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AN ITÔ FORMULA FOR A FRACTIONAL BROWNIAN SHEET WITH ARBITRARY HURST PARAMETERS

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Abstract. By using the white noise theory for a fractional Brownian sheet, we derive an Itô formula for the fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0,1)$.

1. Introduction and preliminaries

For $H_1, H_2 \in (0,1)$, a real-valued fractional Brownian sheet B^H with Hurst parameter $H = (H_1, H_2)$ is a centered Gaussian random field with covariance

$$\mathbb{E}(B^{H_1,H_2}_{t_1,t_2} B^{H_1,H_2}_{s_1,s_2}) = \prod_{i=1}^{2} \frac{1}{2}(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}).$$

The fractional Brownian sheet with Hurst parameter H has the following integral representation:

$$\begin{aligned} B^{H_1,H_2}_{t_1,t_2} &= \frac{C(H)}{\Gamma(H_1 + (1/2))\Gamma(H_2 + (1/2))} \\ &\quad \times \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{i=1}^{2} [((t_i - s_i)_+)^{H_i - (1/2)} - ((-s_i)_+)^{H_i-(1/2)}] dW_{s_1,s_2}, \end{aligned} \tag{1}$$

where W is a standard Brownian sheet, $s_+ = \max\{s, 0\}$ and $C(H)$ is the normalizing constant.

The purpose of this paper is to obtain an Itô formula for the fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0,1)$. The main tool is the white noise theory for the fractional Brownian sheet with $H_1, H_2 \in (0,1)$, given by Hu *et al.* [4]. Our result holds for Hurst parameters $H_1, H_2 \in (0,1)$, whereas that of Tudor and Viens [5] is valid only for Hurst parameters $H_1, H_2 \in (1/2, 1)$.

Now we briefly mention the white noise theory of fractional Brownian sheet, given in [4], to be necessary for our main result. Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space

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of rapidly decreasing smooth functions on $\mathbb{R}^2$. We consider the white noise space $(\Omega, \mathbf{F}, \mathbb{P})$ as the underlying probability space, i.e., $\Omega := \mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions and $\mathbb{P}$ is a unique probability measure such that for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\int_{S'(\mathbb{R}^2)} e^{i\langle\omega, f\rangle} d\mathbb{P}(\omega) = e^{-(1/2)\|f\|^2_{L^2(\mathbb{R}^2)}}.$$

We can extend $\langle \cdot, f\rangle$ to $f \in L^2(\mathbb{R}^2)$ and $\mathbb{E}[\langle\cdot, f\rangle\langle\cdot, g\rangle] = (f, g)_{L^2(\mathbb{R}^2)}$. For $s = (s_1, s_2), t = (t_1, t_2)$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we define $\mathbf{1}_{(s,t)}(x) = \prod_{i=1}^2 \mathbf{1}_{(s_i,t_i)}(x_i)$, where the indicator function $\mathbf{1}_{(s_i,t_i)}(x_i)$ is given by

$$\mathbf{1}_{(s_i,t_i)}(x_i) = \begin{cases} 1 & \text{for } s_i \le x_i < t_i, \\ -1 & \text{for } t_i \le x_i < s_i, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in \mathcal{S}(\mathbb{R}^2)$, we define $I^{H_i}_{\pm} f : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, by

$$I^{H_i}_{\pm} f(x) = \begin{cases} \frac{1}{\Gamma(H_i-(1/2))} \int_0^\infty \frac{f(x\mp u\epsilon_i)}{u^{(3/2)-H_i}} du & \text{for } 1/2 < H_i < 1, \\ f & \text{for } H_i = \frac{1}{2}, \\ \frac{(1/2)-H_i}{\Gamma(H_i+(1/2))} \int_0^\infty \frac{f(x)-f(x\mp u\epsilon_i)}{u^{(3/2)-H_i}} du & \text{for } 0 < H_i < 1/2, \end{cases} \tag{2}$$

where $\epsilon_1 = (1, 0)$ and $\epsilon_2 = (0, 1)$. Let $M^{H_1,H_2}_{\pm\pm} f(x) = I^{H_1}_{\pm}(I^{H_2}_{\pm}) f(x)$. Then a continuous version of $\langle \cdot, M^{H_1,H_2}_{--} \mathbf{1}_{(0,t)}\rangle$ is a fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0, 1)$ on $(\Omega, \mathbf{F}, \mathbb{P})$.

Let $\mathbf{H}_n(x)$ be the n-th Hermite polynomial and h_n the n-th Hermite function, $n = 0, 1, \cdots$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ (with $\mathbb{N} = \{1, 2, \cdots\}$), let us set $\mathbf{e}_\alpha(x_1, x_2) = \prod_{i=1}^2 h_{\alpha_i}(x_i)$. Denote by $\mathbf{A}$ the set of all finite sequences $\mathbf{a} = (a_1, a_2, \cdots, a_m)$ with $a_i \in \{0\} \cup \mathbb{N}$, $m = 1, 2, \cdots$. For $\mathbf{a} \in \mathbf{A}$, we set $\mathbf{a}! = \prod_{i=1}^\infty a_i!$ and $|\mathbf{a}| = \sum_{i=1}^\infty a_i$. Let $\alpha^{(i)}$, $i = 1, 2, \cdots$, be a fixed ordering of $\mathbb{N}^2$ such that for $i < j$, $|\alpha^{(i)}| \le |\alpha^{(j)}|$. With these notations, we define

$$\mathbf{H}_{\mathbf{a}}(\omega) = \prod_{i=1}^\infty \mathbf{H}_{a_i}(\langle\omega, \mathbf{e}_{\alpha^{(i)}}\rangle).$$

We recall the following chaos expansion theorem.

Theorem 1. *Let $F \in \mathbb{L}^2 := L^2(\Omega, \mathbf{F}, \mathbb{P})$. Then there exist constants $c_{\mathbf{a}} \in \mathbb{R}$ for $\mathbf{a} \in \mathbf{A}$ such that*

$$F(\omega) = \sum_{\mathbf{a}\in\mathbf{A}} c_{\mathbf{a}} \mathbf{H}_{\mathbf{a}}(\omega) \text{ limit in } \mathbb{L}^2. \tag{3}$$

Furthermore, we have the isometry $\|F\|^2_{\mathbb{L}^2} = \sum_{\mathbf{a}\in\mathbf{A}} \mathbf{a}! c^2_{\mathbf{a}}$.

Let $\mathbb{Z}$ be the set of all integers. For $p \in \mathbb{Z}$ and F given in (3), we define the norm $\|F\|^2_p = \sum_{\mathbf{a}\in\mathbf{A}} c^2_{\mathbf{a}} \mathbf{a}!(2\mathbb{N})^{p\mathbf{a}}$, where $(2\mathbb{N})^{\mathbf{a}} = \prod_{l=1}^m (2l)^{a_l}$ with $\mathbf{a} = (a_1, \cdots, a_m)$. If $p \in \mathbb{N}$, we define the space $(\mathbb{S})_p = \{F \in \mathbb{L}^2 : \|F\|^2_p < \infty\}$ and endow $(\mathbb{S})_p$ with the norm $\|\cdot\|_p$. Also, define $(\mathbb{S})_{-p} = \{F \in \mathbb{L}^2 : \|F\|^2_{-p} < \infty\}$.

Definition 1. (1) The projective limit of the spaces $(\mathbb{S})_p$, $p \in \mathbb{N}$, is called the space of the stochastic test functions and denoted by $(\mathbb{S})$.

(2) The inductive limit of the spaces $(\mathbb{S})_{-p}$, $p \in \mathbb{N}$, is called the space of stochastic distributions and denoted by $(\mathbb{S})^*$.

We first note that $(f, M_{--}^{H_1,H_2}\mathbf{1}_{(0,t)})_{L^2(\mathbb{R}^2)} = (M_{++}^{H_1,H_2}f, \mathbf{1}_{(0,t)})_{L^2(\mathbb{R}^2)}$. Hence with the above notations, the chaos expansion of fBm is given by

$$B_t^{H_1,H_2}(\omega) = \sum_{l=1}^{\infty}(M_{++}^{H_1,H_2}\mathbf{e}_{\alpha^{(l)}}, \mathbf{1}_{(0,t)})_{L^2(\mathbb{R}^2)}\mathbf{H}_{\epsilon_l}(\omega),$$

where $\epsilon_l = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the l-th unit vector. The two-parameter fractional white noise $W_{t_1,t_2}^{H_1,H_2}(\omega)$ is defined by the derivative in $(\mathbb{S})^*$ of $B_{t_1,t_2}^{H_1,H_2}(\omega)$:

$$W_{t_1,t_2}^{H_1,H_2} = \sum_{l=1}^{\infty}(M_{++}^{H_1,H_2}\mathbf{e}_{\alpha^{(l)}})(t_1,t_2)\mathbf{H}_{\epsilon_l}(\omega). \tag{4}$$

For $\alpha^{(l)} = (\alpha_1^{(l)}, \alpha_2^{(l)})$, we define two partial fractional white noises:

$$\partial_{t_1}B_{t_1,t_2}^{H_1,H_2}(\omega) = \sum_{l=1}^{\infty}(I_+^{H_1}h_{\alpha_1^{(l)}})(t_1)(I_+^{H_2}h_{\alpha_2^{(l)}}, \mathbf{1}_{(0,t_2)})_{L^2(\mathbb{R}^2)}\mathbf{H}_{\epsilon_l}(\omega), \tag{5}$$

$$\partial_{t_2}B_{t_1.t_2}^{H_1,H_2}(\omega) = \sum_{l=1}^{\infty}(I_+^{H_1}h_{\alpha_1^{(l)}}, \mathbf{1}_{(0,t_1)})_{L^2(\mathbb{R}^2)}(I_+^{H_2}h_{\alpha_2^{(l)}})(t_2)\mathbf{H}_{\epsilon_l}(\omega). \tag{6}$$

By the direct extension of the method used by Elliott and Van der Hoek [3] in the one-parameter case, we easily prove that these noises, defined by (4), (5) and (6), satisfy that for each $(t_1, t_2) \in \mathbb{R}^2$,

$$\partial_{t_1}B_{t_1,t_2}^{H_1,H_2},\ \partial_{t_1}B_{t_1,t_2}^{H_1,H_2} \text{ and } W_{t_1,t_2}^{H_1,H_2} \in (\mathbb{S})^*.$$

We give the definition on $(\mathbb{S})^*$-valued integrals for the two-parameter case.

Definition 2. Let $Y : \mathbb{R}^2 \to (\mathbb{S})^*$ be a given function satisfying

$$\int_{\mathbb{R}^2}\langle\langle Y(t_1,t_2), \varphi\rangle\rangle_{(\mathbb{S})^*,(\mathbb{S})}dt_1dt_2 < \infty \text{ for all } \varphi \in (\mathbb{S}), \tag{7}$$

where $\langle\langle\cdot,\cdot\rangle\rangle_{(\mathbb{S})^*,(\mathbb{S})}$ is the bi-pairing $(\mathbb{S})^*$ and $(\mathbb{S})$. Then we define $(\mathbb{S})^*$-valued integral $\int_{\mathbb{R}^2} Y(t_1,t_2)dt_1dt_2$ to be the unique element of $(\mathbb{S})^*$ such that for all $\varphi \in (\mathbb{S})$,

$$\langle\langle\int_{\mathbb{R}^2} Y(t_1,t_2)dt_1dt_2, \varphi\rangle\rangle_{(\mathbb{S})^*,(\mathbb{S})} = \int_{\mathbb{R}^2}\langle\langle Y(t_1,t_2), \varphi\rangle\rangle_{(\mathbb{S})^*,(\mathbb{S})}dt_1dt_2.$$

If (7) holds, we then say that Y is *integrable in* $(\mathbb{S})^*$. If $F, G \in (\mathbb{S})^*$, then their Wick product $F \diamond G \in (\mathbb{S})^*$. Since $\partial_{t_1}B_{t_1,t_2}^{H_1,H_2} \in (\mathbb{S})^*$ and $\partial_{t_1}B_{t_1,t_2}^{H_1,H_2} \in (\mathbb{S})^*$, we define a mixed fractional white noise by

$$\partial_{t_1}B_{t_1,t_2}^{H_1,H_2} \diamond \partial_{t_1}B_{t_1,t_2}^{H_1,H_2} \in (\mathbb{S})^*.$$

So the stochastic integrals of various types can be defined.

Definition 3. Suppose that $Y : \mathbb{R}^2 \to (\mathbb{S})^*$ is a given function such that

$$\begin{aligned} &\int_{\mathbb{R}^2} Y(t_1, t_2) \diamond \left(\partial_{t_1} B^{H_1,H_2}_{t_1,t_2} \diamond \partial_{t_2} B^{H_1,H_2}_{t_1,t_2} \right) dt_1 dt_2, \\ &\int_{\mathbb{R}^2} Y(t_1, t_2) \diamond W^{H_1,H_2}_{t_1,t_2} dt_1 dt_2, \\ &\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1-1} t_2^{2H_2} \diamond \partial_{t_2} B^{H_1,H_2}_{t_1,t_2} dt_1 dt_2, \end{aligned}$$

and

$$\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1} t_2^{2H_2-1} \diamond \partial_{t_1} B^{H_1,H_2}_{t_1,t_2} dt_1 dt_2$$

are integrable in $(\mathbb{S})^*$. Then we define the following $(\mathbb{S})^*$-valued integrals:

$$\begin{aligned} &\int_{\mathbb{R}^2} Y(t_1, t_2) d\tilde{M}(t_1, t_2) \\ = &\int_{\mathbb{R}^2} Y(t_1, t_2) \diamond \left(\partial_{t_1} B^{H_1,H_2}_{t_1,t_2} \diamond \partial_{t_2} B^{H_1,H_2}_{t_1,t_2} \right) dt_1 dt_2, \\ &\int_{\mathbb{R}^2} Y(t_1, t_2) dB^{H_1,H_2}_{t_1,t_2} \\ = &\int_{\mathbb{R}^2} Y(t_1, t_2) \diamond W^{H_1,H_2}_{t_1,t_2} dt_1 dt_2, \\ &\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1-1} t_2^{2H_2} dt_1 d_{t_2} B^{H_1,H_2}_{t_1,t_2} \\ = &\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1-1} t_2^{2H_2} \diamond \partial_{t_2} B^{H_1,H_2}_{t_1,t_2} dt_1 dt_2, \\ &\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1} t_2^{2H_2-1} d_{t_1} B^{H_1,H_2}_{t_1,t_2} dt_2 \\ = &\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1} t_2^{2H_2-1} \diamond \partial_{t_1} B^{H_1,H_2}_{t_1,t_2} dt_1 dt_2. \end{aligned}$$

Remark 1. We adopt the same notation $\tilde{M}(t_1, t_2)$ as that used by Tudor and Viens [5] in which $d\tilde{M}(t_1, t_2)$ is interpreted as $d_{t_1} B^{H_1,H_2}_{t_1,t_2} \cdot d_{t_2} B^{H_1,H_2}_{t_1,t_2}$. Here the notation $d_{t_1} B^{H_1,H_2}_{t_1,t_2}$ is used to denote the differential of the fractional Brownian motion given by $t_1 \to B^{H_1,H_2}_{t_1,t_2}$ for fixed t_2, and similarly for $d_{t_2} B^{H_1,H_2}_{t_1,t_2}$. In our case, by (4), (5) and (6), we have

$$\begin{aligned} \tilde{M}(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} \int_0^{s_2} W^{H_1,H_2}_{s_1,u_2} du_2 \diamond \int_0^{s_1} W^{H_1,H_2}_{u_1,s_2} du_1 ds_1 ds_2 \\ &= \int_{s_1=0}^{t_1} \int_{s_2=0}^{t_2} d_{s_1} B^{H_1,H_2}_{s_1,s_2} \diamond d_{s_2} B^{H_1,H_2}_{s_1,s_2}. \end{aligned}$$

Hence we can also interpret $d\tilde{M}(t_1, t_2)$ as $d_{s_1} B^{H_1,H_2}_{s_1,s_2} \diamond d_{s_2} B^{H_1,H_2}_{s_1,s_2}$, which is the reason for using the notation $\tilde{M}$. □

2. Itô formula

In this section we give the Itô formula for the fractional Brownian sheet. The following theorem is the extension of the one-parameter case given by Bender [1] and Biagini *et al.* [2].

Theorem 2. *Let $f \in C^4(\mathbb{R})$ and let $(B^{H_1,H_2}_{t_1,t_2})$, $(t_1,t_2) \in \mathbb{R}^2$, be a fractional Brownian sheet with $H_1, H_2 \in (0,1)$. Also, we assume that for each $(t_1,t_2) \in \mathbb{R}^2$, $f(B^{H_1,H_2}_{t_1,t_2}) \in \mathbb{L}^2$. Then we have that in $\mathbb{L}^2$,*

$$
\begin{aligned}
f(B^{H_1,H_2}_{t_1,t_2}) \quad = \quad & f(0) + \int_0^{t_1}\int_0^{t_2} f^{(1)}(B^{H_1,H_2}_{s_1,s_2})dB^{H_1,H_2}_{s_1,s_2} \\
& +2H_1H_2\int_0^{t_1}\int_0^{t_2} f^{(2)}(B^{H_1,H_2}_{s_1,s_2})s_1^{2H_1-1}s_2^{2H_2-1}ds_1ds_2 \\
& + \int_0^{t_1}\int_0^{t_2} f^{(2)}(B^{H_1,H_2}_{s_1,s_2})d\tilde{M}(s_1,s_2) \\
& +H_1\int_0^{t_1}\int_0^{t_2} f^{(3)}(B^{H_1,H_2}_{s_1,s_2})s_1^{2H_1-1}s_2^{2H_2}ds_1d_{s_2}B^{H_1,H_2}_{s_1,s_2} \\
& +H_2\int_0^{t_1}\int_0^{t_2} f^{(3)}(B^{H_1,H_2}_{s_1,s_2})s_1^{2H_1}s_2^{2H_2-1}d_{s_1}B^{H_1,H_2}_{s_1,s_2}ds_2 \\
& +H_1H_2\int_0^{t_1}\int_0^{t_2} f^{(4)}(B^{H_1,H_2}_{s_1,s_2})s_1^{4H_1-1}s_2^{4H_2-1}ds_1ds_2,
\end{aligned} \tag{8}
$$

where $f^{(k)}(x) = (d^k/dx^k)f(x)$ for $k = 1, \cdots, 4$.

Proof. For any constant $c \in \mathbb{R}$, we define $g(x) = e^{cx}$ and put $Y(t_1,t_2) = g(B^{H_1,H_2}_{t_1,t_2})$. By definition of Wick exponential, we have

$$
Y(t_1,t_2) = \exp^{\diamond}\left(cB^{H_1,H_2}_{t_1,t_2} + \frac{1}{2}c^2t_1^{2H_1}t_2^{2H_2}\right).
$$

By using Wick calculus in $(\mathbb{S})^*$, we obtain

$$
\begin{aligned}
\frac{d^2}{dt_2dt_1}Y(t_1,t_2) \quad = \quad & cY(t_1,t_2) \diamond \frac{d^2}{dt_2dt_1}B^{H_1,H_2}_{t_1,t_2} \\
& +c^2Y(t_1,t_2)2H_1H_2t_1^{2H_1-1}t_2^{2H_2-1} \\
& +c^2Y(t_1,t_2) \diamond \left(\partial_{t_1}B^{H_1,H_2}_{t_1,t_2} \diamond \partial_{t_2}B^{H_1,H_2}_{t_1,t_2}\right) \\
& +c^2Y(t_1,t_2)H_2t_1^{2H_1}t_2^{2H_2-1} \diamond \frac{d}{dt_1}B^{H_1,H_2}_{t_1,t_2} \\
& +c^3Y(t_1,t_2)H_1t_1^{2H_1-1}t_2^{2H_2} \diamond \partial_{t_2}B^{H_1,H_2}_{t_1,t_2} \\
& +c^4Y(t_1,t_2)2H_1H_2t_1^{4H_1-1}t_2^{4H_2-1}.
\end{aligned} \tag{9}
$$

We write (9) as follows (with $g(x)=e^{cx}$): in $(\mathbb{S})^*$,

$$
\begin{aligned}
g(B^{H_1,H_2}_{t_1,t_2}) &= g(0)+\int_0^{t_1}\int_0^{t_2} g^{(1)}(B^{H_1,H_2}_{s_1,s_2})\diamond \frac{d^2}{ds_2ds_1}B^{H_1,H_2}_{s_1,s_2}ds_1ds_2\\
&\quad +2H_1H_2\int_0^{t_1}\int_0^{t_2} g^{(2)}(B^{H_1,H_2}_{s_1,s_2})s_1^{2H_1-1}s_2^{2H_2-1}ds_1ds_2\\
&\quad +\int_0^{t_1}\int_0^{t_2} g^{(2)}(B^{H_1,H_2}_{s_1,s_2})\diamond\left(\partial_{s_1}B^{H_1,H_2}_{s_1,s_2}\diamond\partial_{s_2}B^{H_1,H_2}_{s_1,s_2}\right)ds_1ds_2\\
&\quad +H_2\int_0^{t_1}\int_0^{t_2} g^{(3)}(B^{H_1,H_2}_{s_1,s_2})s_1^{2H_1}s_2^{2H_2-1}\diamond\partial_{s_1}B^{H_1,H_2}_{s_1,s_2}ds_1ds_2\\
&\quad +H_1\int_0^{t_1}\int_0^{t_2} g^{(3)}(B^{H_1,H_2}_{s_1,s_2})s_1^{2H_1-1}s_2^{2H_2}\diamond\partial_{s_2}B^{H_1,H_2}_{s_1,s_2}ds_1ds_2\\
&\quad +2H_1H_2\int_0^{t_1}\int_0^{t_2} g^{(4)}(B^{H_1,H_2}_{s_1,s_2})s_1^{4H_1-1}s_2^{4H_2-1}ds_1ds_2.
\end{aligned}
\tag{10}
$$

Let $\mathcal{E}$ be the family of linear combination of the functions e^{cx}, i.e.,

$$\mathcal{E}=\left\{\sum_{k=1}^n a_ke^{c_kx}|n\in\mathbb{N},a_k,c_k\in\mathbb{R}\right\}.$$

For given $f\in C^4(\mathbb{R})$, we can find a sequence $f_n\in\mathcal{E}$ such that as $n\to\infty$,

$$f_n(x)\to f(x) \text{ and } f_n^{(k)}(x)\to f^{(k)}(x),\ k=1,\cdots,4. \tag{11}$$

By linearity (10) is true for $f_n\in\mathcal{E}$. Note that the following functions are continuous in $(S)^*$:

$$(s_1,s_2)\to f^{(k)}(B^{H_1,H_2}_{s_1,s_2}),k=1,\cdots,4,W^{H_1,H_2}_{s_1,s_2},\partial_{s_1}B^{H_1,H_2}_{s_1,s_2},\text{ and }\partial_{s_2}B^{H_1,H_2}_{s_1,s_2}.$$

Hence the integrals of the right-hand side in (8) exist. By (11) and the dominated convergence theorem, the asserted equation (8) holds in $(\mathbb{S})^*$. But the left-hand side is in $\mathbb{L}^2$ and the theorem is established. □

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