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A SHORT PROOF OF AN INEQUALITY OF LITTLEWOOD AND PALEY

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ABSTRACT. A very short proof is given of the inequality

$$\int_{|z|<1} |\nabla u(z)|^p (1-|z|)^{p-1}\, dxdy \le C_p \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p\, dt - |u(0)|^p \right),$$

where $p > 2$, and u is the Poisson integral of $f \in L^p(\partial\mathbb{D})$, $\mathbb{D} = \{z : |z| < 1\}$.

Let $\mathbb{D}$ denote the open unit disk of the complex plane, and $\mathbb{T} = \partial\mathbb{D}$. The following theorem was proved by Littlewood and Paley in [2].

Theorem LP. *If f is a real valued function of class $L^p(\mathbb{T})$, $p > 2$, and if u is the Poisson integral of f, then*

$$\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|^2)^{p-1}\, dA(z) \le C_p \|f\|_p^p, \tag{1}$$

where C_p is a constant depending only on p and

$$\|f\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p\, dt.$$

Here dA stands for the Lebesgue measure normalized so that $A(\mathbb{D}) = 1$.

This theorem can easily be proved by using the Riesz-Thorin interpolation theorem. In [3] Luecking gave an elementary but rather long proof based on the formula

$$\|f\|_p^p - |u(0)|^p = \frac{(p^2-p)}{2} \int_{\mathbb{D}} |\nabla u|^2 |u|^{p-2} \log\frac{1}{|z|}\, dA(z). \tag{2}$$

This formula, a consequence of the Green formula, was used by P. Stein [5] to prove the Riesz theorem on conjugate functions (see [1], p. 55). We also start from (2), and reduce to the case of positive harmonic functions, which satisfy the following inequality:

If u is a positive harmonic function on $\mathbb{D}$, then

$$|\nabla u(z)| \le 2(1-|z|^2)^{-1} u(z) \quad (z \in \mathbb{D}). \tag{3}$$

This inequality is obtained by applying the special case $z = 0$ to the function

$$w \mapsto u\left(\frac{z-w}{1-\bar{z}w}\right).$$

In fact, we shall prove a slightly improved version of Theorem LP.

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Theorem 1. *If f is a real valued function of class $L^p(\mathbb{T})$, $p > 2$, and if u is the Poisson integral of f, then*

$$\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|^2)^{p-1}\, dA(z) \le C_p(\|f\|_p^p - |u(0)|^p), \tag{4}$$

where C_p is a constant depending only on p.

Proof. Let $f \in L^p(\mathbb{T}),\ p > 2$. Let u_i $(i = 1, 2)$ denote the Poisson integral of f_i, where $f_1 = \max(f, 0)$ and $f_2 = \max(-f, 0)$. Then $u_i \ge 0$, $u = u_1 - u_2$ and

$$\|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p. \tag{5}$$

Also, since

$$|\nabla u|^p \le 2^{p-1}\Big(|\nabla u_1|^p + |\nabla u_2|^p\Big), \tag{6}$$

the proof reduces to the case where $u > 0$. Then it follows from (2), (3) and the inequality

$$\log\frac{1}{|z|} \ge \frac{1-|z|^2}{2}$$

that

$$\begin{aligned}\|f\|_p^p - |u(0)|^p &\ge \frac{p^2-p}{4}\int_{\mathbb{D}} |\nabla u|^2\, u^{p-2}(1-|z|^2)\, dA(z)\\ &\ge \frac{p^2-p}{4}\int_{\mathbb{D}} |\nabla u|^2\, 2^{2-p}\, |\nabla u|^{p-2}(1-|z|^2)^{p-1}\, dA(z).\end{aligned}$$

This proves (1) for $f > 0$ with $C_p = 2^p/(p^2-p)$. If f is arbitrary, then we use (5), (6) and the inequality

$$|a-b|^p \le a^p + b^p \quad (a \ge 0,\ b \ge 0)$$

to get

$$\begin{aligned}\|f\|_p^p - |u(0)|^p &\ge \|f_1\|_p^p - |u_1(0)|^p + \|f_2\|_p^p - |u_2(0)|^p\\ &\ge (p^2-p)/2^p \int_{\mathbb{D}} \big(|\nabla u_1|^p + |\nabla u_1|^p\big)(1-|z|)^{p-1}\, dA(z)\\ &\ge 2^{1-p}(p^2-p)/2^p \int_{\mathbb{D}} |\nabla u|^p (1-|z|)^{p-1}\, dA(z).\end{aligned}$$

Hence

$$\int_{\mathbb{D}} |\nabla u|^p (1-|z|^2)^{p-1}\, dA(z) \le C_p\big(\|f\|_p^p - |u(0)|^p\big)$$

with

$$C_p = 2^{2p-1}/(p^2-p).$$

This completes the proof. □

Remark 1. Inequality (4) can be written as

$$\|f\|_p^p - |f(0)|^p \ge c_p \int_{\mathbb{D}} |\nabla u|^p (1-|z|^2)^{p-1}\, dA(z),$$

which can be viewed as a refinement of the inequality $\|f\|_p^p - |f(0)|^p \ge 0$, a consequence of the subharmonicity of the function $|u(z)|^p$.

Remark 2. Inequality (4) holds for functions with values in a Hilbert space, but the proof is more delicate. See [4].

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