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# FIXED POINTS OF UNIVALENT FUNCTIONS II

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(Communicated by Juha M. Heinonen)

*To my friend Masakazu Shiba on the occasion of his 60th birthday*

Abstract. For a closed nowhere dense subset $C$ of $\partial\mathbb{D}$ a bounded univalent holomorphic function $f$ on $\mathbb{D}$ is found such that $C$ equals the cluster set of its fixed points.

## 1. Introduction

Let $f$ be a univalent (i.e. holomorphic and injective, but not necessarily normalized) function of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$.

Such a function $f \not\equiv id_{\mathbb{D}}$ may have infinitely many fixed points. In [2] it is proved that for each Blaschke sequence $(z_n)$ there exists a univalent function $f \not\equiv id_{\mathbb{D}}$ with $f(z_n) = z_n$ for all $n \in \mathbb{N}$, provided that the cluster set $F_c(f)$ of $(z_n)$ is an arbitrary Carleson set, i.e. a closed set of Lebesgue measure 0 on $\partial\mathbb{D}$ with $\sum_n \varepsilon_n \log \varepsilon_n > -\infty$, where $\varepsilon_n$ denotes the length of the open components (arcs) of its complement with respect to $\partial\mathbb{D}$. In [4] it is proved that $F_c$ is a nowhere dense subset of the unit circle unless $f \equiv id_{\mathbb{D}}$. It remains open whether $F_c(f)$ can be a proper subset of $\partial\mathbb{D}$ of positive measure. Here we will prove:

**Theorem 1.** *Let $C$ be a nowhere dense, closed subset of $\partial\mathbb{D}$. Then there exists a bounded univalent function $g : \mathbb{D} \to \mathbb{C}$ with the property that the cluster set $F_c(g) \subset \partial\mathbb{D}$ of its fixed points is $C$.*

Cantor sets on $\partial\mathbb{D}$ fulfill the assumption of Theorem 1. Their Lebesgue measure with respect to $\partial\mathbb{D}$ can be arbitrarily close to $2\pi$.

If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $\alpha \in [0, 2\pi[$, then the cluster set $C_\alpha(f)$ of the restriction $f_\alpha(t) := f(te^{i\alpha})$ for $t \to 1-$ is called the radial cluster set of $f$ with respect to $e^{i\alpha}$. It is well known that this cluster set coincides with the cluster set of $f(z)$, where $z$ tends to $e^{i\alpha}$ in an arbitrary triangle $\Delta_\alpha \subset \mathbb{D}$ with $e^{i\alpha} \in \partial\Delta_\alpha$. One can also describe $C_\alpha(f)$ as the set of principal points of the prime end of the image domain $f(\mathbb{D})$ which belongs to the boundary point $e^{i\alpha}$ of $\mathbb{D}$ (see [5] or [6]). An accumulation point of $f(z)$ for $z \to e^{i\alpha}$ which is not radial is called tangential. In terms of prime ends of $f(\mathbb{D})$ these are the points of the corresponding prime end impressions which are not principal points.

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By a result in [4] (Thm. 1), which combines a result of Beurling ([1], Thm. 3.5) and the theorem of F. and M. Riesz ([1], Thm. 2.5), the set $\{e^{i\alpha} : e^{i\alpha} \in C_\alpha(f)\}$ has measure zero if $f : \mathbb{D} \to \mathbb{C}$ is univalent. This shows that the fixed points of the function $f$ in Theorem 1 have to approach their accumulation points on the boundary of $\mathbb{D}$ tangentially up to a zero set.

The crucial tool in the proof is Theorem 2, which states the possibility of fusion of univalent functions. This is a consequence of an improved version of Alice Roth's well-known fusion lemma, due to Masakazu Shiba and the author ([7], [8]).

## 2. FUSION OF UNIVALENT FUNCTIONS

In this section we will prove the following result.

**Theorem 2.** *Let $G_1, G_2$ be simply connected domains, bounded by Jordan curves $\gamma_1$, resp. $\gamma_2$, and $\overline{G_1} \cap \overline{G_2} = \{z_0\}$. Moreover let $f_1$, resp. $f_2$, be continuous on $\overline{G_1}$, resp. on $\overline{G_2}$, univalent on $G_1$, resp. on $G_2$.*

*If $f_1(\overline{G_1}) \cap f_2(\overline{G_2})$ consists of the single point $w_0 = f_1(z_0) = f_2(z_0)$, then, for all $\varepsilon > 0$, there exists a neighborhood $U$ of $\overline{G_1} \cup \overline{G_2}$ and an entire function $f$, which is univalent on $U$ and fulfills $\|f - f_1\|_{G_1} < \varepsilon$ as well as $\|f - f_2\|_{G_2} < \varepsilon$.*

The proof requires some preparation.

**Definition 1.** We call a pair $K_1, K_2$ of compact sets in the plane normal, if

1. $\mathbb{C} \setminus (K_1 \cup K_2)$ is connected.
2. The interiors $K_1^o$, $K_2^o$ of both sets are connected.
3. $K_1 \cap K_2^o = K_2 \cap K_1^o = \emptyset$.
4. $K_1 = \overline{K_1^o}$ and $K_1 = \overline{K_1^o}$.

**Definition 2.** We say that a pair of compact plane sets $K_1, K_2$ has the rational fusion property if there exists some positive number $a = a(K_1, K_2)$ such that the following is true: For each pair $r_1, r_2$ of rational functions and for each compact set $K$ there is some rational function $r$ with

$$|r(z) - r_j(z)| \le a \cdot \sup\{|r_1(w) - r_2(w)| : w \in K \cup (K_1 \cap K_2)\}$$

for all $z \in K_j \cup K$ simultaneously for $j = 1, 2$.

We say that the pair $K_1, K_2$ has the holomorphic fusion property with respect to some domain $G \subset \mathbb{C}$ if we may replace the rational functions $r_1, r_2, r$ by holomorphic functions on $G$.

In [7] the following extension of Alice Roth's famous Fusion Lemma (cf. [3]) is proved, basing on a result of M. Shiba and the author [8].

**Proposition 1.** *A normal pair of compact sets $K_1, K_2$ has the rational fusion property if and only if $\partial K_1 \cup \partial K_2 = \partial(K_1 \cup K_2)$.*

*Proof of Theorem* 2. Let $\gamma_{1,\delta}$ be a Jordan curve in $\mathbb{C} \setminus \overline{G_1}$ with dist $(\gamma_1, \gamma_{1,\delta}) < \delta$ and let $G_{1,\delta}$ denote the bounded component of $\mathbb{C} \setminus \gamma_{1,\delta}$. Then $\overline{G_1} \subset G_{1,\delta}$, and we have a biholomorphic map $\phi_\delta : G_{1,\delta} \to G$. An equicontinuity argument shows that, for each given $\varepsilon$, the function $f_{1,\delta} = f_1 \circ \phi_\delta$ fulfills $\|f_1 - f_{1,\delta}\|_{G_1} < \varepsilon$ if $\delta \le \delta_0(\varepsilon)$. We may also assume that $f_{1,\delta}(z_0) = f_1(z_0)$, otherwise we replace $\varepsilon$ by $\varepsilon/2$ and consider the function $f_{1,\delta}(z_0) + f_1(z_0) - f_{1,\delta}(z_0)$. This shows that we may provide that $f_1$ (as well as $f_2$) is univalent on a neighborhood of $K_1 := \overline{G_1}$ (resp. on $K_2 = \overline{G_2}$).

Without loss of generality we may assume that $z_0 = 0$, $f_1(0) = 0 = f_2(0)$.

In the first part of the proof we consider the case that ${f_1}'(0) = {f_2}'(0)$. Without loss of generality we may assume that $f_1, f_2$ are entire functions, because we may approximate our starting functions uniformly on suitable compact sets containing $K_1$, resp. $K_2$, in its interior. This will preserve the univalence on a neighborhood of $K_j$ (as the argument principle shows) if the approximation bound is sufficiently small.

Obviously $K_1, K_2$ is a normal pair, and we also find a normal pair of compact sets $C_j$, also bounded by a Jordan curve respectively, containing $K_j \setminus \{0\}$ $(j = 1, 2)$ in the interior, such that $C_1 \cap C_2 = \{0\}$, and $f_1, f_2$ are univalent in an open neighborhood of $C_1, C_2$. We may also provide that $C_1, C_2$ is a rational fusion pair by Proposition 1 and, obviously, that it is a holomorphic fusion pair. As the "bridge" $K$ joining $C_1, C_2$ we take a disc $D_\rho := \{|z| \le \rho\}$; its radius will be fixed later. So we find some entire function $f$ with the property that

$$|f'(z) - {f_j}'(z)| \le a \cdot \max\{|{f_1}'(w) - {f_2}'(w)| \,:\, w \in D_\rho\} \quad (z \in C_j \cup D_\rho, j = 1, 2),$$

where $a$ depends only on $C_1, C_2$. This shows that (note that $f_1(0) = f_2(0)$ has been assumed)

$$|f(z) - f_j(z)| \le b \cdot \max\{|{f_1}'(w) - {f_2}'(w)| \,:\, w \in D_\rho\} \quad (z \in C_j \cup D_\rho, j = 1, 2)$$

with some $b$ depending only on geometrical properties of $C_1, C_2$. Thus, by the argument principle, we see that $f$ is univalent on a neighborhood of $C_1$ as well as on a neighborhood of $C_2$ if $\max\{|{f_1}'(w) - {f_2}'(w)| \,:\, w \in D_\rho\}$ is sufficiently small, and thus for small $\rho$, as follows from the assumption that ${f_1}'(0) = {f_2}'(0)$ and from the continuity of ${f_j}'$.

On $D_\rho$ we have for small $\rho$ that $|f'(z) - {f_1}'(0)| < |{f_1}'(0)|$. Since ${f_1}'(0) \ne 0$ this implies that $\Re f' > 0$ on $D_\rho$, and a well-known argument shows that therefore $f$ is univalent on $D_\rho$. Finally we may assume that $f(C_1 \setminus D_\rho) \cap f(C_2 \setminus D_\rho) = \emptyset$, as follows from the assumption that $f_1(K_1) \cap f_2(K_2) = \{0\}$, provided that the approximation was done with a bound sufficiently close to zero.

So $f$ has the desired properties if $\rho$ is small enough.

Now we consider the case that ${f_1}'(0) \ne {f_2}'(0)$.

Let $\sigma(z) = \sigma(z, \eta) = 1 - (1 - z)^\eta$ with some (small) $\eta > 0$. This function is holomorphic on each simply connected domain which does not contain 1. We obtain $\sigma'(1 - \eta^{1/(1-\eta)}, \eta) = 1$. Without loss of generality we may assume that $1 \notin K_1$.

Now we consider the functions $\varphi(z, \eta) = \sigma(z + 1 - \eta^{1/(1-\eta)}, \eta)$ and $g_1(z) = f_1(z) + ({f_2}'(0) - {f_1}'(0))\, \varphi(z, \eta)$ on $K_1$ for small $\eta$. Note that this expression is defined on a neighborhood of $K_1$. Then, by the mentioned properties of $\sigma$, we have that

$$\|\varphi(., \eta)\|_{K_1} = |\varphi(0)| = 1 - \eta^{\eta/(1-\eta)} \to 0 \text{ if } \eta \to 0,$$

provided that $\eta = \eta(K, \varepsilon)$ is small. Then $g_1$ is univalent on a neighborhood of $K_1$ for all small $\eta$, and if some $\varepsilon$ is given, we have that $\|g_1 - f_1\|_{K_1} < \varepsilon$ for those $\eta$. On the other hand we see that ${g_1}'(0) = {f_2}'(0)$. The remaining argumentation is as in the first case, if we replace $f_1$ by $g_1$. □

## 3. A PREPARATORY LEMMA ON CONFORMAL MAPPINGS

We will need the following statement on conformal mappings.

**Lemma 1.** *Let* $D = D_\kappa = \{|z-1| < \kappa\}$ $(\kappa > 0)$ *and let* $H$ *be a (simply connected) Jordan domain with* $H \subset \mathbb{D} \cup D$ *and* $\mathbb{D} \setminus D \subset H$. *The set* $H \cap \partial\mathbb{D} \subset D$ *is assumed to be nonempty and connected.*

*Let* $\Theta$ *denote the Riemann map of* $\mathbb{D}$ *onto* $H$ *with* $\Theta(0) = 0$ *and let some* $\varepsilon > 0$ *be given. With respect to the continuous extension of* $\Theta$ *on* $\overline{\mathbb{D}}$ *we define the set*

$$E = \{z : |z| = 1 \text{ and } |\Theta(z)| > 1\}.$$

*Then* $E$ *is an interval on* $\partial\mathbb{D}$, *and there exists some* $\kappa_0 > 0$ *such that the length of* $E$ *is less than* $\varepsilon$ *if* $\kappa < \kappa_0$.

*Proof.* That $E$ is an interval on $\partial\mathbb{D}$ follows from the assumption on $H \cap \partial\mathbb{D} \subset D$.

If the assertion were not true, we could find some positive $\varepsilon$ and a sequence $\delta_k \to 0, \kappa_k \to 0$ and simply connected domains $H_k$ with the described properties, such that the lengths of the intervals $E_k$ (defined with respect to the associated conformal maps $\Theta_k$ with $\Theta_k(0) = 0$) all are at least $\varepsilon$. Let $V$ be a disc centered at 1. Then it follows by standard arguments that the sequence of the conformal maps $\Theta_k$ converges uniformly on $\overline{\mathbb{D}} \setminus V$ to the identity map. Thus $E_k$ must be contained in $V$ for all sufficiently large $k$. This leads to a contradiction if we take $V$ so small that the length of $\partial\mathbb{D} \cap V$ is less than $\varepsilon$. □

## 4. A SEQUENCE OF FUNCTIONS

Let $C \subset \partial\mathbb{D}$ be a nowhere dense closed set. Then the complement with respect to $\partial\mathbb{D}$ is open, and thus is the countable union of pairwise disjoint open intervals $I_j$ on $\partial\mathbb{D}$. Let $e^{i\alpha_j}$ be the center point of $I_j$. For each $j \in \mathbb{N}$ we fix some open disk $V_j$ with $e^{i\alpha_j} \in V_j$ and $V_j \cap \partial\mathbb{D} \subset I_j$. Moreover we may assume that these $V_j$ are pairwise disjoint.

*Remark* 1. If a sequence $w_j \in \overline{V_j} \cap \mathbb{D}$ $(j \in \mathbb{N})$ is given, then the cluster set of $(w_j)$ consists of all nonisolated points of $C$, as follows from the properties of $C$ as a closed and nowhere dense set. In order to construct a function $g$ which also fulfills the assertion of Theorem 1 with respect to possible isolated points of $C$, we can enlarge this set by adding a discrete subset of $\partial\mathbb{D}$, which contains for each isolated point $c_1 \in C$ a sequence $c_1^\nu \in \partial\mathbb{D} \setminus C$ tending to $c_1$. Then the fixed point cluster $F_c(g)$ of the function $g$ we will construct below contains $c_1$, but not the additional points $c_1^\nu$. We may assume in the following that $C$ has been prepared in this way.

Now we describe a sequence of positive numbers $\varepsilon_j$ $(j \geq 2)$ with $\varepsilon_2 > \varepsilon_3 > \dots$ and $\sum_{j=1}^\infty \varepsilon_j < \frac{1}{3}$.

The aim of this section is to construct a sequence of functions $g_n : \mathbb{D} \to \mathbb{C}$ which fulfills:

(i) $g_n$ is holomorphic and univalent on $\overline{\mathbb{D}}$.
(ii) $|g_n(z) - g_{n-1}(z)| < \varepsilon_n$ for all $z \in \mathbb{D} \setminus V_n$ if $n > 1$.
(iii) $\mathbb{C} \setminus \overline{g_n(\mathbb{D})}$ is connected.
(iv) $e^{i\alpha_j} \notin g_n(\mathbb{D})$ for $j > n$.
(v) For each $n \in \mathbb{N}$ the function $g_n$ has fixed points $z_{n,k} \in V_k$ $(k = 1, \dots, n)$, and it holds that $|z_{n,k} - z_{k,k}| < \frac{1}{2}\,\mathrm{dist}(z_{k,k}, \partial V_k)$ for all $k = 1, \dots, n-1$.

(vi) $g_n$ has in $\overline{\mathbb{D}} \setminus V_n$ the same number of fixed points as $g_{n-1}$ for all $n \geq 2$.
(vii) $g_n$ has no fixed point on $\partial\mathbb{D}\,(n \in \mathbb{N})$.

We try $g_1(z) := \frac{z}{4} + e^{i\alpha_1} \cdot \sigma(\frac{ze^{-i\alpha_1}}{1+\eta}, \eta)$ with $0 < \eta < 1$ (the functions $\sigma(., \eta)$ have been defined close to the end of section 2). The derivative of this function has, for each choice of the parameters, positive real part in $\mathbb{D}$, and therefore it is univalent. The number $\eta$ can be taken in such a way that $g_1$ has a fixed point $z_{1,1} = \rho_1 e^{i\alpha_1}$ in $V_1$. Concerning (v) we having nothing more to check in this first step. Note that $\sigma(.\eta)$ tends uniformly to 0 on each compact subset of $\mathbb{C} \setminus [1, \infty[$. The conditions (i), (iii), (iv) and (vii) are obviously fulfilled for all sufficiently small positive $\eta$, while (ii) and (vi) are empty in this case. If some number $b$ bigger than $\frac{1}{4}$ is given we can arrange that $|g_1| < b$ on $\mathbb{D} \setminus V_1$.

Let, for some $n \in \mathbb{N}$, a function $g_n$ with the mentioned properties be found, and we will construct the function $g_{n+1}$.

We start with this part of (v), which claims that there are appropriate fixed points $z_{n+1,1}, \ldots, z_{n+1,n}$ of our candidate function $F$. The "new" fixed point $z_{n+1,n+1}$ we will discuss later. From (v) we know that $g_n$ has a fixed point $z_{n,k}$ in each of the the discs

$$\Delta_k := \left\{ |z - z_{k,k}| < \frac{1}{2} \operatorname{dist}(z_{k,k}, \partial V_k) \right\} \quad (k = 1, \ldots, n).$$

Since a fixed point of a function $f$ is a solution of the equation $f(z) - z = 0$, we know that 0 is an inner point of every domain $W_k := (g_n - id)(\Delta_k)\,(k = 1, \ldots, n)$. So, if $\varepsilon < \operatorname{dist}(z_{n,k}, \partial W_k)$ for all $k = 1, \ldots, n$, we indeed find fixed points $z_{n+1,k} \in \Delta_k \subset V_k\,(k = 1, \ldots, n)$ of $F$, provided that $|F - g_n| < \varepsilon$ on $\Delta_1 \cup \cdots \cup \Delta_n$. From (vii) we conclude that the fixed point set of $g_n$ in $\overline{\mathbb{D}}$ has positive distance $d$ to $\partial\mathbb{D}$. Therefore we will approximate $g_n$ with a bound

$$\varepsilon < \min \left\{ \varepsilon_{n+1}, \min_{k=1,\ldots,n} \operatorname{dist}(z_{n,k}, \partial W_k), d \right\}.$$

Now we fix some $\varepsilon$ according to this condition. Note that each holomorphic $F : \overline{\mathbb{D}} \to \mathbb{C}$ with $\|F - g_n\|_{\mathbb{D} \setminus V_{n+1}} < \varepsilon$ will have fixed points $z_{n+1,1}, \ldots, z_{n+1,n}$ with the properties mentioned in (v). Moreover, the number of fixed points in $\overline{\mathbb{D}} \setminus V_{n+1}$ of such a function $F$ is the same as of $g_n$ in this set. Thus (vi) is automatically fulfilled if we take $F$ as described.

In particular, $F$ cannot have any fixed point in $\partial\mathbb{D} \setminus V_{n+1}$, as follows from (vii).

Now we consider the point $z_0 := e^{i\alpha_{n+1}}$. Then $z_0 \in \mathbb{C} \setminus \overline{g_n(\mathbb{D})}$ by (v). Since $z_0 \notin V_1 \cup \cdots \cup V_n$ it follows from (ii) that $|g_n(z_0)| \leq |g_1(z_0)| + \sum_{j=2}^{n} \varepsilon_j \leq \frac{1}{3} + \frac{1}{3}$.

By (iii) we find some smooth Jordan arc

$$\Gamma : [0, 1] \to \mathbb{C} \setminus \bigcup_{j \in \mathbb{N} \setminus \{n+1\}} \overline{V_j}$$

with $\Gamma(0) = g_n(z_0)$, $\Gamma(1) = z_0$ and $\Gamma(t) \in \mathbb{C} \setminus \overline{g_n(\mathbb{D})}$ for all $t \in ]0, 1]$.

Moreover there exists some simply connected domain

$$V \subset \mathbb{C} \setminus \left( \overline{g_n(\mathbb{D})} \cup \bigcup_{j \in \mathbb{N} \setminus \{n+1\}} \overline{V_j} \right) \tag{1}$$

(think of $V$ as a small tube containing $\Gamma(\,]0, 1]\,)$) with $\partial V \cap \partial g_n(\mathbb{D}) = \{g_n(z_0)\}$.

*Remark* 2. The domain $V$ depends only on $g_n$ and on $z_0$, and it contains some closed disc $D^* = \{|z - z_0| \leq r\}$ of positive radius.

We find, by Riemann's mapping theorem, some surjective univalent function

$$f_1 : G_1 := \{z \,:\, \big|z - \big(1 + \frac{\kappa}{4}\big)z_0\big| < \frac{\kappa}{4}\} \to V$$

with $f_1(z_0) = g_n(z_0)$, where the parameter $\kappa > 0$ is arbitrary in the moment, but it will be fixed below. With the notations $K_1 = \overline{G_1}$, $G_2 = \mathbb{D}$, $K_2 = \overline{G_2}$ and $f_2 = g_n$ we obtain:

(a) $\Gamma([0,1]) \subset f_1(K_1)$,
(b) $G_1$ and $G_2$ are Jordan domains,
(c) $K_1 \cap K_2 = \{z_0\}$,
(d) $f_1$, resp. $f_2$, is continuous on $K_1$, resp. on $K_2$, and univalent on $G_1$, resp. on $G_2$,
(e) $f_1(K_1) \cap f_2(K_2) = \{g_n(z_0)\}$ and $g_n(z_0) = f_1(z_0) = f_2(z_0)$.

We apply Theorem 2 and find an entire function $f$, which is univalent on a neighborhood $U$ of $K_1 \cup K_2$ and fulfills

$$\|f_1 - f\|_{K_1} < \frac{\varepsilon}{2} \text{ and } \|f_2 - f\|_{K_2} < \frac{\varepsilon}{2}. \tag{2}$$

Let $H \subset \mathbb{D} \cup \{|z - z_0| < \kappa\}$ be a simply connected Jordan domain such that $\mathbb{C} \setminus \overline{H}$ is connected and that $f$ is univalent on $H$.

Now we apply Lemma 1 and consider the conformal map $\Theta : \mathbb{D} \to H$. We will have that

$$\|\Theta - id\|_{\mathbb{D}} < \frac{\varepsilon}{2} \text{ and } G_1 \subset \Theta(V_{n+1} \cap \mathbb{D}), \tag{3}$$

if $\kappa$ is small enough, and for small $\kappa$ we also see that $F^{-1}(D^*) \subset D^*$, where $F := f \circ \Theta$ (which is our candidate for the desired function $g_{n+1}$). We fix such a number $\kappa$.

$F$ is continuous on $\overline{\mathbb{D}}$ and univalent on $\mathbb{D}$ by construction. This gives (i).

On $\mathbb{D} \setminus V_{n+1}$ we see that $|F(z) - g_n(z)| < \varepsilon \le \varepsilon_{n+1}$ by (2) and (3). So (ii) is fulfilled (with $F$ as the candidate for $g_{n+1}$).

$\mathbb{C} \setminus \overline{F(\mathbb{D})}$ is the complement of the closure of the domain $H$, and thus this set is connected by construction. This shows that (iii) holds.

Since $e^{i\alpha_j} \in V_j$ $(j \in \mathbb{N})$ we verify (iv) by (1).

Now we prove the part of (v) concerning the desired fixed point $z_{n+1,n+1}$ of $F$ in $V_{n+1}$.

It has been remarked that the closed disc $D^*$ is mapped under $F^{-1}$ into itself. By Brouwer's fixed point theorem, $F^{-1}$ has a fixed point $z_1$ in $D^*$. But then $z_1 = F^{-1}(z_1) \in V_{n+1}$ by (3) and by $D^* \subset V$.

Finally we discuss property (vii) with respect to $F$. We discuss the $h(z) := F(z) - z$. For each $\delta > 0$ there is of course some complex number $w_0$ with $w_0 \notin h(\partial\mathbb{D})$. If we consider, for suitably small $\delta$, the function $F(z) - w_0$ instead of $F$, then (vii) is fulfilled and (i),...,(vi) remain true.

Now we are ready to define $g_{n+1} := F$ and $z_{n+1,n+1} := z_1$.

The construction of the functions $g_n$ can be done in such a way that, for example, $|g_n(z)| < 2$ for all $n \in \mathbb{N}$ and for all $z \in \mathbb{D}$. The bound 2 could be replaced by each number bigger than 1.

## 5. Proof of Theorem 1

From (ii) and $\sum_{j=1}^{\infty} \varepsilon_j < \frac{1}{3}$ we see that $g = \lim_{n\to\infty} g_n$ exists, and $g$ can be neither the identity nor a constant function. By standard arguments we conclude that $g$ is univalent on $\mathbb{D}$, and it follows from the construction that $|g|$ is bounded. The inequality

$$|z_{n,k} - z_{k,k}| < \frac{1}{2}\,\mathrm{dist}(z_{k,k}, \partial V_k)$$

for all $1 \leq k < n$ in (v) guarantees that none of the sequences $s_k = (z_{n,k})_{n\geq k}$ can tend to $\partial\mathbb{D}$. Thus $g$ has in each $V_k$ a fixed point $z_k$, which obviously fulfills $|z_k - z_{k,k}| \leq \frac{1}{2}\,\mathrm{dist}(z_{k,k}, \partial V_k)$.

The fixed point cluster $F_c(g)$ of the function $g$ contains the cluster set of the sets $V_k$, which is $C$ (or, to be more precise, the set of all nonisolated points of $C$, compare Remark 1).

The validity of the opposite inclusion can be obtained as follows. Let some $\zeta \in \partial\mathbb{D}$ be given, which is not a nonisolated point of $C$ (compare Remark 1). Then there is some open disk $U(\zeta)$, centered at $\zeta$, and some $n_0 \in \mathbb{N}$ with $U(\zeta) \cap V_n = \emptyset$ for all $n \geq n_0$. Therefore, by (vi), we see that the number of fixed points of $g_n$ in $U(\zeta) \cap \mathbb{D}$ has a (with respect to $n$) uniform finite bound. Thus $\zeta$ does not belong to $F_c(g)$. □

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