TOEPLITZ OPERATORS ON BLOCH-TYPE SPACES

ZHIJIAN WU, RUHAN ZHAO, AND NINA ZORBOSKA

Abstract. We characterize complex measures $\mu$ on the unit disk for which the Toeplitz operator $T^\alpha_\mu$, $\alpha > 0$, is bounded or compact on the Bloch type spaces $B^\alpha$.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the unit disk on the complex plane. Let $dA(z) = \frac{1}{\pi} \, dx \, dy$ be the normalized Lebesgue measure on $\mathbb{D}$.

For a complex measure $\mu$, $\alpha > 0$, and $b \in L^1$, define a Toeplitz operator as follows:

$$T^\alpha_\mu(b)(z) = \alpha \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha - 1} b(w)}{(1 - \overline{w}z)^{\alpha + 1}} \, d\mu(w).$$

We also define the general Bergman projection of the measure $\mu$ and for $\alpha > -1$ as follows:

$$P_\alpha(\mu)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha b(w)}{(1 - \overline{w}z)^{\alpha + 2}} \, d\mu(w).$$

The general Bergman projection of the function $b$ is

$$P_\alpha(b)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha b(w)}{(1 - \overline{w}z)^{\alpha + 2}} \, dA(w).$$

Thus $T^\alpha_\mu(b)(z) = P_{\alpha - 1}(\mu b)(z)$, where $d\mu_b(z) = b(z) \, d\mu(z)$. Note that these choices of the indexes provide the standard notation for the general Bergman projections and for the standard case $\alpha = 1$ as follows:

When $\alpha = 1$ and the measure $\mu$ is such that $d\mu(z) = f(z) \, dA(z)$, with $f \in L^1$, we have that $T^1_\mu$ is the standard Toeplitz operator defined by

$$T^1_\mu(b) = T_f(b) = P(f b), \quad \forall b \in L^1.$$

Here $P = P_0$ denotes the standard Bergman projection. Recall that if $f$ is in $L^\infty$, then $T_f$ is bounded on the Bergman spaces $L^p_\alpha$, $p > 1$.

Toeplitz operators have been studied extensively on the Bergman spaces by many authors. For references, see for example [9]. In this paper we study the boundedness and compactness of general Toeplitz operators on the $\alpha$-Bloch spaces.

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For $\alpha > 0$, the $\alpha$-\textit{Bloch spaces} $\mathcal{B}^{\alpha}$ are the spaces of analytic functions $f$ on $\mathbb{D}$ such that
\[
M^{\alpha}(f) = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^{\alpha} < \infty.
\]
Each $\mathcal{B}^{\alpha}$ is a Banach space with norm of $f$ equal to $\|f\|_{\mathcal{B}^{\alpha}} = M^{\alpha}(f) + |f(0)|$.

An analytic function on $\mathbb{D}$ belongs to the \textit{little $\alpha$-Bloch space} $\mathcal{B}_{0}^{\alpha}$, $\alpha > 0$, whenever
\[
\lim_{|z| \to 1} |f'(z)|(1 - |z|^2)^{\alpha} = 0.
\]

The spaces $\mathcal{B}_{0}^{\alpha}$ are the subspaces of $\mathcal{B}^{\alpha}$ that are the closure of the polynomials with respect to the $\mathcal{B}^{\alpha}$ norm.

The $\alpha$-Bloch spaces and operators on them have been studied in many different contexts. For more references and details on the next few facts stated below see [9] and [10].

For $\alpha = 1$, $\mathcal{B}^{1} = \mathcal{B}$ is the classical Bloch space. There is a natural connection between the Bloch space and the Toeplitz operators via the fact that $P(L^{\infty}) = \mathcal{B}$. A necessary condition for boundedness of $T_{f}$ on $\mathcal{B}$ is that $Pf \in \mathcal{B}$, and so the condition is satisfied whenever $f \in L^{\infty}$. As we will see later, this is not a sufficient condition for boundedness of $T_{f}$ on $\mathcal{B}$.

In general, the growth condition of a function $f$ in $\mathcal{B}^{\alpha}$ with $f(0) = 0$ is determined by $|f(z)| \leq \|f\|_{\mathcal{B}^{\alpha}} \int_{0}^{1} \frac{|z|dt}{(1 - |z|^2)^{\alpha}}$. Thus, for $\alpha = 1$, we get that
\[
|f(z)| \leq \|f\|_{\mathcal{B}} \log \frac{1}{1 - |z|},
\]
while for $\alpha > 1$ we have that $|f(z)| \leq \frac{1}{\alpha - 1} \|f\|_{\mathcal{B}^{\alpha}} \frac{1}{(1 - |z|)^{\alpha - 1}}$. For $0 < \alpha < 1$, the spaces $\mathcal{B}^{\alpha}$ are Lipschitz spaces $\text{Lip}_{1 - \alpha}$ and $\mathcal{B}^{\alpha} \subset H^{\infty}$.

For $0 < \alpha < \beta$, we have that $\mathcal{B}^{\alpha} \subset \mathcal{B}^{\beta}$. The Bloch space $\mathcal{B}$ is included in all Bergman spaces $L^{p}_{\alpha}$, $p \geq 1$, but for large $\alpha$, such as $\alpha \geq 2$, $\mathcal{B}^{\alpha}$ gets so large that, for example, it includes the Bergman space $L^{2}_{\alpha}$.

In presenting our results we will also refer to the logarithmic Bloch space.

The \textit{logarithmic Bloch space} $\text{LB}$ is the space of analytic functions $f$ on $\mathbb{D}$ such that
\[
\sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} < \infty.
\]

Correspondingly, the \textit{little logarithmic Bloch space} $\text{LB}_{0}$ is the space of analytic functions $f$ on $\mathbb{D}$ such that
\[
\lim_{|z| \to 1} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} = 0.
\]

Throughout the paper, in order to have the operator $T_{\mu}^{\alpha}$, $\alpha > 0$ well defined, we will assume that the complex measure $\mu$ is such that, for all $g$ in $\mathcal{B}^{\alpha}$
\[
\int_{\mathbb{D}} |g(w)|(1 - |w|^2)^{\alpha - 1}|d\mu(w)| < \infty.
\]

Note that in the case when $d\mu(z) = f(z)dA(z)$, a sufficient condition on $\mu$, so that the integral above is finite, is that $f \in L^{1}$, when $\alpha > 1$, $f \in L^{1}(\log \frac{1}{1 - |w|^2}dA(w))$, when $\alpha = 1$ and $f \in L^{1}((1 - |w|^2)^{\alpha - 1}dA(w))$, when $0 < \alpha < 1$. 

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2. Bounded Toeplitz operators

In this section we will present our main characterization of bounded Toeplitz operators on general $\alpha$-Bloch spaces. The result includes a restriction on the inducing measure that is a generalization of several known cases.

For a complex measure $\mu$ and $\alpha > 0$, we will say that $\mu$ satisfies the condition $R_\alpha$ if

$$R_\alpha(\mu)(w) = \alpha(1 - |w|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{(z - w)(1 - w \bar{z})^{\alpha+1}} \, d\mu(z) \in L^\infty.$$  

We get another form of $R_\alpha$ using the identity $(1 - |w|^2) = 1 - w \bar{z} + w(\bar{z} - \bar{w})$:

$$R_\alpha(\mu)(w) = \alpha \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{(z - w)(1 - w \bar{z})^{\alpha}} \, d\mu(z) + w P_{\alpha-1}(\mu)(w).$$

The special case when the measure $\mu$ is such that $d\mu(z) = f(z)d\mu(z)$, with $f \in H^\infty$, is the case when the Toeplitz operator is a multiplication operator. For $f \in B^\alpha$, as we will see below, $\mu$ satisfies the condition $R_\alpha$ if and only if $f \in H^\infty$.

Arazy’s ([1]) and Zhu’s ([8] and [10]) results provide a complete characterization of bounded multiplication operators on $\alpha$-Bloch spaces. We will state them in terms of multipliers.

**Theorem A ([10]).** Let $M(X)$ denote the space of multipliers of the Banach space of functions $X$, i.e., $M(X) = \{f \in X : fg \in X, \forall g \in X\}$.

(i) If $0 < \alpha < 1$, then $M(B^\alpha) = B^\alpha$ and $M(B^\alpha_0) = B^\alpha_0$.

(ii) If $\alpha = 1$, then $M(B^\alpha) = M(B^\alpha_0) = H^\infty(D) \cap LB$.

(iii) If $\alpha > 1$, then $M(B^\alpha) = M(B^\alpha_0) = H^\infty(D)$.

As a consequence of the above theorem we can see that, for example, not every bounded function $f$ induces a bounded Toeplitz operator $T_f$ on the Bloch space $B$. As we will see later, for $f \in L^\infty$, $T_f$ is bounded on $B$ if and only if $Pf \in LB$, which is a generalization of the analytic case of the above theorem. We will also see that there exist unbounded $L^1$ functions that induce bounded Toeplitz operators on the Bloch spaces.

In the next proposition we give more details about the condition $R_\alpha$ in some special cases.

**Proposition 2.1.** Let $\mu$ be a measure such that $d\mu(z) = f(z)d\mu(z)$, with $f \in L^1$. Then:

(i) If $f \in L^\infty$, $\mu$ satisfies the condition $R_\alpha$, $\alpha > 0$.

(ii) If $f$ is analytic and in $B^\alpha$, then $R_\alpha(\mu)(w) = w f(w)$.

(iii) If $f$ is conjugate analytic, $\hat{f}$ is in $B^\alpha$ and $f(0) = 0$, then $R_\alpha(\mu)(w) = \frac{1}{\pi} \hat{f}(\bar{w})$.

**Proof.** Let $\psi_\omega(z) = \frac{\omega - z}{\omega - \bar{z}}$, for $z, \omega$ in $\mathbb{D}$.

(i) For $\alpha > 0$ and $f \in L^\infty$,

$$|R_\alpha(\mu)(w)| = \alpha(1 - |w|^2) \left| \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1} f(z)}{(z - w)(1 - w \bar{z})^{\alpha+1}} \, dA(z) \right| \\ \leq \alpha \|f\|_{L^\infty} (1 - |w|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{|z - w||1 - w \bar{z}|^{\alpha+1}} \, dA(z) \\ \leq \alpha c \|f\|_{L^\infty} (1 - |w|^2)(1 - |w|^2)^{-1} = \alpha c \|f\|_{L^\infty}. $$
The second inequality follows from the fact that \( \int_{\mathbb{D}} \frac{(1-|z|^2)^\beta}{|z-w|^\gamma} dA(z) \) is equivalent to \((1-|w|^2)^{\beta-\gamma+1}\), whenever \( \beta - \gamma + 1 < 0 \), which can be easily derived from the well-known Forelli-Rudin estimates (see [6], page 17, or [5], page 53). The half-plane version of the proof of this equivalency can be found in [7].

Thus \( R_\alpha(\mu)(w) \in L^\infty \), and (i) is proved.

(ii) Let \( f \) be in \( B^\alpha \). With a change of variable \( z = \psi_w(u) \), we get that

\[
R_\alpha(\mu)(w) = \alpha \int_{\mathbb{D}} \left( 1 - |\psi_w(u)|^2 \right)^{\alpha-1} \frac{1}{|1-wu|^\alpha+1} f \circ \psi_w(u) dA(u) + wP_{\alpha-1}(f)(w)
\]

\[
= -\alpha \int_{\mathbb{D}} \left( 1 - |u|^2 \right)^{\alpha-1} \frac{1}{u(1-\overline{w}u)^\alpha+1} f \circ \psi_w(u) dA(u) + w f(w) = I + w f(w).
\]

For a fixed \( w \) we have that \( \frac{f \circ \psi_w(u)}{(1-\overline{w}u)^{\alpha+1}} \) belongs to \( B^\alpha \) whenever \( f \in B^\alpha \), and so \( \frac{(1-|u|^2)^{\alpha-1} f \circ \psi_w(u)}{(1-\overline{w}u)^{\alpha+1}} \) must be in \( L^1 \). Using Taylor series expansion it is not hard to see that then \( I = 0 \). Hence \( R_\alpha(\mu)(w) = w f(w) \).

(iii) Let \( f \) be conjugate analytic and let \( f(0) = 0 \). Then \( P_{\alpha-1}(f)(w) = 0 \), and as in the proof of part (ii), we have that

\[
\overline{R_\alpha(\mu)(w)} = -\alpha \int_{\mathbb{D}} \left( 1 - |u|^2 \right)^{\alpha-1} \frac{1}{u(1-\overline{w}u)^\alpha+1} \overline{f \circ \psi_w(u)} dA(u) = \frac{\overline{f(0)} - \overline{f}(w)}{w} = \frac{1}{w} \overline{f}(w).
\]

\[ \square \]

Remark. If \( f \) from the previous proposition is harmonic, then \( R_\alpha(\mu) \) is also harmonic. If also \( f \) is a harmonic extension of an \( L^1(T) \) function, then \( R_\alpha(\mu)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}) \). Thus, for \( f \) harmonic and in \( L^\infty \), we have that \( R_\alpha(\mu) \) is also harmonic and in \( L^\infty \), with \( \|R_\alpha(\mu)\|_\infty = \|f\|_\infty \).

We will use the following lemma in the proof of our next theorem. The result is known in a more general form and can be found, for example, in [2]. We include a proof for completeness.

**Lemma 2.1.** Let \( h \in L^1_a \) and \( g \in B^\alpha, \alpha > 0 \). Then we have that

\[
H_\alpha^a h(z) = (I - P_\alpha) (\bar{g} h)(z) = \int_{\mathbb{D}} \frac{g'(w)}{w} \frac{h(w) (1 - |w|^2)^{\alpha+1}}{(z - \overline{w})^{\alpha+1}} dA(w).
\]
Proof. We will use the following three formulas which can be found in [3] and [10].

For \( g \in B^\alpha \) and \( h \in L^1_u \):

(a) \( H^\alpha_g h(z) = (\alpha + 1) \int_D (\bar{g}(z) - \bar{g}(u)) \frac{(1 - |u|^2)^\alpha}{(1 - \bar{u}z)^{\alpha+2}} h(u) \, dA(u) \),

(b) \( g(z) = g(0) + \int_D \frac{(1 - |u|^2)^\alpha g'(u)}{u(1 - \bar{u}z)^{\alpha+2}} dA(u) \),

(c) \( (H^\alpha_g h) \circ \psi_z \) \( (\psi'_z) \frac{dw}{dz} = H^\alpha_{\bar{g} \circ \psi_z} \) \( (h \circ \psi_z) \) \( (\psi'_z) \frac{dw}{dz} \).

Without loss of generality, let us assume that \( g(0) = 0 \). Then

\[
H^\alpha_g h(0) = (\alpha + 1) \int_D (\bar{g}(u))(1 - |u|^2)^\alpha h(u) \, dA(u)
= -(\alpha + 1) \int_D \left( \int_D \frac{(1 - |u|^2)^{\alpha+1} g'(w)}{w(1 - \bar{u}w)^{\alpha+2}} \, dA(w) \right) (1 - |u|^2)^\alpha h(u) \, dA(u)
= -\int_D \frac{(1 - |w|^2)^{\alpha+1} g'(w)}{w} h(w) \, dA(w).
\]

Since \( 1 - |\psi_z(v)|^2 = (1 - |v|^2) |\psi'_z(v)| \) and \( |\psi'_z(v)| = \frac{|z|^2 - 1}{(1 - zv)^2} = \frac{1}{\psi_z'(\psi_z(v))} \), we have:

\[
H^\alpha_g h(z) = \frac{1}{(|z|^2 - 1)^{\frac{\alpha+2}{2}}} \left( (H^\alpha_g h) \circ \psi_z \right) (0) \left( \psi'_z(0) \right)^{\frac{\alpha+2}{2}}
= \frac{1}{(|z|^2 - 1)^{\frac{\alpha+2}{2}}} \left( H^\alpha_{\bar{g} \circ \psi_z} \left( (h \circ \psi_z)(\psi'_z(w)) \frac{dw}{dz} \right) \right) (0)
= -\int_D \frac{\bar{g}'(v) h(v) (1 - |v|^2)^{\alpha+1} (1 - \bar{z}v)(1 - \bar{v}z)^2}{(z - v)(1 - \bar{v}z)^{\alpha+2}(|z|^2 - 1)|1 - \bar{v}z|^2} \, dA(v)
= \int_D \frac{\bar{g}'(v) h(v) (1 - |v|^2)^{\alpha+1}}{(z - v)(1 - \bar{v}z)^{\alpha+1}} \, dA(v).
\]

We have the following theorem:

**Theorem 2.1.** Suppose \( \mu \) satisfies the condition \( R_\alpha \), i.e.,

\[
(1) \quad R_\alpha(\mu)(w) = \alpha(1 - |w|^2) \int_D \frac{(1 - |z|^2)^{\alpha-1}}{(z - w)(1 - w\bar{z})^{\alpha+1}} \, d\mu(z) \in L^\infty.
\]

Then we have:

(i) If \( 0 < \alpha < 1 \), then \( T^\alpha_\mu \) is bounded on \( B^\alpha \) if and only if \( P_{\alpha-1}(\mu) \in B^\alpha \).

(ii) If \( \alpha = 1 \), then \( T^\alpha_\mu \) is bounded on \( B^\alpha \) if and only if \( P_{\alpha-1}(\mu) \in LB \).

(iii) If \( \alpha > 1 \), then \( T^\alpha_\mu \) is bounded on \( B^\alpha \) if and only if \( P_{\alpha-1}(\mu) \in B \).

**Proof.** Let \( g \in B^\alpha \) and \( h \in L^1_u \). It is known that \( (L^1_u)^* = B^\alpha \) under the integral pairing \( \langle h, g \rangle_\alpha = \alpha \int_D h(z) \bar{g}(z) (1 - |z|^2)^{\alpha-1} \, dA(z) \). See [10] Theorem 14 for more details.
Thus, by Fubini’s Theorem,
\[
\langle h, T^\alpha_\mu (g) \rangle = \alpha \int_D h(z)\overline{\mu}(g)(z)(1 - |z|^2)^{\alpha - 1} \, dA(z)
\]
\[
= \alpha \int_D h(z)\overline{\mu}(g)(z)(1 - |z|^2)^{\alpha - 1} \, d\mu(z)
\]
\[
= \alpha \int_D P_\alpha (h\overline{g})(z)(1 - |z|^2)^{\alpha - 1} \, d\mu(z)
\]
\[
+ \alpha \int_D [(I - P_\alpha )(h\overline{g})](z)(1 - |z|^2)^{\alpha - 1} \, d\mu(z) = I_1 + I_2.
\]
From Lemma 2.1 we have that
\[
[(I - P_\alpha )(h\overline{g})](z) = \int_D \frac{g(w)h(w)(1 - |w|^2)^{\alpha + 1}}{(z - w)(1 - \overline{w}z)^{\alpha + 1}} \, dA(w).
\]
Hence, by (1),
\[
|I_2| = \left| \alpha \int_D \frac{g(w)h(w)(1 - |w|^2)^{\alpha + 1}}{(z - w)(1 - \overline{w}z)^{\alpha + 1}} \, dA(w)(1 - |z|^2)^{\alpha - 1} \, d\mu(z) \right|
\]
\[
\leq \|g\|_{B^\alpha} \|h\|_1 \|R_\alpha (\mu)\|_\infty \leq C < \infty.
\]
On the other hand, by Fubini’s Theorem,
\[
I_1 = \int_D h(w)\overline{g}(w)Q_\alpha (\mu)(w)(1 - |w|^2)^\alpha \, dA(w),
\]
where \(Q_\alpha (\mu)(w) = \alpha (\alpha + 1) \int_D \frac{(1 - |z|^2)^{\alpha - 1}}{(1 - \overline{w}z)^{\alpha + 1}} \, d\mu(z) \).

Hence \(T^\alpha_\mu \in B^\alpha \) if and only if
\[
(1 - |w|^2)^\alpha g(w)Q_\alpha (\mu)(w) \in L^\infty.
\]
The relation between \(Q_\alpha (\mu)\) and \(P_{\alpha - 1}(\mu)\) is
\[
Q_\alpha (\mu)(w) = (\alpha + 1)P_{\alpha - 1}(\mu)(w) + wP'_{\alpha - 1}(\mu)(w).
\]
It is easy to see that \(P_{\alpha - 1}(\mu)\) is in \(B^\alpha\), \(LB\) and \(B\) as \(0 < \alpha < 1\), \(\alpha = 1\) and \(\alpha > 1\), respectively, implies that \(Q_\alpha (\mu)\) satisfies, respectively:
\[
Q_\alpha (\mu)(w)(1 - |w|^2)^\alpha \in L^\infty, \text{ if } 0 < \alpha < 1;
\]
\[
Q_\alpha (\mu)(w)(1 - |w|^2)^\alpha \log \frac{2}{1 - |w|^2} \in L^\infty, \text{ if } \alpha = 1;
\]
\[
Q_\alpha (\mu)(w)(1 - |w|^2)^\alpha \in L^\infty, \text{ if } \alpha > 1.
\]

Notice that \(B^\alpha \subset H^\infty\) as \(0 < \alpha < 1\) and using the fact that, as \(\alpha > 1\), \(g \in B^\alpha\) if and only if \((1 - |w|^2)^{\alpha - 1} g(w) \in L^\infty\), we easily see that (2) is true for all these cases. Thus, \(T^\alpha_\mu\) is bounded on \(B^\alpha\).

Conversely, if \(T^\alpha_\mu\) is bounded on \(B^\alpha\), then \(T^\alpha_\mu(1) = P_{\alpha - 1}(\mu) \in B^\alpha\). If \(R_\alpha (\mu) \in L^\infty\), as above, we get that there is a constant \(c > 0\), independent of \(g\), such that
\[
(1 - |w|^2)^\alpha |g(w)Q_\alpha (\mu)(w)| \leq c\|g\|_{B^\alpha}\.
\]
For \(0 < \alpha < 1\), we have already shown that \(P_{\alpha - 1}(\mu) \in B^\alpha\).
Let \( \alpha \geq 1 \), using test functions \( g_\alpha(w) = \log \frac{2}{1 - \bar{w}w} \) and \( g_\alpha(w) = (1 - \bar{w}w)^{1-\alpha} \), as \( \alpha = 1 \) and \( \alpha > 1 \), respectively, we get

\[
Q_\alpha(\mu)(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \in L^\infty \quad \text{if } \alpha = 1;
\]

\[
Q_\alpha(\mu)(w)(1 - |w|^2) \in L^\infty \quad \text{if } \alpha > 1.
\]

If \( \alpha = 1 \), \( P_{\alpha - 1}(\mu) \in B \) implies that \( P_{\alpha - 1}(\mu)(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \in L^\infty \). Thus, using (3) and (4), we easily see that \( P_{\alpha - 1}(\mu)'(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \in L^\infty \).

If \( \alpha > 1 \), then \( P_{\alpha - 1}(\mu) \in B^\alpha \) implies that

\[
P_{\alpha - 1}(\mu)(w)(1 - |w|^2)^{\alpha - 1} \in L^\infty.
\]

For \( \alpha - 1 \leq 1 \), this implies that \( P_{\alpha - 1}(\mu)(w)(1 - |w|^2) \in L^\infty \), and using (3) and (5), we get that \( P_{\alpha - 1}(\mu) \in B \).

For \( \alpha - 1 > 1 \), combining (3), (5) and (6) we get \( P_{\alpha - 1}(\mu)'(w)(1 - |w|^2)^{\alpha - 1} \in L^\infty \), which is equivalent to

\[
P_{\alpha - 1}(\mu)(w)(1 - |w|^2)^{\alpha - 2} \in L^\infty.
\]

If \( \alpha - 2 > 1 \), combining (3), (5) and (7) we get that \( P_{\alpha - 1}(\mu)'(w)(1 - |w|^2)^{\alpha - 2} \) is in \( L^\infty \), and we continue the process from above until we reach a positive integer \( n \) such that \( \alpha - n \leq 1 \) and \( P_{\alpha - n}(\mu)'(w)(1 - |w|^2)^{\alpha - n} \in L^\infty \).

Then, \( P_{\alpha - 1}(\mu)'(w)(1 - |w|^2) \in L^\infty \), and so \( P_{\alpha - 1}(\mu) \in B \). This completes the proof.

Note that in the case \( d\mu(z) = f(z)dA(z) \), with \( f \in B^\alpha \), it follows from part (ii) of Proposition 2.1 that \( R_\alpha(\mu)(w) = wf(w) \). Thus \( \mu \) satisfies the condition \( R_\alpha \) if and only if \( f \in H^\infty \).

**Corollary 2.1.** Let \( f \in L^\infty \) and let \( T^\mu_f(g) = P_{\alpha - 1}(fg) \) for \( g \in B^\alpha \). Then we have:

(i) If \( 0 < \alpha < 1 \), then \( T^\mu_f \) is bounded on \( B^\alpha \) if and only if \( P_{\alpha - 1}(f) \in B^\alpha \).

(ii) If \( \alpha = 1 \), then \( T^\mu_f \) is bounded on \( B^\alpha \) if and only if \( P_{\alpha - 1}(f) \in LB \).

(iii) If \( \alpha > 1 \), then \( T^\mu_f \) is bounded on \( B^\alpha \) if and only if \( P_{\alpha - 1}(f) \in B \).

**Proof.** Follows from Proposition 2.1 and Theorem 2.1.

Since the Bloch space \( B \) is the dual of \( L^1_\alpha \) and \( T^*_\mu \) is the dual of \( T^\mu \), under this duality, we get the following corollary. Similar results including a few other restrictions can be found in \[1\] and \[8\].

**Corollary 2.2.** Let \( \mu \) be a complex measure satisfying condition \( R_\alpha \). Then \( T^\mu \) is bounded on \( L^1_\alpha \) if and only if \( P(\mu) \in LB \).

**Remark.** The proof of Theorem 2.1 implies that the condition \( R_\alpha \) in the theorem can be replaced by a more general condition involving Carleson measures.

Recall that for a positive Borel measure \( \nu \) on \( \mathbb{D} \) we say that \( \nu \) is a Carleson measure on the Bergman space if there exists \( c > 0 \) such that, for all \( h \in L^2_\alpha \),

\[
\int_{\mathbb{D}} |h(z)|^2 d\nu(z) \leq c \int_{\mathbb{D}} |h(z)|^2 dA(z).
\]

Let \( \nu_\alpha(\mu) \) be the positive measure defined by \( d\nu_\alpha(\mu)(z) = |R_\alpha(\mu)(z)|dA(z) \).

If \( \mu \) satisfies condition \( R_\alpha \), then it is easy to see that \( \nu_\alpha(\mu) \) is a Carleson measure. By inspecting the proof of Theorem 2.1 we can see that, whenever \( R_\alpha(\mu) \) is in \( L^1 \),

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the statements (i), (ii) and (iii) of Theorem 2.1 are still true if the condition $R_\alpha$ is replaced by a weaker one, namely, by requiring only that the measure $\nu_\alpha(\mu)$ is a Carleson measure for the Bergman space.

Note that the measure $\nu_\alpha(\mu)$ is a Carleson measure for the Bergman space if and only if there exists $r > 0$, such that
\[
\sup_{z \in \mathbb{D}} \frac{1}{|D(z,r)|} \int_{D(z,r)} |R_\alpha(\mu)(w)|dA(w) < \infty,
\]
where $D(z,r)$ denotes the hyperbolic disk with center $z$ and radius $r$. See [9, Theorem 6.2.2] for details.

Next we give an example of an unbounded $L^1$ function that induces a bounded Toeplitz operator on $B^\alpha$. We present part of the calculations as a separate lemma that covers a wider class of functions.

**Lemma 2.2.** Let $f(z) = \sum_{n=0}^{\infty} a_n (1 - |z|^2)^n$ be a function in $L^1((1 - |z|^2)^{\alpha-1}dA(z))$, for $\alpha > 0$, and such that $\sum_{n=0}^{\infty} \frac{|a_n|}{n+\alpha}$ is finite. Let $d_\alpha = \alpha \sum_{n=0}^{\infty} \frac{a_n}{n+\alpha}$ and let $F_\alpha(f) = \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\alpha-1}dA(z)$. Then, for $d\mu(z) = (z) dA(z)$ we have
\[
R_\alpha(\mu)(w) = d_\alpha w - \frac{\alpha}{w} \left( F_\alpha(f) - \frac{1}{(1 - |w|^2)^{\alpha}} \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\alpha-1}dA(z) \right),
\]
where $D_w = \{ z : |w| < |z| < 1 \}$.

**Proof.** As in the proof of part (ii) of Proposition 2.1 we have that, for $w \neq 0$,
\[
R_\alpha(\mu)(w) = wP_{\alpha-1}(f)(w) - \alpha \int_{\mathbb{D}} \frac{(1 - |u|^2)^{\alpha-1}}{\bar{u}(1 - \bar{u}w)^{n+\alpha+1}} f \circ \psi_w(u) dA(u).
\]

Using the given series expansion of $f$ and the identity $1 - |\psi_w(u)|^2 = \frac{(1 - |w|^2)^2}{1 - \bar{w}u}$, we get that $P_{\alpha-1}(f)(w) = \alpha \sum_{n=0}^{\infty} \frac{a_n}{n+\alpha} = d_\alpha$, and that the above integral equals
\[
\int_{\mathbb{D}} \sum_{n=0}^{\infty} a_n (1 - |w|^2)^n \frac{(1 - |u|^2)^{n+\alpha-1}}{(1 - \bar{w}u)^{n+\alpha+1}} dA(u) = \frac{(1 - |w|^2)^{-\alpha} - 1}{\bar{u}}.
\]

Since $\frac{(1 - \bar{u}w)^{-\alpha-1}}{u}$ is a conjugate analytic, integrable function, we have that
\[
(n + \alpha) \int_{\mathbb{D}} \frac{(1 - |u|^2)^{-n}}{u} dA(u) = 0,
\]
since also $\int_{\mathbb{D}} (1 - |u|^2)^{n+\alpha} dA(u) = 0$, we have that, furthermore,
\[
R_\alpha(\mu)(w) = d_\alpha w - \alpha \sum_{n=0}^{\infty} \frac{a_n}{n+\alpha} (1 - |w|^2)^n \frac{(1 - |w|^2)^{-\alpha} - 1}{\bar{u}}
\]
\[
= d_\alpha w - \frac{\alpha}{\bar{w}} \left( \sum_{n=0}^{\infty} \frac{a_n}{n+\alpha} - \sum_{n=0}^{\infty} \frac{a_n}{n+\alpha} (1 - |w|^2)^n \right)
\]
\[
= d_\alpha w - \frac{\alpha}{\bar{w}} \left( F_\alpha(f) - \frac{1}{(1 - |w|^2)^{\alpha}} \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\alpha-1}dA(z) \right).
\]

□
Example. For $\alpha > 0$, let $c_\alpha = \frac{1}{\alpha} \left( \sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)} \right)^{-1}$ and for $z \neq 0$, let

$$f_\alpha(z) = 1 - c_\alpha \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - |z|^2 \right)^n = 1 + c_\alpha \log(|z|^2).$$

The function $f_\alpha$ is a radial, unbounded, $L^1$ function. Since $P_{\alpha-1}(f_\alpha)$ is a radial analytic function, it must be a constant. It is easy to check that $P_{\alpha-1}(f_\alpha)(0) = 0$ and so $P_{\alpha-1}(f_\alpha) = 0$.

In order to show that $T_\alpha$ is bounded on $B^\alpha$ we only have to prove that $\mu_\alpha$, with $d\mu_\alpha(z) = f_\alpha(z)dA(z)$, satisfies the condition $R_\alpha$. Using Lemma 2.2 and the fact that $d_\alpha = 0$, we get that

$$R_\alpha(\mu_\alpha)(w) = \frac{1}{w} \left( 1 + \alpha c_\alpha \frac{1}{(1-|w|^2)^\alpha} \int_{|w|^2}^{1} (1-t)^{\alpha-1} \log t dt \right).$$

It is easy to see that $\lim_{|w| \to 1} |R_\alpha(\mu)(w)| = 1$. Since $\int_{|w| \to 0} (1-t)^{\alpha-1} \log t dt = -\frac{1}{\alpha c_\alpha}$, we also have that $\lim_{|w| \to 0} |R_\alpha(\mu)(w)| = \alpha$. Thus, $R_\alpha(\mu_\alpha) \in L^\infty$, and by Theorem 2.1, $T_\alpha$ is bounded on $B^\alpha$.

Note that we can also use Lemma 2.2 to show that $R_\alpha(\mu)$ is not a necessary condition for the boundedness of the Toeplitz operator $T_\mu$. For example, let $d\mu(z) = f(z)dA(z)$ with $f(z) = \frac{1}{|z|^\alpha}$. Note that for this radial, positive function $P_\alpha(f)$ is equal to $\frac{\Gamma(1/4)\Gamma(\alpha)}{\Gamma(1/4+\alpha)}$, and $d_\alpha$ is equivalent to $\sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)}$. Using Lemma 2.2 we can get that $|R_\alpha(\mu)(w)| \geq \frac{c}{|w|^\alpha}$ and so $R_\alpha(\mu)$ is not in $L^\infty$. Since the measure $\nu_\alpha(\mu)$ is a Carleson measure, $T_\mu$ is still bounded on $B^\alpha$.

3. Compact Toeplitz operators

For the next result, we need the following lemma from [5].

Lemma 3.1. Let $0 < \alpha < 1$ and let $T$ be a bounded linear operator from $B^\alpha$ into a normed linear space $Y$. Then $T$ is compact if and only if $\|Tg_n\|_Y \to 0$, whenever $(g_n)$ is a bounded sequence in $B^\alpha$ that converges to 0 uniformly on $\mathbb{D}$.

Theorem 3.1. Suppose

$$\lim_{|w| \to 1} R_\alpha(\mu)(w) = 0. \quad (8)$$

Then we have:

(i) If $0 < \alpha < 1$, then $T^\alpha_\mu$ is compact on $B^\alpha$ if and only if $P_{\alpha-1}(\mu) \in B^\alpha$.

(ii) If $\alpha = 1$, then $T^\alpha_\mu$ is compact on $B^\alpha$ if and only if $P_{\alpha-1}(\mu) \in L^\infty$.

(iii) If $\alpha > 1$, then $T^\alpha_\mu$ is compact on $B^\alpha$ if and only if $P_{\alpha-1}(\mu) \in B_0$.

Proof. For the case $\alpha = 1$ or $\alpha > 1$, let $(g_n)$ be a sequence in $B^\alpha$ such that $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $h$ be in the
unit ball of $L^1_\mu$. Similarly, as in the proof of Theorem 2.1, we have

$$
\langle h, T^\alpha_\mu (g_n) \rangle_\alpha = \alpha \int_\mathbb{D} P_\alpha (h \tilde{g}_n) (z) (1 - |z|^2)^{\alpha - 1} d\mu (z) + \alpha \int_\mathbb{D} [(I - P_\alpha) (h \tilde{g}_n)] (z) (1 - |z|^2)^{\alpha - 1} d\mu (z) = I_1 + I_2.
$$

For $0 < r < 1$, and $\mathbb{D}_r = \{ z : |z| \leq r \}$,

$$
I_2 = \alpha \int_{\mathbb{D}_r} \frac{g_n (w)}{\mu (w)} h (w) (1 - |w|^2)^{\alpha} R_\alpha (\mu) (w) dA (w)
= \alpha \left( \int_{\mathbb{D}_r} + \int_{\mathbb{D} \setminus \mathbb{D}_r} \right) \frac{g_n (w)}{\mu (w)} h (w) (1 - |w|^2)^{\alpha} R_\alpha (\mu) (w) dA (w) = K_1 + K_2.
$$

For a fixed $\varepsilon > 0$, using condition (8), let $r$ be sufficiently close to 1 so that $|R_\alpha (\mu) (w)| < \varepsilon$ as $w \in \mathbb{D} \setminus \mathbb{D}_r$. Then $|K_2| \leq \varepsilon \| g_n \|_{B^\alpha} \| h \|_1 \leq \varepsilon$.

Since $g_n (z) \to 0$ as $n \to \infty$, we can choose $n$ big enough so that $|g_n (z)| (1 - |z|^2)^{\alpha} < \varepsilon$. Therefore, $|K_1| \leq \varepsilon \| R_\alpha (\mu) \|_\infty \| h \|_1 \leq \varepsilon \| R_\alpha (\mu) \|_\infty$.

Hence $|I_2| < C \varepsilon$, where $\varepsilon$ does not depend on $h$, and so $\lim_{n \to \infty} \sup_{\| h \|_1, \leq 1} |I_2| = 0$. Thus $T^\alpha_\mu$ is compact on $B^\alpha$ if and only if $\sup_{\| h \|_1, \leq 1} |I_1| \to 0$ as $n \to \infty$.

Similarly, as in the proof of Theorem 2.1, we have

$$
I_1 = \int_{\mathbb{D}} h (w) \frac{g_n (w)}{\mu (w)} Q_\alpha (\mu) (w) (1 - |w|^2)^{\alpha} dA (w)
= \left( \int_{\mathbb{D}_r} + \int_{\mathbb{D} \setminus \mathbb{D}_r} \right) h (w) g_n (w) Q_\alpha (\mu) (w) (1 - |w|^2)^{\alpha} dA (w) = M_1 + M_2.
$$

It is easy to see that if $P_{\alpha - 1} (\mu)$ is in $L B_0$ and $B_0$ as $\alpha = 1$ and $\alpha > 1$, respectively, then, as $|w| \to 1$, $Q_\alpha$ satisfies, respectively:

$$
Q_\alpha (\mu) (w) (1 - |w|^2) \log \frac{2}{1 - |w|^2} \to 0, \text{ if } \alpha = 1;
Q_\alpha (\mu) (w) (1 - |w|^2) \to 0, \text{ if } \alpha > 1.
$$

Again, notice that, as $\alpha > 1$, $g \in B^\alpha$ if and only if $(1 - |z|^2)^{\alpha - 1} g (z) \in L^\infty$, and we may choose $r$ sufficiently close to 1 such that whenever $w \in \mathbb{D} \setminus \mathbb{D}_r$, $|Q_\alpha (\mu) (w)| (1 - |w|^2) \log \frac{2}{1 - |w|^2} < \varepsilon$ as $\alpha = 1$; and $|Q_\alpha (\mu) (w)| (1 - |w|^2)^{\alpha} < \varepsilon$ as $\alpha > 1$, respectively.

We see that $|M_2| \leq \varepsilon \alpha \| g_n \|_{B^\alpha} \| h \|_1 \leq C \varepsilon$, where $\varepsilon$ does not depend on $h$.

Since $g_n (z) \to 0$ uniformly on compact subsets of $\mathbb{D}$, we can choose $n$ big enough so that $|g_n (w)| < \varepsilon$. Hence we easily see that, as $n$ is big enough, $|M_1| < C \varepsilon$, where $\varepsilon$ does not depend on $h$. Therefore, $\sup_{\| h \|_1, \leq 1} |I_1| \to 0$ as $n \to \infty$, and so $T^\alpha_\mu$ is compact on $B^\alpha$ as $\alpha = 1$ or $\alpha > 1$.

Now let $0 < \alpha < 1$. Let $\{ g_n \}$ be a sequence in $B^\alpha$ such that $\| g_n \|_{B^\alpha} \leq 1$ and $g_n (z) \to 0$ uniformly on $\mathbb{D}$. Let $h$ be in the unit ball of $L^1_\mu$. By a similar discussion as above, $T^\alpha_\mu$ is compact on $B^\alpha$ if and only if $\sup_{\| h \|_1, \leq 1} |I_1| \to 0$ as $n \to \infty$. However,
since \( \|g_n\|_{B^\alpha} \leq 1 \) and \( g_n(z) \to 0 \) uniformly on \( \mathbb{D} \), we can choose \( n \) big enough such that \( |g_n(w)| < \varepsilon \) uniformly for \( w \in \mathbb{D} \).

Thus, as \( n \) is big enough, \( |I_1| \leq \varepsilon \|P_{\alpha-1}(\mu)\|_{B^\alpha} \|h\|_1 \).

Therefore, \( \sup_{\|h\|_1 \leq 1} |I_1| \to 0 \) as \( n \to \infty \), and so \( T_\mu^* \) is compact on \( B^\alpha \).

Conversely, if \( T_\mu^* \) is compact on \( B^\alpha \). Suppose first that \( \alpha = 1 \) or \( \alpha > 1 \). Then \( \sup_{\|h\|_1 \leq 1} |I_1| \to 0 \) as \( n \to \infty \) for any sequence \( g_n \) in \( B^\alpha \) such that \( \|g_n\|_{B^\alpha} \leq 1 \) and \( g_n(z) \to 0 \) uniformly on compact subsets of \( \mathbb{D} \).

Let \( h_z(w) = \frac{(1-|z|^2)^\alpha}{(1-\bar{w}z)^{2+\alpha}} \). Then \( \|h_z\|_1 \leq C \), and

\[
I_1 = \int_{\mathbb{D}} \frac{g_n(w)Q_\alpha(\mu)(w)(1-|w|^2)^\alpha}{(1-\bar{w}z)^{2+\alpha}} \, dA(w)
\]

\[
= (\alpha + 1)^{-1}(1-|z|^2)^\alpha g_n(z)Q_\alpha(\mu)(z).
\]

Thus, \( \sup_{\|h\|_1 \leq 1} |I_1| \to 0 \) as \( n \to \infty \) implies

\[
(9) \quad \lim_{n \to \infty} \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |g_n(z)||Q_\alpha(\mu)(z)| = 0.
\]

As \( \alpha = 1 \) or \( \alpha > 1 \), using testing functions \( g_\alpha(z) = \left(\log \frac{1}{|z|^2}\right)^{-1}\left(\log \frac{2}{1-|z|}\right)^2 \) and \( g_\alpha(z) = (1-|z|^2)(1-\bar{z}z)^{-\alpha} \) respectively in (9), as in the proof of Theorem 2.1, we easily see that \( P_{\alpha-1}(\mu) \) is in \( LB_0 \) and \( B_0 \), respectively.

Now let \( 0 < \alpha < 1 \) and let \( T_\mu^* \) be compact on \( B^\alpha \). Then \( T_\mu^* \) is bounded on \( B^\alpha \).

By Theorem 2.1, \( P_{\alpha-1}(\mu) \in B^\alpha \). The proof is completed.

Remark. The requirement that \( \lim_{|w| \to 1} R_\alpha(\mu)(w) = 0 \), in the proof of Theorem 3.1, can be replaced by a more general requirement that the measure \( \nu_\alpha(\mu) \), defined in Section 2, is a compact Carleson measure.

Recall that for a positive Borel measure \( \nu \) on \( \mathbb{D} \) we say that \( \nu \) is a compact Carleson measure whenever

\[
\lim_{|z| \to 1} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w)| \, d\nu(w) = 0,
\]

where \( D(z,r) \) denotes the hyperbolic disk with center \( z \) and radius \( r \). See [9, Theorem 6.2.5] for more details. If the measure \( \mu \) is such that \( \lim_{|z| \to 1} R_\alpha(\mu)(w) = 0 \), then it is easy to see that \( \nu_\alpha(\mu) \) is a compact Carleson measure.

Note that in the case \( d\mu(z) = f(z)\,dA(z) \), with \( f \in B^\alpha \), it follows from part (ii) of Proposition 2.1 that \( R_\alpha(\mu)(w) = w\,f(w) \). Thus the condition (8) in Theorem 3.1 is satisfied only when \( f = 0 \).

References


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