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A SIMPLE PROOF OF ZAGIER DUALITY FOR HILBERT MODULAR FORMS

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ABSTRACT. In this paper, we give a simple proof of an identity between the Fourier coefficients of the weakly holomorphic modular forms of weight 0 arising from Borcherds products of Hilbert modular forms and those of the weakly holomorphic modular forms of weight 2 satisfying a certain property.

1. Introduction and results

A certain sequence of modular forms of weight 1/2 arises in the theory of Borcherds products for modular forms with Heegner divisors. Zagier proved that there exists a duality, called Zagier duality, between the Fourier coefficients of these modular forms and those of some modular forms of weight 3/2. In [2] Rouse obtained an analog of Zagier duality between the Fourier coefficients of the weakly holomorphic modular forms of weight 0 arising from Borcherds products of Hilbert modular forms and those of the weakly holomorphic modular forms of weight 2 satisfying a certain property. In this paper, we give a simple proof of the identity proved in [2], by constructing a linear relation among the Fourier coefficients of weakly holomorphic modular forms.

Let χ_p be the Dirichlet character $\left(\frac{\cdot}{p}\right)$. For an integer $k \geq 0$ and $\epsilon = \pm 1$ let $A_k^\epsilon(\Gamma_0(p), \chi_p)$ denote the subspace of weakly holomorphic modular forms $F(z)$ of weight k and character χ_p such that if $F(z) = \sum_{n\in\mathbb{Z}} c(n)q^n$, then $c(n) = 0$ for $\chi_p(n) = -\epsilon$.

Proposition 1.1. *Suppose that* $f = \sum_{n\in\mathbb{Z}} a_f(n)q^n \in A^\epsilon_{k(f)}(\Gamma_0(p), \chi_p)$ *and* $g = \sum_{n\in\mathbb{Z}} a_g(n)q^n \in A^\epsilon_{k(g)}(\Gamma_0(p), \chi_p)$. *For* $t \in \{0, \infty\}$ *let*

$$f|_2\gamma_t = \sum_{n\in\mathbb{Z}} a_f^t(n)q^n \text{ and } g|_0\gamma_t = \sum_{n\in\mathbb{Z}} a_g^t(n)q^n,$$

where $\gamma_0 = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ *and* $\gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *If* $k(f) + k(g) = 2$, *then*

$$\sum_{i+j=0} (a_f^0(i)a_g^0(j) + a_f^\infty(i)a_g^\infty(j)) = 0.$$

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Let p be a prime and

$$s(m) = \begin{cases} 2 & \text{if } m \equiv 0 \pmod{p}, \\ 1 & \text{if } m \not\equiv 0 \pmod{p}. \end{cases}$$

From Proposition 1.1 we obtain the following theorem that gives a connection between a weakly holomorphic modular form of weight 0 arising in Borcherds products for Hilbert modular forms (see Theorem 6 in [1]) and a certain weakly holomorphic modular form of weight 2.

Theorem 1.2. *Suppose that, for a nonnegative integer m and a positive integer d with $\chi(m) \neq -\epsilon$ and $\chi(d) \neq -\epsilon$, there is a function $F^{\epsilon}_{d,p}(z) \in A^{\epsilon}_0(\Gamma_0(p), \chi_p)$ such that*

$$F^{\epsilon}_{d,p}(z) = \frac{1}{s(d)}q^{-d} + O(1) = \sum_{n\in\mathbb{Z}} A_{d,p}(n)q^n$$

and $G^{\epsilon}_{m,p}(z) \in A^{\epsilon}_2(\Gamma_0(p), \chi_p)$ such that

$$G^{\epsilon}_{m,p}(z) = \frac{1}{s(m)}q^{-m} + O(q) = \sum_{n\in\mathbb{Z}} B_{m,p}(n)q^n.$$

Then

$$A_{d,p}(m) = -B_{m,p}(d).$$

Remark 1.3. Let $S^{+}_k(\Gamma_0(p), \chi_p)$ denote the subspace of holomorphic cusp forms in $A^{+}_k(\Gamma_0(p), \chi_p)$. When $\epsilon = 1$ and $p \in \{5, 13, 17\}$, Theorem 1.2 was proved in [2] by the arithmetic of operators for modular forms.

Remark 1.4. If $p = 5, 13$ or 17, then for each $m \geq 1$ with $\chi_p(m) \neq -1$, there exists a unique $F^{+}_{m,p}(z) \in A^{+}_0(\Gamma_0(p), \chi_p)$ and a unique $G^{+}_{m,p}(z) \in A^{+}_2(\Gamma_0(p), \chi_p)$.

Proof of Proposition 1.1. Suppose G is a meromorphic modular form of weight 2 on $\Gamma_0(p)$ for a prime p. We denote the set of distinct cusps of $\Gamma_0(p)$ as $S_p = \{\infty, 0\}$. For $\tau \in \mathbb{H} \cup S_p$, let D_τ be the image of τ under the canonical map from $\mathbb{H} \cup S_p$ to $X_0(p)$, where $\mathbb{H}$ denotes the complex upper half-plane. The residue of G at D_τ on $X_0(p)$, denoted by $\mathrm{Res}_{D_\tau} G dz$, is well defined since we have a canonical correspondence between meromorphic modular forms of weight 2 on $\Gamma_0(p)$ and meromorphic 1-forms of $X_0(p)$. If $Res_\tau G$ denotes the residue of G at τ on $\mathbb{H}$, then we obtain

$$\mathrm{Res}_{D_\tau} G dz = \frac{1}{l_\tau} Res_\tau G.$$

Here, λ_τ is the order of the isotropy group at τ. The residue of G at each cusp t in S_p is

$$\mathrm{Res}_{D_t} G dz = \frac{a_t(0)}{2\pi i},$$

where $G(z) \mid_2 \gamma_t = \sum_{n=m_t}^{\infty} a_t(n)q^n$ at ∞.

To prove Proposition 1.1 we take $G = fg$. Since χ_p is a quadratic character, G is a meromorphic modular form of weight 2 on $\Gamma_0(p)$. We have

$$fg|_2\gamma_t = \left(\sum_{n\in\mathbb{Z}} a^t_f(n)q^n\right)\left(\sum_{n\in\mathbb{Z}} a^t_g(n)q^n\right).$$

Therefore, by the residue theorem we give a proof of Proposition 1.1. □

Proof of Theorem 1.2. We begin by stating the following lemma.

Lemma 1.5 (Lemma 3 of [1]). *Let* $F(z) = \sum_{n\in\mathbb{Z}} A(n)q^n \in A_k(\Gamma_0(p),\chi_p)$ *and* $\epsilon \in \{1,-1\}$. *Then* $F(z) \in A_k^{\epsilon}(\Gamma_0(p),\chi_p)$ *if and only if*

$$p^{1-k/2}(F|U_p) = \epsilon\sqrt{p}(F|_k\gamma_0),$$

where $F|U_p = \sum_{n\in\mathbb{Z}} A(pn)q^n$.

From Lemma 1.5 we have

$$\epsilon \cdot (F^{\epsilon}_{d,p}|_0\gamma_0) = p^{\frac{1}{2}}F^{\epsilon}_{d,p}|U_p = p^{\frac{1}{2}}\frac{s(d)-1}{s(d)}q^{-d/p} + p^{\frac{1}{2}}\sum_{n\geq 0} A_{d,p}(pn)q^n$$

and

$$\epsilon \cdot (G^{\epsilon}_{m,p}|_2\gamma_0) = p^{-\frac{1}{2}}G^{\epsilon}_{m,p}|U_p = p^{-\frac{1}{2}}\frac{s(m)-1}{s(m)}q^{-m/p} + p^{-\frac{1}{2}}\sum_{n\geq 1} B_{m,p}(pn)q^n.$$

Using Proposition 1.1, we have

$$\left(\frac{1}{s(m)} + \frac{s(m)-1}{s(m)}\right)A_{d,p}(m) + \left(\frac{1}{s(d)} + \frac{s(d)-1}{s(d)}\right)B_{m,p}(d) = 0.$$

This proves Theorem 1.2. □

Remark 1.6. Following the method used in the proof of Proposition 1.1, one can also obtain the similar identity between the Fourier coefficients of f_{k_1} in $A^{\epsilon}_{k_1}(\Gamma_0(p),\chi_p)$ and those of f_{k_2} in $A^{\epsilon}_{k_2}(\Gamma_0(p),\chi_p)$ where $k_1 + k_2 = 2$.

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