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ALMOST SIMPLE GROUPS OF SUZUKI TYPE ACTING ON POLYTOPES

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ABSTRACT. Let $S = Sz(q)$, with $q \neq 2$ an odd power of two. For each almost simple group G such that $S < G \leq Aut(S)$, we prove that G is not a C-group and therefore is not the automorphism group of an abstract regular polytope. For $G = Sz(q)$, we show that there is always at least one abstract regular polytope $\mathcal{P}$ such that $G = Aut(\mathcal{P})$. Moreover, if $\mathcal{P}$ is an abstract regular polytope such that $G = Aut(\mathcal{P})$, then $\mathcal{P}$ is a polyhedron.

1. INTRODUCTION

In [1], Leemans and Vauthier built an atlas of abstract regular polytopes for small groups. The groups $Sz(8)$ and $Aut(Sz(8)) = Sz(8) : 3$ were among the groups analysed by Leemans and Vauthier. It turns out that $Sz(8)$ has seven polytopes, all of rank three, and that $Aut(Sz(8))$ has no polytopes.

In this short note, we prove that if $G = Sz(q)$ with $q \neq 2$ an odd power of 2, then all the abstract regular polytopes having G as automorphism group are of rank three (and there exists at least one such polytope for each value of q). Moreover, if $Sz(q) < G \leq Aut(Sz(q))$, we show that G is not a C-group and therefore that there cannot exist an abstract regular polytope having G as automorphism group.

2. PREMILINARIES

Thin regular residually connected geometries with a linear diagram, abstract polytopes and string C-groups are the same mathematical objects. The link between these objects may be found for instance in [2]. Here we take the viewpoint of string C-groups because it is the easiest and the most efficient one to define abstract regular polytopes.

As defined for instance in [2], a C-group is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the following property, called the *intersection property*:

$$\forall J, K \subseteq \{0, \ldots, n-1\}, \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle.$$

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A C-group $(G, \{\rho_0, \dots, \rho_{n-1}\})$ is a string C-group if its generators satisfy the following relations:

$$(\rho_j \rho_k)^2 = 1_G \forall j,k \in \{0, \dots, n-1\} \text{ with } \mid j-k \mid \geq 2.$$

In this short note, we prove the following result.

Theorem 1. *Let $Sz(q) \leq G \leq Aut(Sz(q))$ with $q = 2^{2e+1}$ and $e > 0$ a positive integer. Then G is a C-group if and only if $G = Sz(q)$. Moreover, if $(G, \{\rho_0, \dots, \rho_{n-1}\})$ is a string C-group, then $n = 3$.*

We may translate this theorem in abstract regular polytopes theory. It means that

- if $Sz(q) < G \leq Aut(Sz(q))$, then G is not the automorphism group of an abstract regular polytope;
- if $G = Sz(q)$, there exists an abstract regular polytope $\mathcal{P}$ such that $G = Aut(\mathcal{P})$. Moreover, if $\mathcal{P}$ is an abstract regular polytope such that $G = Aut(\mathcal{P})$, then $\mathcal{P}$ must be an abstract polyhedron, i.e. a rank three polytope.

3. Almost simple groups of Suzuki type and C-groups

We first recall an easy lemma which will be used in the proof of Theorem 1. The proof is left to the reader.

Lemma 1. *Let G be a group and let H be a proper subgroup of G such that all involutions of G are in H. Then G is not a C-group.*

Obviously, if a group is not a C-group, then it is not a string C-group, and therefore it does not act regularly on an abstract polytope.

Lemma 2. *Let $Sz(q) < G \leq Aut(Sz(q))$ with $q \neq 2$ an odd power of two. Then G is not a C-group.*

Proof. Let $q = 2^{2e+1}$ with $e > 0$ an integer. Since $Aut(Sz(q)) \cong Sz(q) : C_{2e+1}$ where C_{2e+1} denotes a cyclic group of order $2e+1$, and since $2e+1$ is odd, the groups $Sz(q)$ and $Aut(Sz(q))$ have the same number of involutions. Therefore, applying Lemma 1, we conclude that G is not a C-group. □

Lemma 3. *Let $G = Sz(q)$ with $q \neq 2$ an odd power of two. There exists a set $\{\rho_0, \rho_1, \rho_2\}$ of involutions of G such that $(G, \{\rho_0, \rho_1, \rho_2\})$ is a string C-group.*

Proof. We may assume that G is a permutation group acting two-transitively on a set Ω of q^2+1 points. Take two involutions $\rho_0, \rho_1 \in G$ such that $\langle \rho_0, \rho_1 \rangle = D_{2(q-1)}$, where $D_{2(q-1)}$ denotes a dihedral group of order $2(q-1)$. Each of these two involutions fixes exactly one point of Ω. Let $p_i \in \Omega$ be the point fixed by ρ_i ($i = 0, 1$). It is well known that $p_0 \neq p_1$. Take an involution $\rho_2 \in G$ such that $\rho_2 \in G_{p_0}$. Then $\langle \rho_0, \rho_2 \rangle = D_{2n}$ for some n, and since ρ_0 and ρ_2 both fix the same point of Ω, we have $n = 2$. Moreover, we have $\langle \rho_1, \rho_2 \rangle = D_{2m}$ for some m with $m \neq 2$ and the order of $\rho_1 \rho_2$ is m. The group $G = \langle \rho_0, \rho_1, \rho_2 \rangle$ is thus a string C-group. The corresponding polyhedron is of type $\{q-1, m\}$. The number m depends on the choice of ρ_2. □

Lemma 4. *Let $(G, \{\rho_0, \dots, \rho_{n-1}\})$ be a string C-group such that $G \cong Sz(q)$ with $q \neq 2$ an odd power of two. Then $n = 3$.*

Proof. We have $n > 2$, for if $n = 2$, the group G must be a dihedral group. Suppose that $n > 3$. Then, G possesses a subgroup $H = \langle \rho_0, \rho_1, \rho_3, \ldots, \rho_{n-1} \rangle \cong \langle \rho_0, \rho_1 \rangle \times \langle \rho_3, \ldots, \rho_{n-1} \rangle = D_{2m} \times K$, where $D_{2m} = \langle \rho_0, \rho_1 \rangle$ and $K = \langle \rho_3, \ldots, \rho_{n-1} \rangle$. Looking at the maximal subgroups of $Sz(q)$ as given for instance by Michio Suzuki in [3], we readily see that the only subgroups of the form $D_{2m} \times K$ in $Sz(q)$ are with $m = 2$. If $H = D_{2m} \times K = 2^2 \times K$, then $G = \langle H, \rho_2 \rangle$, and the subgroup $\langle \rho_0 \rangle$ is thus a normal subgroup of G, a contradiction. Therefore, $n = 3$. □

Proof of Theorem 1. The proof is obtained by putting together Lemmas 2, 3 and 4. □

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