

TOEPLITZ OPERATORS ON BLOCH-TYPE SPACES

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ABSTRACT. We characterize complex measures μ on the unit disk for which the Toeplitz operator T_μ^α , $\alpha > 0$, is bounded or compact on the Bloch type spaces B^α .

1. INTRODUCTION AND PRELIMINARIES

Let \mathbb{D} be the unit disk on the complex plane. Let $dA(z) = \frac{1}{\pi} dx dy$ be the normalized Lebesgue measure on \mathbb{D} .

For a complex measure μ , $\alpha > 0$, and $b \in L^1$, define a Toeplitz operator as follows:

$$T_\mu^\alpha(b)(z) = \alpha \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha-1} b(w)}{(1 - \bar{w}z)^{\alpha+1}} d\mu(w).$$

We also define the general Bergman projection of the measure μ and for $\alpha > -1$ as follows:

$$P_\alpha(\mu)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{(1 - \bar{w}z)^{\alpha+2}} d\mu(w).$$

The general Bergman projection of the function b is

$$P_\alpha(b)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha b(w)}{(1 - \bar{w}z)^{\alpha+2}} dA(w).$$

Thus $T_\mu^\alpha(b)(z) = P_{\alpha-1}(\mu_b)(z)$, where $d\mu_b(z) = b(z)d\mu(z)$. Note that these choices of the indexes provide the standard notation for the general Bergman projections and for the standard case $\alpha = 1$ as follows:

When $\alpha = 1$ and the measure μ is such that $d\mu(z) = f(z)dA(z)$, with $f \in L^1$, we have that T_μ^1 is the standard Toeplitz operator defined by

$$T_\mu^1(b) = T_f(b) = P(fb), \quad \forall b \in L^1.$$

Here $P = P_0$ denotes the standard Bergman projection. Recall that if f is in L^∞ , then T_f is bounded on the Bergman spaces L_a^p , $p > 1$.

Toeplitz operators have been studied extensively on the Bergman spaces by many authors. For references, see for example [9]. In this paper we study the boundedness and compactness of general Toeplitz operators on the α -Bloch spaces.

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For $\alpha > 0$, the α -Bloch spaces B^α are the spaces of analytic functions f on \mathbb{D} such that

$$M^\alpha(f) = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

Each B^α is a Banach space with norm of f equal to $\|f\|_{B^\alpha} = M^\alpha(f) + |f(0)|$. An analytic function on \mathbb{D} belongs to the *little α -Bloch space* B_0^α , $\alpha > 0$, whenever

$$\lim_{|z| \rightarrow 1} |f'(z)|(1 - |z|^2)^\alpha = 0.$$

The spaces B_0^α are the subspaces of B^α that are the closure of the polynomials with respect to the B^α norm.

The α -Bloch spaces and operators on them have been studied in many different contexts. For more references and details on the next few facts stated below see [9] and [10].

For $\alpha = 1$, $B^1 = B$ is the classical Bloch space. There is a natural connection between the Bloch space and the Toeplitz operators via the fact that $P(L^\infty) = B$. A necessary condition for boundedness of T_f on B is that $Pf \in B$, and so the condition is satisfied whenever $f \in L^\infty$. As we will see later, this is not a sufficient condition for boundedness of T_f on B .

In general, the growth condition of a function f in B^α with $f(0) = 0$ is determined by $|f(z)| \leq \|f\|_{B^\alpha} \int_0^1 \frac{|z|dt}{(1-|z|t)^\alpha}$. Thus, for $\alpha = 1$, we get that

$$|f(z)| \leq \|f\|_B \log \frac{1}{1 - |z|},$$

while for $\alpha > 1$ we have that $|f(z)| \leq \frac{1}{\alpha-1} \|f\|_{B^\alpha} \frac{1}{(1-|z|)^{\alpha-1}}$. For $0 < \alpha < 1$, the spaces B^α are Lipschitz spaces $Lip_{1-\alpha}$ and $B^\alpha \subset H^\infty$.

For $0 < \alpha < \beta$, we have that $B^\alpha \subset B^\beta$. The Bloch space B is included in all Bergman spaces L_a^p , $p \geq 1$, but for large α , such as $\alpha \geq 2$, B^α gets so large that, for example, it includes the Bergman space L_a^2 .

In presenting our results we will also refer to the logarithmic Bloch space.

The *logarithmic Bloch space* LB is the space of analytic functions f on \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} < \infty.$$

Correspondingly, the *little logarithmic Bloch space* LB_0 is the space of analytic functions f on \mathbb{D} such that

$$\lim_{|z| \rightarrow 1} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} = 0.$$

Throughout the paper, in order to have the operator T_μ^α , $\alpha > 0$ well defined, we will assume that the complex measure μ is such that, for all g in B^α

$$\int_{\mathbb{D}} |g(w)|(1 - |w|^2)^{\alpha-1} |d\mu(w)| < \infty.$$

Note that in the case when $d\mu(z) = f(z)dA(z)$, a sufficient condition on μ , so that the integral above is finite, is that $f \in L^1$, when $\alpha > 1$, $f \in L^1(\log \frac{1}{1-|w|^2} dA(w))$, when $\alpha = 1$ and $f \in L^1((1 - |w|^2)^{\alpha-1} dA(w))$, when $0 < \alpha < 1$.

2. BOUNDED TOEPLITZ OPERATORS

In this section we will present our main characterization of bounded Toeplitz operators on general α -Bloch spaces. The result includes a restriction on the inducing measure that is a generalization of several known cases.

For a complex measure μ and $\alpha > 0$, we will say that μ satisfies the condition R_α if

$$R_\alpha(\mu)(w) = \alpha(1 - |w|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{(z - w)(1 - \bar{w}\bar{z})^{\alpha+1}} d\mu(z) \in L^\infty.$$

We get another form of R_α using the identity $(1 - |w|^2) = 1 - w\bar{z} + w(\bar{z} - \bar{w})$:

$$R_\alpha(\mu)(w) = \alpha \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{(z - w)(1 - w\bar{z})^\alpha} d\mu(z) + wP_{\alpha-1}(\mu)(w).$$

The special case when the measure μ is such that $d\mu(z) = f(z)dA(z)$, with $f \in H^\infty$, is the case when the Toeplitz operator is a multiplication operator. For $f \in B^\alpha$, as we will see below, μ satisfies the condition R_α if and only if $f \in H^\infty$.

Arazy's ([1]) and Zhu's ([8] and [10]) results provide a complete characterization of bounded multiplication operators on α -Bloch spaces. We will state them in terms of multipliers.

Theorem A ([10]). *Let $M(X)$ denote the space of multipliers of the Banach space of functions X , i.e., $M(X) = \{f \in X : fg \in X, \forall g \in X\}$.*

- (i) *If $0 < \alpha < 1$, then $M(B^\alpha) = B^\alpha$ and $M(B_0^\alpha) = B_0^\alpha$.*
- (ii) *If $\alpha = 1$, then $M(B^\alpha) = M(B_0^\alpha) = H^\infty(\mathbb{D}) \cap LB$.*
- (iii) *If $\alpha > 1$, then $M(B^\alpha) = M(B_0^\alpha) = H^\infty(\mathbb{D})$.*

As a consequence of the above theorem we can see that, for example, not every bounded function f induces a bounded Toeplitz operator T_f on the Bloch space B . As we will see later, for $f \in L^\infty$, T_f is bounded on B if and only if $Pf \in LB$, which is a generalization of the analytic case of the above theorem. We will also see that there exist unbounded L^1 functions that induce bounded Toeplitz operators on the Bloch spaces.

In the next proposition we give more details about the condition R_α in some special cases.

Proposition 2.1. *Let μ be a measure such that $d\mu(z) = f(z)dA(z)$, with $f \in L^1$. Then:*

- (i) *If $f \in L^\infty$, μ satisfies the condition R_α , $\alpha > 0$.*
- (ii) *If f is analytic and in B^α , then $R_\alpha(\mu)(w) = wf(w)$.*
- (iii) *If f is conjugate analytic, \bar{f} is in B^α and $f(0) = 0$, then $R_\alpha(\mu)(w) = \frac{1}{\bar{w}}f(w)$.*

Proof. Let $\psi_w(z) = \frac{w-z}{1-\bar{w}z}$, for z, w in \mathbb{D} .

- (i) For $\alpha > 0$ and $f \in L^\infty$,

$$\begin{aligned} |R_\alpha(\mu)(w)| &= \alpha(1 - |w|^2) \left| \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1} \bar{f}(z)}{(z - w)(1 - \bar{w}\bar{z})^{\alpha+1}} dA(z) \right| \\ &\leq \alpha \|f\|_\infty (1 - |w|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{|z - w| |1 - \bar{w}\bar{z}|^{\alpha+1}} dA(z) \\ &\leq c\alpha \|f\|_\infty (1 - |w|^2)(1 - |w|^2)^{-1} = c\alpha \|f\|_\infty. \end{aligned}$$

The second inequality follows from the fact that $\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta}{|z-w| |1-\bar{w}z|^\gamma} dA(z)$ is equivalent to $(1-|w|^2)^{\beta-\gamma+1}$, whenever $\beta - \gamma + 1 < 0$, which can be easily derived from the well-known Forelli-Rudin estimates (see [6], page 17, or [8], page 53). The half-plane version of the proof of this equivalency can be found in [7].

Thus $R_\alpha(\mu)(w) \in L^\infty$, and (i) is proved.

(ii) Let f be in B^α . With a change of variable $z = \psi_w(u)$, we get that

$$\begin{aligned} R_\alpha(\mu)(w) &= \alpha \int_{\mathbb{D}} \frac{(1-|\psi_w(u)|^2)^{\alpha-1} |\psi'_w(u)|^2}{-\bar{u}(1-w\bar{\psi}_w(u))^{\alpha+2}} f \circ \psi_w(u) dA(u) + w P_{\alpha-1}(f)(w) \\ &= -\alpha \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{\bar{u}(1-\bar{w}u)^{\alpha+1}} f \circ \psi_w(u) dA(u) + wf(w) = I + wf(w). \end{aligned}$$

For a fixed w we have that $\frac{f \circ \psi_w(u)}{(1-\bar{w}u)^{\alpha+1}}$ belongs to B^α whenever $f \in B^\alpha$, and so $\frac{(1-|u|^2)^{\alpha-1} f \circ \psi_w(u)}{(1-\bar{w}u)^{\alpha+1}}$ must be in L^1 . Using Taylor series expansion it is not hard to see that then $I = 0$. Hence $R_\alpha(\mu)(w) = wf(w)$.

(iii) Let f be conjugate analytic and let $f(0) = 0$. Then $P_{\alpha-1}(f)(w) = 0$, and as in the proof of part (ii), we have that

$$\begin{aligned} \overline{R_\alpha(\mu)(w)} &= -\alpha \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{u(1-\bar{u}w)^{\alpha+1}} \overline{f \circ \psi_w(u)} dA(u) \\ &= -\alpha \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{(1-\bar{u}w)^{\alpha+1}} \frac{\overline{f \circ \psi_w(u)} - \bar{f}(w)}{u} dA(u), \end{aligned}$$

since $\int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{u(1-\bar{u}w)^{\alpha+1}} dA(u) = 0$. Using the fact that $\frac{\overline{f \circ \psi_w(u)} - \bar{f}(w)}{u}$ is an analytic integrable function, we get that

$$\overline{R_\alpha(\mu)(w)} = -\frac{\overline{f \circ \psi_w(w)} - \bar{f}(w)}{w} = -\frac{\bar{f}(0) - \bar{f}(w)}{w} = \frac{1}{w} \bar{f}(w).$$

□

Remark. If f from the previous proposition is harmonic, then $R_\alpha(\mu)$ is also harmonic. If also f is a harmonic extension of an $L^1(T)$ function, then $R_\alpha(\mu)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$. Thus, for f harmonic and in L^∞ , we have that $R_\alpha(\mu)$ is also harmonic and in L^∞ , with $\|R_\alpha(\mu)\|_\infty = \|f\|_\infty$.

We will use the following lemma in the proof of our next theorem. The result is known in a more general form and can be found, for example, in [2]. We include a proof for completeness.

Lemma 2.1. *Let $h \in L^1_a$ and $g \in B^\alpha$, $\alpha > 0$. Then we have that*

$$H_g^\alpha h(z) = (I - P_\alpha)(\bar{g}h)(z) = \int_{\mathbb{D}} \frac{\overline{g'(w)} h(w) (1-|w|^2)^{\alpha+1}}{(z-w)(1-\bar{w}z)^{\alpha+1}} dA(w).$$

Proof. We will use the following three formulas which can be found in [3] and [10]. For $g \in B^\alpha$ and $h \in L_a^1$:

$$\begin{aligned} \text{(a)} \quad & H_{\bar{g}}^\alpha h(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(\bar{g}(z) - \bar{g}(u)) (1 - |u|^2)^\alpha}{(1 - \bar{u}z)^{\alpha+2}} h(u) dA(u), \\ \text{(b)} \quad & g(z) = g(0) + \int_{\mathbb{D}} \frac{(1 - |u|^2)^{\alpha+1} g'(u)}{\bar{u}(1 - \bar{u}z)^{\alpha+2}} dA(u), \\ \text{(c)} \quad & ((H_{\bar{g}}^\alpha h) \circ \psi_z) (\psi'_z)^{\frac{\alpha+2}{2}} = H_{\bar{g} \circ \psi_z}^\alpha \left((h \circ \psi_z) (\psi'_z)^{\frac{\alpha+2}{2}} \right). \end{aligned}$$

Without loss of generality, let us assume that $g(0) = 0$. Then

$$\begin{aligned} H_{\bar{g}}^\alpha h(0) &= (\alpha + 1) \int_{\mathbb{D}} (-\bar{g}(u))(1 - |u|^2)^\alpha h(u) dA(u) \\ &= -(\alpha + 1) \int_{\mathbb{D}} \overline{\left(\int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1} g'(w)}{\bar{w}(1 - \bar{w}u)^{\alpha+2}} dA(w) \right)} (1 - |u|^2)^\alpha h(u) dA(u) \\ &= -(\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1} \bar{g}'(w)}{w} \left(\int_{\mathbb{D}} \frac{(1 - |u|^2)^\alpha h(u)}{(1 - \bar{u}w)^{\alpha+2}} dA(u) \right) dA(w) \\ &= - \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1} \bar{g}'(w)}{w} h(w) dA(w). \end{aligned}$$

Since $1 - |\psi_z(v)|^2 = (1 - |v|^2) |\psi'_z(v)|$ and $\psi'_z(v) = \frac{|z|^2 - 1}{(1 - \bar{z}v)^2} = \frac{1}{\psi'_z(\psi_z(v))}$, we have:

$$\begin{aligned} H_{\bar{g}}^\alpha h(z) &= \frac{1}{(|z|^2 - 1)^{\frac{\alpha+2}{2}}} ((H_{\bar{g}}^\alpha h) \circ \psi_z) (0) (\psi'_z(0))^{\frac{\alpha+2}{2}} \\ &= \frac{1}{(|z|^2 - 1)^{\frac{\alpha+2}{2}}} \left(H_{\bar{g} \circ \psi_z}^\alpha \left((h \circ \psi_z) (\psi'_z)^{\frac{\alpha+2}{2}} \right) \right) (0) \\ &= \frac{-1}{(|z|^2 - 1)^{\frac{\alpha+2}{2}}} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1} \overline{(g \circ \psi_z)'(w)}}{w} h \circ \psi_z(w) (\psi'_z(w))^{\frac{\alpha+2}{2}} dA(w) \\ &= - \int_{\mathbb{D}} \frac{\bar{g}'(v) h(v) (1 - |v|^2)^{\alpha+1} (1 - |z|^2) (1 - \bar{z}v) (1 - \bar{v}z)^2}{(z - v) (1 - \bar{v}z)^{\alpha+2} (|z|^2 - 1) |1 - \bar{z}v|^2} dA(v) \\ &= \int_{\mathbb{D}} \frac{\bar{g}'(v) h(v) (1 - |v|^2)^{\alpha+1}}{(z - v) (1 - \bar{v}z)^{\alpha+1}} dA(v). \end{aligned}$$

□

We have the following theorem:

Theorem 2.1. *Suppose μ satisfies the condition R_α , i.e.,*

$$(1) \quad R_\alpha(\mu)(w) = \alpha(1 - |w|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{(z - w)(1 - w\bar{z})^{\alpha+1}} d\mu(z) \in L^\infty.$$

Then we have:

- (i) *If $0 < \alpha < 1$, then T_μ^α is bounded on B^α if and only if $P_{\alpha-1}(\mu) \in B^\alpha$.*
- (ii) *If $\alpha = 1$, then T_μ^α is bounded on B^α if and only if $P_{\alpha-1}(\mu) \in LB$.*
- (iii) *If $\alpha > 1$, then T_μ^α is bounded on B^α if and only if $P_{\alpha-1}(\mu) \in B$.*

Proof. Let $g \in B^\alpha$ and $h \in L_a^1$. It is known that $(L_a^1)^* = B^\alpha$ under the integral pairing $\langle h, g \rangle_\alpha = \alpha \int_{\mathbb{D}} h(z) \bar{g}(z) (1 - |z|^2)^{\alpha-1} dA(z)$. See [10, Theorem 14] for more details.

Thus, by Fubini's Theorem,

$$\begin{aligned} \langle h, T_\mu^\alpha(g) \rangle_\alpha &= \alpha \int_{\mathbb{D}} h(z) \overline{T_\mu^\alpha(g)(z)} (1 - |z|^2)^{\alpha-1} dA(z) \\ &= \alpha \int_{\mathbb{D}} h(z) \overline{g(z)} (1 - |z|^2)^{\alpha-1} d\bar{\mu}(z) \\ &= \alpha \int_{\mathbb{D}} P_\alpha(h\bar{g})(z) (1 - |z|^2)^{\alpha-1} d\bar{\mu}(z) \\ &\quad + \alpha \int_{\mathbb{D}} [(I - P_\alpha)(h\bar{g})](z) (1 - |z|^2)^{\alpha-1} d\bar{\mu}(z) = I_1 + I_2. \end{aligned}$$

From Lemma 2.1 we have that

$$[(I - P_\alpha)(h\bar{g})](z) = \int_{\mathbb{D}} \frac{\overline{g'(w)}h(w)(1 - |w|^2)^{\alpha+1}}{(z - w)(1 - \bar{w}z)^{\alpha+1}} dA(w).$$

Hence, by (1),

$$\begin{aligned} |I_2| &= \left| \alpha \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\overline{g'(w)}h(w)(1 - |w|^2)^{\alpha+1}}{(z - w)(1 - \bar{w}z)^{\alpha+1}} dA(w) (1 - |z|^2)^{\alpha-1} d\bar{\mu}(z) \right| \\ &= \left| \int_{\mathbb{D}} \overline{g'(w)}h(w)(1 - |w|^2)^\alpha \overline{R_\alpha(\mu)(w)} dA(w) \right| \\ &\leq \|g\|_{B^\alpha} \|h\|_1 \|R_\alpha(\mu)\|_\infty \leq C < \infty. \end{aligned}$$

On the other hand, by Fubini's Theorem,

$$I_1 = \int_{\mathbb{D}} h(w) \overline{g(w)Q_\alpha(\mu)(w)} (1 - |w|^2)^\alpha dA(w),$$

where $Q_\alpha(\mu)(w) = \alpha(\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha-1}}{(1 - \bar{z}w)^{\alpha+2}} d\mu(z)$.

Hence $T_\mu^\alpha(g) \in B^\alpha$ if and only if

$$(2) \quad (1 - |w|^2)^\alpha g(w) Q_\alpha(\mu)(w) \in L^\infty.$$

The relation between $Q_\alpha(\mu)$ and $P_{\alpha-1}(\mu)$ is

$$(3) \quad Q_\alpha(\mu)(w) = (\alpha + 1)P_{\alpha-1}(\mu)(w) + wP'_{\alpha-1}(\mu)(w).$$

It is easy to see that $P_{\alpha-1}(\mu)$ is in B^α , LB and B as $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$, respectively, implies that $Q_\alpha(\mu)$ satisfies, respectively:

$$\begin{aligned} Q_\alpha(\mu)(w)(1 - |w|^2)^\alpha &\in L^\infty, \text{ if } 0 < \alpha < 1; \\ Q_\alpha(\mu)(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} &\in L^\infty, \text{ if } \alpha = 1; \\ Q_\alpha(\mu)(w)(1 - |w|^2) &\in L^\infty, \text{ if } \alpha > 1. \end{aligned}$$

Notice that $B^\alpha \subset H^\infty$ as $0 < \alpha < 1$ and using the fact that, as $\alpha > 1$, $g \in B^\alpha$ if and only if $(1 - |w|^2)^{\alpha-1}g(w) \in L^\infty$, we easily see that (2) is true for all these cases. Thus, T_μ^α is bounded on B^α .

Conversely, if T_μ^α is bounded on B^α , then $T_\mu^\alpha(1) = P_{\alpha-1}(\mu) \in B^\alpha$. If $R_\alpha(\mu) \in L^\infty$, as above, we get that there is a constant $c > 0$, independent of g , such that

$$(1 - |w|^2)^\alpha |g(w)Q_\alpha(\mu)(w)| \leq c\|g\|_{B^\alpha}.$$

For $0 < \alpha < 1$, we have already shown that $P_{\alpha-1}(\mu) \in B^\alpha$.

For $\alpha \geq 1$, using test functions $g_a(w) = \log \frac{2}{1-\bar{a}w}$ and $g_a(w) = (1 - \bar{a}w)^{1-\alpha}$, as $\alpha = 1$ and $\alpha > 1$, respectively, we get

$$(4) \quad Q_\alpha(\mu)(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \in L^\infty \quad \text{if } \alpha = 1;$$

$$(5) \quad Q_\alpha(\mu)(w)(1 - |w|^2) \in L^\infty \quad \text{if } \alpha > 1.$$

If $\alpha = 1$, $P_{\alpha-1}(\mu) \in B$ implies that $P_{\alpha-1}(\mu)(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \in L^\infty$. Thus, using (3) and (4), we easily see that $P_{\alpha-1}(\mu)'(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \in L^\infty$.

If $\alpha > 1$, then $P_{\alpha-1}(\mu) \in B^\alpha$ implies that

$$(6) \quad P_{\alpha-1}(\mu)(w)(1 - |w|^2)^{\alpha-1} \in L^\infty.$$

For $\alpha - 1 \leq 1$, this implies that $P_{\alpha-1}(\mu)(w)(1 - |w|^2) \in L^\infty$, and using (3) and (5), we get that $P_{\alpha-1}(\mu) \in B$.

For $\alpha - 1 > 1$, combining (3), (5) and (6) we get $P_{\alpha-1}(\mu)'(w)(1 - |w|^2)^{\alpha-1} \in L^\infty$, which is equivalent to

$$(7) \quad P_{\alpha-1}(\mu)(w)(1 - |w|^2)^{\alpha-2} \in L^\infty.$$

If $\alpha - 2 > 1$, combining (3), (5) and (7) we get that $P_{\alpha-1}(\mu)'(w)(1 - |w|^2)^{\alpha-2}$ is in L^∞ , and we continue the process from above until we reach a positive integer n such that $\alpha - n \leq 1$ and $P_{\alpha-1}(\mu)'(w)(1 - |w|^2)^{\alpha-n} \in L^\infty$.

Then, $P_{\alpha-1}(\mu)'(w)(1 - |w|^2) \in L^\infty$, and so $P_{\alpha-1}(\mu) \in B$.

This completes the proof. □

Note that in the case $d\mu(z) = f(z)dA(z)$, with $f \in B^\alpha$, it follows from part (ii) of Proposition 2.1 that $R_\alpha(\mu)(w) = wf(w)$. Thus μ satisfies the condition R_α if and only if $f \in H^\infty$.

Corollary 2.1. *Let $f \in L^\infty$ and let $T_f^\alpha(g) = P_{\alpha-1}(fg)$ for $g \in B^\alpha$. Then we have:*

- (i) *If $0 < \alpha < 1$, then T_f^α is bounded on B^α if and only if $P_{\alpha-1}(f) \in B^\alpha$.*
- (ii) *If $\alpha = 1$, then T_f^α is bounded on B^α if and only if $P_{\alpha-1}(f) \in LB$.*
- (iii) *If $\alpha > 1$, then T_f^α is bounded on B^α if and only if $P_{\alpha-1}(f) \in B$.*

Proof. Follows from Proposition 2.1 and Theorem 2.1. □

Since the Bloch space B is the dual of L_a^1 and $T_\mu^* = T_{\bar{\mu}}$ under this duality, we get the following corollary. Similar results including a few other restrictions can be found in [4] and [8].

Corollary 2.2. *Let μ be a complex measure satisfying condition R_α . Then $T_{\bar{\mu}}$ is bounded on L_a^1 if and only if $P(\mu) \in LB$.*

Remark. The proof of Theorem 2.1 implies that the condition R_α in the theorem can be replaced by a more general condition involving Carleson measures.

Recall that for a positive Borel measure ν on \mathbb{D} we say that ν is a Carleson measure on the Bergman space if there exists $c > 0$ such that, for all $h \in L_a^2$,

$$\int_{\mathbb{D}} |h(z)|^2 d\nu(z) \leq c \int_{\mathbb{D}} |h(z)|^2 dA(z).$$

Let $\nu_\alpha(\mu)$ be the positive measure defined by $d\nu_\alpha(\mu)(z) = |R_\alpha(\mu)(z)|dA(z)$.

If μ satisfies condition R_α , then it is easy to see that $\nu_\alpha(\mu)$ is a Carleson measure. By inspecting the proof of Theorem 2.1 we can see that, whenever $R_\alpha(\mu)$ is in L^1 ,

the statements (i), (ii) and (iii) of Theorem 2.1 are still true if the condition R_α is replaced by a weaker one, namely, by requiring only that the measure $\nu_\alpha(\mu)$ is a Carleson measure for the Bergman space.

Note that the measure $\nu_\alpha(\mu)$ is a Carleson measure for the Bergman space if and only if there exists $r > 0$, such that

$$\sup_{z \in \mathbb{D}} \frac{1}{|D(z, r)|} \int_{D(z, r)} |R_\alpha(\mu)(w)| dA(w) < \infty,$$

where $D(z, r)$ denotes the hyperbolic disk with center z and radius r . See [9, Theorem 6.2.2] for details.

Next we give an example of an unbounded L^1 function that induces a bounded Toeplitz operator on B^α . We present part of the calculations as a separate lemma that covers a wider class of functions.

Lemma 2.2. *Let $f(z) = \sum_{n=0}^\infty a_n(1 - |z|^2)^n$ be a function in $L^1((1 - |z|^2)^{\alpha-1} dA(z))$, for $\alpha > 0$, and such that $\sum_{n=0}^\infty \frac{|a_n|}{n+\alpha}$ is finite. Let $d_\alpha = \alpha \sum_{n=0}^\infty \frac{a_n}{n+\alpha}$ and let $F_\alpha(f) = \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\alpha-1} dA(z)$. Then, for $d\mu(z) = f(z)dA(z)$ we have*

$$R_\alpha(\mu)(w) = d_\alpha w - \frac{\alpha}{\bar{w}} \left(F_\alpha(f) - \frac{1}{(1 - |w|^2)^\alpha} \int_{D_w} f(z)(1 - |z|^2)^{\alpha-1} dA(z) \right),$$

where $D_w = \{z : |w| < |z| < 1\}$.

Proof. As in the proof of part (ii) of Proposition 2.1 we have that, for $w \neq 0$,

$$R_\alpha(\mu)(w) = wP_{\alpha-1}(f)(w) - \alpha \int_{\mathbb{D}} \frac{(1 - |u|^2)^{\alpha-1}}{\bar{u}(1 - \bar{w}u)^{\alpha+1}} f \circ \psi_w(u) dA(u).$$

Using the given series expansion of f and the identity $1 - |\psi_w(u)|^2 = \frac{(1 - |w|^2)(1 - |u|^2)}{|1 - \bar{w}u|^2}$, we get that $P_{\alpha-1}(f)(w) = \alpha \sum_{n=0}^\infty \frac{a_n}{n+\alpha} = d_\alpha$, and that the above integral equals

$$\int_{\mathbb{D}} \sum_{n=0}^\infty a_n(1 - |w|^2)^n \frac{(1 - |u|^2)^{n+\alpha-1}}{(1 - \bar{w}u)^{n+\alpha+1}} \frac{(1 - \bar{w}u)^{-n}}{\bar{u}} dA(u).$$

Since $\frac{(1 - \bar{w}u)^{-n-1}}{\bar{u}}$ is a conjugate analytic, integrable function, we have that

$$(n + \alpha) \int_{\mathbb{D}} \frac{(1 - \bar{w}u)^{-n} - 1}{\bar{u}} \frac{(1 - |u|^2)^{n+\alpha-1}}{(1 - \bar{w}u)^{n+\alpha+1}} dA(u) = \frac{(1 - |w|^2)^{-n} - 1}{\bar{w}}.$$

Since also $\int_{\mathbb{D}} \frac{(1 - |u|^2)^{n+\alpha-1}}{\bar{u}(1 - \bar{w}u)^{n+\alpha+1}} dA(u) = 0$, we have that, furthermore,

$$\begin{aligned} R_\alpha(\mu)(w) &= d_\alpha w - \alpha \sum_{n=1}^\infty \frac{a_n}{n + \alpha} (1 - |w|^2)^n \frac{(1 - |w|^2)^{-n} - 1}{\bar{w}} \\ &= d_\alpha w - \frac{\alpha}{\bar{w}} \left(\sum_{n=0}^\infty \frac{a_n}{n + \alpha} - \sum_{n=0}^\infty \frac{a_n}{n + \alpha} (1 - |w|^2)^n \right) \\ &= d_\alpha w - \frac{\alpha}{\bar{w}} \left(F_\alpha(f) - \frac{1}{(1 - |w|^2)^\alpha} \int_{D_w} f(z)(1 - |z|^2)^{\alpha-1} dA(z) \right). \end{aligned}$$

□

Example. For $\alpha > 0$, let $c_\alpha = \frac{1}{\alpha} \left(\sum_{n=1}^\infty \frac{1}{n(n+\alpha)} \right)^{-1}$ and for $z \neq 0$, let

$$f_\alpha(z) = 1 - c_\alpha \sum_{n=1}^\infty \frac{1}{n} (1 - |z|^2)^n = 1 + c_\alpha \log(|z|^2).$$

The function f_α is a radial, unbounded, L^1 function. Since $P_{\alpha-1}(f_\alpha)$ is a radial analytic function, it must be a constant. It is easy to check that $P_{\alpha-1}(f_\alpha)(0) = 0$ and so $P_{\alpha-1}(f_\alpha) = 0$.

In order to show that T_{f_α} is bounded on B^α we only have to prove that μ_α , with $d\mu_\alpha(z) = f_\alpha(z)dA(z)$, satisfies the condition R_α . Using Lemma 2.2 and the fact that $d_\alpha = 0$, we get that

$$R_\alpha(\mu_\alpha)(w) = \frac{1}{\bar{w}} \left(1 + \alpha c_\alpha \frac{1}{(1 - |w|^2)^\alpha} \int_{|w|^2}^1 (1 - t)^{\alpha-1} \log t dt \right).$$

It is easy to see that $\lim_{|w| \rightarrow 1} |R_\alpha(\mu)(w)| = 1$. Since $\int_0^1 (1 - t)^{\alpha-1} \log t dt = \frac{-1}{\alpha c_\alpha}$, we also have that $\lim_{|w| \rightarrow 0} |R_\alpha(\mu)(w)| = \alpha$. Thus, $R_\alpha(\mu_\alpha) \in L^\infty$, and by Theorem 2.1, T_{f_α} is bounded on B^α .

Note that we can also use Lemma 2.2 to show that $R_\alpha(\mu) \in L^\infty$ is not a necessary condition for the boundedness of the Toeplitz operator T_μ . For example, let $d\mu(z) = f(z)dA(z)$ with $f(z) = \frac{1}{|z|^{3/2}}$. Note that for this radial, positive function $F_\alpha(f)$ is equal to $\frac{\Gamma(1/4)\Gamma(\alpha)}{\Gamma(\alpha+1/4)}$, and d_α is equivalent to $\alpha \sum_{n=1}^\infty \frac{1}{n^{1/4}(n+\alpha)}$. Using Lemma 2.2 we can get that $|R_\alpha(\mu)(w)| \geq \frac{c}{|w|^{3/2}}$ and so $R_\alpha(\mu)$ is not in L^∞ . Since the measure $\nu_\alpha(\mu)$ is a Carleson measure, T_μ is still bounded on B^α .

3. COMPACT TOEPLITZ OPERATORS

For the next result, we need the following lemma from [5].

Lemma 3.1. *Let $0 < \alpha < 1$ and let T be a bounded linear operator from B^α into a normed linear space Y . Then T is compact if and only if $\|Tg_n\|_Y \rightarrow 0$, whenever $\{g_n\}$ is a bounded sequence in B^α that converges to 0 uniformly on \mathbb{D} .*

Theorem 3.1. *Suppose*

$$(8) \quad \lim_{|w| \rightarrow 1} R_\alpha(\mu)(w) = 0.$$

Then we have:

- (i) *If $0 < \alpha < 1$, then T_μ^α is compact on B^α if and only if $P_{\alpha-1}(\mu) \in B^\alpha$.*
- (ii) *If $\alpha = 1$, then T_μ^α is compact on B^α if and only if $P_{\alpha-1}(\mu) \in LB_0$.*
- (iii) *If $\alpha > 1$, then T_μ^α is compact on B^α if and only if $P_{\alpha-1}(\mu) \in B_0$.*

Proof. For the case $\alpha = 1$ or $\alpha > 1$, let $\{g_n\}$ be a sequence in B^α such that $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \rightarrow 0$ uniformly on compact subsets of \mathbb{D} . Let h be in the

unit ball of L^1_a . Similarly, as in the proof of Theorem 2.1, we have

$$\begin{aligned} & \langle h, T_\mu^\alpha(g_n) \rangle_\alpha \\ &= \alpha \int_{\mathbb{D}} P_\alpha(h\bar{g}_n)(z)(1 - |z|^2)^{\alpha-1} d\bar{\mu}(z) \\ & \quad + \alpha \int_{\mathbb{D}} [(I - P_\alpha)(h\bar{g}_n)](z)(1 - |z|^2)^{\alpha-1} d\bar{\mu}(z) \\ &= I_1 + I_2. \end{aligned}$$

For $0 < r < 1$, and $\mathbb{D}_r = \{z : |z| \leq r\}$,

$$\begin{aligned} I_2 &= \alpha \int_{\mathbb{D}} \overline{g'_n(w)} h(w) (1 - |w|^2)^\alpha R_\alpha(\mu)(w) dA(w) \\ &= \alpha \left(\int_{\mathbb{D}_r} + \int_{\mathbb{D} \setminus \mathbb{D}_r} \right) \overline{g'_n(w)} h(w) (1 - |w|^2)^\alpha R_\alpha(\mu)(w) dA(w) = K_1 + K_2. \end{aligned}$$

For a fixed $\varepsilon > 0$, using condition (8), let r be sufficiently close to 1 so that $|R_\alpha(\mu)(w)| < \varepsilon$ as $w \in \mathbb{D} \setminus \mathbb{D}_r$. Then $|K_2| \leq \varepsilon \|g_n\|_{B^\alpha} \|h\|_1 \leq \varepsilon$.

Since $g_n(z) \rightarrow 0$ as $n \rightarrow \infty$, we can choose n big enough so that $|g_n(z)|(1 - |w|^2)^\alpha < \varepsilon$. Therefore, $|K_1| \leq \varepsilon \|R_\alpha(\mu)\|_\infty \|h\|_1 \leq \varepsilon \|R_\alpha(\mu)\|_\infty$.

Hence $|I_2| < C\varepsilon$, where C does not depend on h , and so $\lim_{n \rightarrow \infty} \sup_{\|h\|_1 \leq 1} |I_2| = 0$. Thus T_μ^α is compact on B^α if and only if $\sup_{\|h\|_1 \leq 1} |I_1| \rightarrow 0$ as $n \rightarrow \infty$.

Similarly, as in the proof of Theorem 2.1, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{D}} h(w) \overline{g_n(w) Q_\alpha(\mu)(w)} (1 - |w|^2)^\alpha dA(w) \\ &= \left(\int_{\mathbb{D}_r} + \int_{\mathbb{D} \setminus \mathbb{D}_r} \right) h(w) \overline{g_n(w) Q_\alpha(\mu)(w)} (1 - |w|^2)^\alpha dA(w) = M_1 + M_2. \end{aligned}$$

It is easy to see that if $P_{\alpha-1}(\mu)$ is in LB_0 and B_0 as $\alpha = 1$ and $\alpha > 1$, respectively, then, as $|w| \rightarrow 1$, Q_α satisfies, respectively:

$$\begin{aligned} & Q_\alpha(\mu)(w)(1 - |w|^2) \log \frac{2}{1 - |w|^2} \rightarrow 0, \text{ if } \alpha = 1; \\ & Q_\alpha(\mu)(w)(1 - |w|^2) \rightarrow 0, \text{ if } \alpha > 1. \end{aligned}$$

Again, notice that, as $\alpha > 1$, $g \in B^\alpha$ if and only if $(1 - |z|^2)^{\alpha-1} g(z) \in L^\infty$, and we may choose r sufficiently close to 1 such that whenever $w \in \mathbb{D} \setminus \mathbb{D}_r$, $|Q_\alpha(\mu)(w)|(1 - |w|^2) \log \frac{2}{1 - |w|^2} < \varepsilon$ as $\alpha = 1$; and $|Q_\alpha(\mu)(w)|(1 - |w|^2)^\alpha < \varepsilon$ as $\alpha > 1$, respectively.

We see that $|M_2| \leq \varepsilon \alpha \|g_n\|_{B^\alpha} \|h\|_1 \leq C\varepsilon$, where C does not depend on h .

Since $g_n(z) \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , we can choose n big enough so that $|g_n(w)| < \varepsilon$. Hence we easily see that, as n is big enough, $|M_1| < C\varepsilon$, where C does not depend on h . Therefore, $\sup_{\|h\|_1 \leq 1} |I_1| \rightarrow 0$ as $n \rightarrow \infty$, and so T_μ^α is compact on B^α as $\alpha = 1$ or $\alpha > 1$.

Now let $0 < \alpha < 1$. Let $\{g_n\}$ be a sequence in B^α such that $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \rightarrow 0$ uniformly on \mathbb{D} . Let h be in the unit ball of L^1_a . By a similar discussion as above, T_μ^α is compact on B^α if and only if $\sup_{\|h\|_1 \leq 1} |I_1| \rightarrow 0$ as $n \rightarrow \infty$. However,

since $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \rightarrow 0$ uniformly on $\overline{\mathbb{D}}$, we can choose n big enough such that $|g_n(w)| < \varepsilon$ uniformly for $w \in \overline{\mathbb{D}}$.

Thus, as n is big enough, $|I_1| \leq \varepsilon \alpha \|P_{\alpha-1}(\mu)\|_{B^\alpha} \|h\|_1$.

Therefore, $\sup_{\|h\|_1 \leq 1} |I_1| \rightarrow 0$ as $n \rightarrow \infty$, and so T_μ^α is compact on B^α .

Conversely, if T_μ^α is compact on B^α . Suppose first that $\alpha = 1$ or $\alpha > 1$. Then $\sup_{\|h\|_1 \leq 1} |I_1| \rightarrow 0$ as $n \rightarrow \infty$ for any sequence g_n in B^α such that $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \rightarrow 0$ uniformly on compact subsets of \mathbb{D} .

Let $h_z(w) = \frac{(1 - |z|^2)^\alpha}{(1 - \bar{z}w)^{2+\alpha}}$. Then $\|h_z\|_1 \leq C$, and

$$\begin{aligned} I_1 &= (1 - |z|^2)^\alpha \int_{\mathbb{D}} \frac{\overline{g_n(w)Q_\alpha(\mu)(w)}(1 - |w|^2)^\alpha}{(1 - \bar{z}w)^{2+\alpha}} dA(w) \\ &= (\alpha + 1)^{-1} (1 - |z|^2)^\alpha \overline{g_n(z)Q_\alpha(\mu)(z)}. \end{aligned}$$

Thus, $\sup_{\|h\|_1 \leq 1} |I_1| \rightarrow 0$ as $n \rightarrow \infty$ implies

$$(9) \quad \lim_{n \rightarrow \infty} \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g_n(z)| |Q_\alpha(\mu)(z)| = 0.$$

As $\alpha = 1$ or $\alpha > 1$, using testing functions $g_a(z) = \left(\log \frac{1}{1-|a|^2}\right)^{-1} \left(\log \frac{2}{1-\bar{a}z}\right)^2$ and $g_a(z) = (1 - |a|^2)(1 - \bar{a}z)^{-\alpha}$ respectively in (9), as in the proof of Theorem 2.1, we easily see that $P_{\alpha-1}(\mu)$ is in LB_0 and B_0 , respectively.

Now let $0 < \alpha < 1$ and let T_μ^α be compact on B^α . Then T_μ^α is bounded on B^α . By Theorem 2.1, $P_{\alpha-1}(\mu) \in B^\alpha$. The proof is completed. \square

Remark. The requirement that $\lim_{|w| \rightarrow 1} R_\alpha(\mu)(w) = 0$, in the proof of Theorem 3.1, can be replaced by a more general requirement that the measure $\nu_\alpha(\mu)$, defined in Section 2, is a compact Carleson measure.

Recall that for a positive Borel measure ν on \mathbb{D} we say that ν is a compact Carleson measure whenever

$$\lim_{|z| \rightarrow 1} \frac{1}{|D(z, r)|} \int_{D(z, r)} d\nu(w) = 0,$$

where $D(z, r)$ denotes the hyperbolic disk with center z and radius r . See [9, Theorem 6.2.5] for more details. If the measure μ is such that $\lim_{|w| \rightarrow 1} R_\alpha(\mu)(w) = 0$, then it is easy to see that $\nu_\alpha(\mu)$ is a compact Carleson measure.

Note that in the case $d\mu(z) = f(z)dA(z)$, with $f \in B^\alpha$, it follows from part (ii) of Proposition 2.1 that $R_\alpha(\mu)(w) = wf(w)$. Thus the condition (8) in Theorem 3.1 is satisfied only when $f = 0$.

REFERENCES

[1] J. Arazy, *Multipliers of Bloch Functions*, University of Haifa Mathem. Public. Series 54 (1982).
 [2] J. Arazy, S. Fisher, S. Janson, J. Peetre, *An Identity for Reproducing Kernels in a Planar Domain and Hilbert-Schmidt Hankel Operators*, J. Reine Angew. Math. 406 (1990), 179-199.
 [3] J. Arazy, S. Fisher, J. Peetre, *Hankel Operators on Weighted Bergman Spaces*, American J. Math. 110 (1988), 989-1054. MR0970119 (90a:47067)
 [4] K. R. M. Attele, *Toeplitz and Hankel Operators on Bergman One Space*, Hokkaido Math. J. 21 (1992), 279-293. MR1169795 (93d:47051)
 [5] S. Ohno, K. Stroethoff, R. Zhao, *Weighted Composition Operators between Bloch-Type spaces*, Rocky Mountain J. Math. 33 (2003), 191-215. MR1994487 (2004d:47058)

- [6] W. Rudin, *Function Theory in the Unit Ball in \mathbb{C}^n* , Springer-Verlag, New York, 1980. MR0601594 (82i:32002)
- [7] J. L. Wang, Z. Wu, *Minimum solutions of $\bar{\delta}^{k+1}$ and Middle Hankel Operators*, J. Funct. Anal. 118 (1993), 167-187. MR1245601 (94j:47042)
- [8] K. Zhu, *Multipliers of BMO in the Bergman Metric with Applications to Toeplitz Operators*, J. Funct. Anal. 87 (1989), 31-50. MR1025882 (90m:42024)
- [9] K. Zhu, *Operator Theory on Function Spaces*, Marcel Dekker, New York, 1990. MR1074007 (92c:47031)
- [10] K. Zhu, *Bloch-Type Spaces of Analytic Functions*, Rocky Mountain J. Math. 23 (1993), 1143-1177. MR1245472 (95d:46020)

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