ALMOST SIMPLE GROUPS OF SUZUKI TYPE
ACTING ON POLYTOPES

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Abstract. Let \( S = Sz(q) \), with \( q \neq 2 \) an odd power of two. For each almost simple group \( G \) such that \( S < G \leq \text{Aut}(S) \), we prove that \( G \) is not a C-group and therefore is not the automorphism group of an abstract regular polytope. For \( G = Sz(q) \), we show that there is always at least one abstract regular polytope \( P \) such that \( G = \text{Aut}(P) \). Moreover, if \( P \) is an abstract regular polytope such that \( G = \text{Aut}(P) \), then \( P \) is a polyhedron.

1. Introduction

In [1], Leemans and Vauthier built an atlas of abstract regular polytopes for small groups. The groups \( Sz(8) \) and \( \text{Aut}(Sz(8)) = Sz(8) : 3 \) were among the groups analysed by Leemans and Vauthier. It turns out that \( Sz(8) \) has seven polytopes, all of rank three, and that \( \text{Aut}(Sz(8)) \) has no polytopes.

In this short note, we prove that if \( G = Sz(q) \) with \( q \neq 2 \) an odd power of two, then all the abstract regular polytopes having \( G \) as automorphism group are of rank three (and there exists at least one such polytope for each value of \( q \)). Moreover, if \( Sz(q) < G \leq \text{Aut}(Sz(q)) \), we show that \( G \) is not a C-group and therefore that there cannot exist an abstract regular polytope having \( G \) as automorphism group.

2. Preliminaries

Thin regular residually connected geometries with a linear diagram, abstract polytopes and string C-groups are the same mathematical objects. The link between these objects may be found for instance in [2]. Here we take the viewpoint of string C-groups because it is the easiest and the most efficient one to define abstract regular polytopes.

As defined for instance in [2], a C-group is a group \( G \) generated by pairwise distinct involutions \( \rho_0, \ldots, \rho_{n-1} \) which satisfy the following property, called the intersection property:

\[
\forall J, K \subseteq \{0, \ldots, n-1\}, \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle.
\]
A C-group \((G, \{\rho_0, \ldots, \rho_{n-1}\})\) is a string C-group if its generators satisfy the following relations:

\[(\rho_j \rho_k)^2 = 1_G \forall j, k \in \{0, \ldots, n-1\} \text{ with } |j - k| \geq 2.\]

In this short note, we prove the following result.

**Theorem 1.** Let \(Sz(q) \leq G \leq \text{Aut}(Sz(q))\) with \(q = 2^{2e+1}\) and \(e > 0\) a positive integer. Then \(G\) is a C-group if and only if \(G = Sz(q)\). Moreover, if \((G, \{\rho_0, \ldots, \rho_{n-1}\})\) is a string C-group, then \(n = 3\).

We may translate this theorem in abstract regular polytopes theory. It means that

- if \(Sz(q) < G \leq \text{Aut}(Sz(q))\), then \(G\) is not the automorphism group of an abstract regular polytope;
- if \(G = Sz(q)\), there exists an abstract regular polytope \(\mathcal{P}\) such that \(G = \text{Aut}(\mathcal{P})\). Moreover, if \(\mathcal{P}\) is an abstract regular polytope such that \(G = \text{Aut}(\mathcal{P})\), then \(\mathcal{P}\) must be an abstract polyhedron, i.e. a rank three polytope.

3. **Almost simple groups of Suzuki type and C-groups**

We first recall an easy lemma which will be used in the proof of Theorem 1. The proof is left to the reader.

**Lemma 1.** Let \(G\) be a group and let \(H\) be a proper subgroup of \(G\) such that all involutions of \(G\) are in \(H\). Then \(G\) is not a C-group.

Obviously, if a group is not a C-group, then it is not a string C-group, and therefore it does not act regularly on an abstract polytope.

**Lemma 2.** Let \(Sz(q) < G \leq \text{Aut}(Sz(q))\) with \(q \neq 2\) an odd power of two. Then \(G\) is not a C-group.

**Proof.** Let \(q = 2^{2e+1}\) with \(e > 0\) an integer. Since \(\text{Aut}(Sz(q)) \cong Sz(q) : C_{2e+1}\) where \(C_{2e+1}\) denotes a cyclic group of order \(2e + 1\), and since \(2e + 1\) is odd, the groups \(Sz(q)\) and \(\text{Aut}(Sz(q))\) have the same number of involutions. Therefore, applying Lemma 1 we conclude that \(G\) is not a C-group. \(\square\)

**Lemma 3.** Let \(G = Sz(q)\) with \(q \neq 2\) an odd power of two. There exists a set \(\{\rho_0, \rho_1, \rho_2\}\) of involutions of \(G\) such that \((G, \{\rho_0, \rho_1, \rho_2\})\) is a string C-group.

**Proof.** We may assume that \(G\) is a permutation group acting two-transitively on a set \(\Omega\) of \(q^2 + 1\) points. Take two involutions \(\rho_0, \rho_1 \in G\) such that \(\langle \rho_0, \rho_1 \rangle = D_{2(q-1)}\), where \(D_{2(q-1)}\) denotes a dihedral group of order \(2(q-1)\). Each of these two involutions fixes exactly one point of \(\Omega\). Let \(p_i \in \Omega\) be the point fixed by \(\rho_i\) \((i = 0, 1)\). It is well known that \(\rho_0 \neq \rho_1\). Take an involution \(\rho_2 \in G\) such that \(\rho_2 \in G_{p_0}\). Then \(\langle \rho_0, \rho_2 \rangle = D_{2n}\) for some \(n\), and since \(\rho_0\) and \(\rho_2\) both fix the same point of \(\Omega\), we have \(n = 2\). Moreover, we have \(\langle \rho_1, \rho_2 \rangle = D_{2m}\) for some \(m\) with \(m \neq 2\) and the order of \(\rho_1\rho_2\) is \(m\). The group \(G = \langle \rho_0, \rho_1, \rho_2 \rangle\) is thus a string C-group. The corresponding polyhedron is of type \(\{q - 1, m\}\). The number \(m\) depends on the choice of \(\rho_2\). \(\square\)

**Lemma 4.** Let \((G, \{\rho_0, \ldots, \rho_{n-1}\})\) be a string C-group such that \(G \cong Sz(q)\) with \(q \neq 2\) an odd power of two. Then \(n = 3\).
Proof. We have \( n > 2 \), for if \( n = 2 \), the group \( G \) must be a dihedral group. Suppose that \( n > 3 \). Then, \( G \) possesses a subgroup \( H = \langle \rho_0, \rho_1, \rho_3, \ldots, \rho_{n-1} \rangle \cong \langle \rho_0, \rho_1 \rangle \times \langle \rho_3, \ldots, \rho_{n-1} \rangle = D_{2m} \times K \), where \( D_{2m} = \langle \rho_0, \rho_1 \rangle \) and \( K = \langle \rho_3, \ldots, \rho_{n-1} \rangle \). Looking at the maximal subgroups of \( Sz(q) \) as given for instance by Michio Suzuki in [3], we readily see that the only subgroups of the form \( D_{2m} \times K \) in \( Sz(q) \) are with \( m = 2 \). If \( H = D_{2m} \times K = 2^2 \times K \), then \( G = \langle H, \rho_2 \rangle \), and the subgroup \( \langle \rho_0 \rangle \) is thus a normal subgroup of \( G \), a contradiction. Therefore, \( n = 3 \). □

Proof of Theorem 1. The proof is obtained by putting together Lemmas 2, 3 and 4. □

References


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