_{n-1}$ for any $n \geq 2$.*

Proof. We note $I^2 = (\Delta_{12}, \Delta_{34}, \Delta_{14})I + (\Delta_{13}, \Delta_{23}, \Delta_{24})^2$. The relations (7), (8), (9), (10) and (11), respectively, imply $\Delta_{23}\Delta_{13}$, Δ_{23}^2, $\Delta_{23}\Delta_{24}$, Δ_{13}^2 and Δ_{24}^2 are contained in $(\Delta_{12}, \Delta_{34}, \Delta_{14})I + \delta A$. Hence we have $I^2 \subseteq (\Delta_{12}, \Delta_{34}, \Delta_{14})I + (\delta, \Delta_{13}\Delta_{24})$. Now we use the Plücker relation $\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} = 0$, and get $\Delta_{13}\Delta_{24} \in (\Delta_{12}, \Delta_{14})I$. Therefore, $I^2 \subseteq (\Delta_{12}, \Delta_{34}, \Delta_{14})I + \delta A$. Then we immediately get the required equalities. □

Lemma 4.2. *Let $\mathfrak{p} \in \mathrm{Min}_A\, A/I$. Then $F_nA_\mathfrak{p} = \delta F_{n-2}A_\mathfrak{p} + (\Delta_{12}, \Delta_{34})F_{n-1}A_\mathfrak{p}$ for any $n \geq 2$.*

Proof. Because $\mathfrak{p}$ contains none of $x_1, x_2, \ldots, x_5$, we see $IA_\mathfrak{p} = (\Delta_{12}, \Delta_{34}, \Delta_{14})A_\mathfrak{p}$ using (3), (4) and (5). On the other hand, substituting the relation $\Delta_{13} = (x_4\Delta_{14} - x_2\Delta_{34})/x_5$ in $A_\mathfrak{p}$ to (10), we get $\Delta_{14}^2 \in \delta A_\mathfrak{p} + (\Delta_{12}, \Delta_{34})IA_\mathfrak{p}$. Therefore, $I^2A_\mathfrak{p} \subseteq \delta A_\mathfrak{p} + (\Delta_{12}, \Delta_{34})IA_\mathfrak{p}$, which implies the required equalities. □

Lemma 4.3. *$\delta, \Delta_{12}, \Delta_{34}$ is an A-regular sequence.*

Proof. Let us take a prime ideal P containing $(\delta, \Delta_{12}, \Delta_{34}, x_1, x_5)A$. Because $\Delta_{12} \equiv -x_2^2 \bmod (x_1)$ and $\Delta_{34} \equiv -x_4^2 \bmod (x_5)$, it follows that $x_2 \in P$ and $x_4 \in P$. Then $\Delta_{23} \equiv -x_3^2 \bmod P$, and so by (8) we get $x_3 \in P$. Hence P is the maximal ideal of A. Thus we see that $\delta, \Delta_{12}, \Delta_{34}$ is part of an sop for A, and so the required assertion follows. □

We put $K = (\Delta_{12}, \Delta_{34}, \Delta_{14})A$.

Lemma 4.4. *A/KI and $A/K :_A I$ are 2-dimensional Cohen-Macaulay rings.*

Proof. Because K is generated by a regular sequence, K/K^2 is A/K-free, and so $K/KI \cong A/I \otimes_A K/K^2$ is A/I-free. Then, looking at the exact sequence $0 \to K/KI \to A/KI \to A/K \to 0$, we get $\operatorname{depth} A/KI = 2$. On the other hand, by [11, 1.3] we have the Cohen-Mcaulayness of $A/K : I$. □

Lemma 4.5. $K \cap I^{(2)} = KI$.

Proof. Take any $\mathfrak{q} \in \mathrm{Ass}_A A/KI$. It is enough to show $KA_\mathfrak{q} \cap I^{(2)}A_\mathfrak{q} = KIA_\mathfrak{q}$. This is obvious when $I \not\subseteq \mathfrak{q}$. Suppose $I \subseteq \mathfrak{q}$. Then $\mathfrak{q} \in \mathrm{Min}_A A/I$ as $\dim A/\mathfrak{q} = 2$ by Lemma 4.4. Hence $KA_\mathfrak{q} = IA_\mathfrak{q}$, and so we get the required equality. □

Lemma 4.6. $KI :_A \delta = K :_A I$.

Proof. Take any $a \in K : I$. Then $a\delta \in K \cap I^{(2)} = KI$ by Lemma 4.5. Hence $KI : \delta \supseteq K : I$. Now we take any $\mathfrak{q} \in \mathrm{Ass}_A A/K : I$. It is enough to show $[KI : \delta]A_\mathfrak{q} = [K : I]A_\mathfrak{q}$. If $I \subseteq \mathfrak{q}$, then $\mathfrak{q} \in \mathrm{Min}_A A/I$ since $\dim A/\mathfrak{q} = 2$ by Lemma 4.4, which means $KA_\mathfrak{q} = IA_\mathfrak{q}$. However, this contradicts $K : I \subseteq \mathfrak{q}$. Hence $I \not\subseteq \mathfrak{q}$. Then $\delta \notin \mathfrak{q}$ as $I^2 \subseteq K + \delta A$, and so $[KI : \delta]A_\mathfrak{q} = K_\mathfrak{q} = [K : I]A_\mathfrak{q}$. Thus we get the required equality. □

Lemma 4.7. $\operatorname{depth} A/F_2 > 0$.

Proof. We have $F_2 = KI + \delta A$ by Lemma 4.1. As $\delta A/KI \cap \delta A \cong A/KI : \delta = A/K : I$, we have an exact sequence $0 \to A/K : I \to A/KI \to A/F_2 \to 0$. Applying the depth lemma to this exact sequence, we get the required assertion. □

Proof of Theorem 1.3. By [10, 1.2] it follows that $G := \mathrm{G}(\mathcal{F})$ is a Cohen-Macaulay ring from Lemmas 4.1, 4.2, 4.3 and 4.7. Since $\delta \in F_2, \Delta_{12} \in F_1, \Delta_{34} \in F_1, \Delta_{14} \in F_1$ is a reduction system of $\mathcal{F}$ by Lemma 4.1, we have $\ell(\mathcal{F}) \leq 4$. Therefore, for any n, we have $\operatorname{depth} A/F_n > 0$ by Theorem 1.1, and so $\mathrm{Ass}_A A/F_n = \mathrm{V}(I) \setminus \{\mathfrak{m}\}$. On the other hand, if $\mathfrak{m} \neq \mathfrak{p} \in \mathrm{V}(I)$, $IA_\mathfrak{p}$ is generated by a regular sequence, and so $I^{(n)}A_\mathfrak{p} = I^nA_\mathfrak{p}$ for any n. Thus we get $F_n = I^{(n)}$ for any $n > 0$. Moreover, we have $\mathrm{a}(G) < 0$ by Theorem 1.2. Therefore, $\mathrm{R}(\mathcal{F}) = \mathrm{R}_s(I)$ is a Cohen-Macaulay ring by [5, (1.2) of Part II]. □

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