

## CONFORMAL MAPPING AND ELLIPSES

ARNE LEDET AND ALEXANDER YU. SOLYNIN

(Communicated by Juha M. Heinonen)

ABSTRACT. We answer a question raised by M. Chuaqui, P. Duren, and B. Osgood by showing that a conformal mapping of a simply connected domain cannot take *two* circles onto two proper ellipses.

The goal of this note is to prove the following.

**Proposition 1.** *Let  $\Omega$  be a simply connected domain on  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  containing circles (on  $\overline{\mathbb{C}}$ )  $C_1$  and  $C_2$ ,  $C_1 \neq C_2$ . Let  $f$  be a conformal mapping from  $\Omega$  into  $\mathbb{C}$ . If  $f$  maps each of the circles  $C_1$  and  $C_2$  onto an ellipse, then  $f$  is a Möbius transformation and therefore the ellipses are actually circles.*

Thus this proposition answers affirmatively a question raised by M. Chuaqui, P. Duren, and B. Osgood in [1] that a conformal mapping of a simply connected domain cannot take *two* circles onto two ellipses.

Precomposing  $f$  with a Möbius transformation and postcomposing it with a linear transformation, we may assume that  $C_1 = \{z : |z| = 1\}$  and that  $f$  maps the unit disc  $\mathbb{D} = \{z : |z| < 1\}$  onto the interior of an ellipse  $\mathcal{L}_1$  with foci  $\pm 1$  such that  $f(r) = 1$ ,  $f(-r) = -1$  for some  $0 < r < 1$ . Then it is a standard exercise in complex analysis to verify that  $w = f(z)$  is defined by

$$(1) \quad w = \frac{1}{2}(\zeta + 1/\zeta) \quad \text{with} \quad \zeta = i \exp \left\{ -\frac{\pi i}{2\mathbf{K}(r^2)} \int_0^{z/r} \frac{dt}{\sqrt{(1-t^2)(1-r^4t^2)}} \right\},$$

where  $\mathbf{K}(\cdot)$  denotes the complete elliptic integral of the first kind. Since the image  $f(\mathbb{D})$  is symmetric with respect to the coordinate axes, it follows that  $f$  is an odd function having real coefficients, i.e.,

$$f(z) = pz + c_3z^3 + c_5z^5 + \dots \quad \text{with} \quad p = \pi/(2r\mathbf{K}(r^2)) > 0.$$

We want to stress that our proof below *does not require* the explicit form (1) of the mapping function, which is included here for completeness.

Instead, we will use the fact that every ellipse  $\mathcal{L}$  having foci  $w_1$  and  $w_2$  is a trajectory of the quadratic differential

$$(2) \quad Q_1(w) dw^2 = -\frac{dw^2}{(w-w_1)(w-w_2)};$$

---

Received by the editors June 6, 2005.

2000 *Mathematics Subject Classification.* Primary 30C35.

*Key words and phrases.* Conformal mapping, ellipse.

The research of the second author was supported in part by NSF grant DMS-0412908.

©2006 American Mathematical Society  
Reverts to public domain 28 years from publication

we refer to [2] for the definition and necessary properties of quadratic differentials. Let  $w = f(z)$  map a disc  $D$  on  $\overline{\mathbb{C}}$  conformally onto the interior of  $\mathcal{L}$ . Let  $z_1$  and  $z_2$  be the preimages in  $D$  of  $w_1$  and  $w_2$  and let  $z_3, z_4$  be the reflections of  $z_1, z_2$  in the circle  $C = \partial D$ . Pulling the quadratic differential (2) back to the disc  $D$  we obtain a quadratic differential  $Q_2(z) dz^2 = Q_1(f(z)) f'^2(z) dz^2$  positive on  $C$  whose only singularities in  $D$  are simple poles at  $z_1$  and  $z_2$ . The latter implies that

$$(3) \quad Q_2(z) dz^2 = -k \frac{dz^2}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)},$$

with some nonzero  $k \in \mathbb{C}$ . In (3), we assume that all singularities are finite. If one of them, say  $z_j$ , is infinite, the corresponding term  $(z - z_j)$  in (3) must be replaced by 1.

Comparing (2) and (3) we find that the mapping  $w = f(z)$  from  $D$  onto the interior of  $\mathcal{L}$  satisfies the differential equation

$$(4) \quad w'^2 = k \frac{(w - w_1)(w - w_2)}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}.$$

For a particular choice when  $D = \mathbb{D}$ ,  $w_1 = 1$ ,  $w_2 = -1$ ,  $z_1 = r$ , and  $z_2 = -r$  with  $0 < r < 1$ , (4) becomes

$$(5) \quad w'^2 = -p^2 \frac{(w^2 - 1)}{(z^2 - r^2)(z^2 - 1/r^2)}, \quad \text{where } p = w'(0) > 0.$$

Now we assume that  $\Omega$  is a simply connected domain on  $\overline{\mathbb{C}}$  containing the discs  $\mathbb{D}$  and  $D \neq \mathbb{D}$ . If there is a function  $w = f(z)$  univalent in  $\Omega$ , which maps the circles  $C_1 = \partial \mathbb{D}$  and  $C_2 = \partial D$  onto ellipses, then, by the principle of analytic continuation,  $w = f(z)$  will satisfy near the origin both equations (4) and (5). Therefore the Taylor coefficients of  $f$  at  $z = 0$  can be found from either one of these equations.

To simplify our computation, we first take logarithmic derivatives in (4) and (5), then expand them into power series at  $z = 0$ . For our purposes we need the first eight coefficients, so all our computations can be easily verified with *Maple* or *Mathematica*. Equating the first eight Taylor coefficients of the results and assuming that  $z_1, z_2, z_3, z_4, w_1, w_2$  all are nonzero and finite we obtain in this generic case the following system of eight polynomial equations, which we call *system* (\*):

$$\begin{aligned} a + b + c + d &= p(A + B), \\ a^2 + b^2 + c^2 + d^2 &= p^2(A^2 + B^2) + 2((r^2 + 1/r^2) - p^2), \\ a^3 + b^3 + c^3 + d^3 &= p^3(A^3 + B^3) + \frac{1}{2}p(A + B)((r^2 + 1/r^2) - p^2), \\ a^4 + b^4 + c^4 + d^4 &= p^4(A^4 + B^4) + \frac{2}{3}p^2(A^2 + B^2 - 2)((r^2 + 1/r^2) - p^2) \\ &\quad + 2((r^4 + 1/r^4) - p^4), \\ a^5 + b^5 + c^5 + d^5 &= p^5(A^5 + B^5) + \frac{1}{4}p(A + B) + \frac{3}{8}p(A + B)((r^4 + 1/r^4) - p^4) \\ &\quad + \frac{5}{12}p^3[2(A^3 + B^3) - (A + B)]((r^2 + 1/r^2) - p^2), \\ a^6 + b^6 + c^6 + d^6 &= p^6(A^6 + B^6) + 2((r^6 + 1/r^6) - p^6) + \frac{7}{15}p^2(A^2 + B^2 - 2) \\ &\quad + \frac{8}{15}p^2(A^2 + B^2 - 2)((r^4 + 1/r^4) - p^4) \\ &\quad + p^4[(A^4 + B^4) - \frac{2}{3}(A^2 + B^2 + 1)]((r^2 + 1/r^2) - p^2), \end{aligned}$$

$$\begin{aligned}
 a^7 + b^7 + c^7 + d^7 &= p^7(A^7 + B^7) + \frac{5}{16}p(A+B)((r^6 + 1/r^6) - p^6) \\
 &\quad + \frac{259}{720}p^3[2(A^3 + B^3) - (A+B)]((r^4 + 1/r^4) - p^4) \\
 &\quad + \left\{ \frac{7}{144}p^5[24(A^5 + B^5) - 20(A^3 + B^3) + (A+B)] \right. \\
 &\quad \left. + \frac{3}{16}p(A+B) \right\}((r^2 + 1/r^2) - p^2) \\
 &\quad + \frac{133}{180}p^3[(A^3 + B^3) - (A+B)] + \frac{401}{720}p^3(A+B), \\
 a^8 + b^8 + c^8 + d^8 &= p^8(A^8 + B^8) + 2((r^8 + 1/r^8) - p^8) \\
 &\quad + \frac{16}{35}p^2(A^2 + B^2 - 2)((r^6 + 1/r^6) - p^6) \\
 &\quad + \frac{14}{45}p^4[3(A^4 + B^4) - 2(A^2 + B^2) - 2]((r^4 + 1/r^4) - p^4) \\
 &\quad + p^6\left[\frac{4}{3}((A^6 + B^6) - (A^4 + B^4)) \right. \\
 &\quad \left. + \frac{8}{45}(A^2 + B^2) - \frac{16}{45}\right]((r^2 + 1/r^2) - p^2) \\
 &\quad + \frac{8}{21}p^2(A^2 + B^2 - 2)((r^2 + 1/r^2) - p^2),
 \end{aligned}$$

where  $a = 1/z_1, b = 1/z_2, c = 1/z_3, d = 1/z_4, A = 1/w_1,$  and  $B = 1/w_2.$

From this point, our argument is purely algebraic. Our goal is to show that the only solution to system (\*) is  $a = r, b = -r, c = 1/r, d = -1/r$  (up to permutation), and  $A = 1, B = -1$  (again up to permutation).

Since the left-hand sides of the equations in system (\*) are Newton power sums in  $a, b, c$  and  $d,$  they are, in particular, symmetric polynomials, and can therefore be written in terms of the elementary symmetric symbols  $s_1 = a + b + c + d, \dots, s_4 = abcd.$  For necessary properties of the Newton power sums and symmetric polynomials, one may consult [3, Section 5.7]. Similarly,  $A$  and  $B$  only occur on the right-hand sides of system (\*) in the form of Newton power sums, which can be expressed in terms of the elementary symmetric symbols  $S_1 = A + B$  and  $S_2 = AB.$  The goal is then to show that the only solution to the rewritten system is  $s_1 = 0, s_2 = -(r^2 + 1/r^2), s_3 = 0, s_4 = 1, S_1 = 0,$  and  $S_2 = -1.$

The first equation of system (\*) now takes the form  $s_1 = pS_1,$  which determines  $s_1.$  The second equation becomes

$$s_1^2 - 2s_2 = p^2(S_1^2 - 2S_2) + 2((r^2 + 1/r^2) - p^2),$$

which we can solve to find

$$s_2 = p^2S_2 - ((r^2 + 1/r^2) - p^2).$$

The third equation is

$$s_1^3 - 3s_1s_2 + 3s_3 = p^3(S_1^3 - 3S_1S_2) + 3qS_1,$$

and we solve this to find

$$s_3 = -\frac{5pS_1}{6}((r^2 + 1/r^2) - p^2).$$

The fourth equation becomes

$$\begin{aligned}
 s_1^4 + 4s_1s_3 - 4s_1^2s_2 + 2s_2^2 - 4s_4 \\
 = 4pq(S_1^2 - 2S_2) + p^4(S_1^4 - 4S_1^2S_2 + 2S_2^2) - 8pq + 2((r^4 + 1/r^4) - p^4),
 \end{aligned}$$

which we solve to find

$$s_4 = 1 - \frac{2}{3}p^2(S_2 + 1)((r^2 + 1/r^2) - p^2).$$

This determines  $s_1, s_2, s_3, s_4$  as functions of  $S_1$  and  $S_2$ . Also, for  $S_1 = 0$  and  $S_2 = -1$ , this gives the required values for  $s_1, s_2, s_3, s_4$ . Thus, the only thing remaining is to show that  $S_1 = 0$  and  $S_2 = -1$  is the only solution to the last four equations of system (\*).

The fifth equation becomes

$$pS_1(61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64) = 0,$$

where  $R = r + 1/r$ . Therefore since  $p \neq 0$  by (5), we must either have  $S_1 = 0$  or

$$(6) \quad 61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64 = 0.$$

Now, assume  $S_1 \neq 0$ . Then (6) is satisfied.

The sixth equation of system (\*) written in terms of symmetric symbols  $S_1$  and  $S_2$  becomes

$$(7) \quad 60p^2[3(61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64)S_1^2 + 8(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)(S_2 + 1)] = 0.$$

So assuming (6), we must in fact have

$$(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)(S_2 + 1) = 0.$$

Hence,  $S_2 = -1$  or

$$-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4 = 0.$$

In the second case, we obtain

$$\begin{aligned} & (61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64) \\ & + 5(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4) \\ & = 24(R^4 - 4R^2 + (1 - p^4)) = 0, \end{aligned}$$

meaning that

$$R = \sqrt{2 \pm \sqrt{3 + p^4}}.$$

Plugging this into (6), we get

$$25(3 + 2p^4 \pm 2p^2\sqrt{3 + p^4}) = 0,$$

which is easily seen to be impossible. Therefore, we must have  $S_2 = -1$ .

The seventh equation of system (\*) can be written as

$$(8) \quad pS_1[42(61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64)p^2S_1^2 + (1246R^4 - 4984R^2 + 3220p^2R^2 - 3584 - 6440p^2 - 4466p^4)p^2S_2 + (5355p^4R^2 + 3690R^4 - 2040R^2 - 615R^6 - 3081p^6 - 12264p^2 + 6636p^2R^2 - 10710p^4 - 1659p^2R^4 - 5760)] = 0,$$

which simplifies to

$$-2(49p^4 + 44)R^2 + 98p^6 + 196p^4 + 235p^2 + 176 = 0,$$

when we use (6) and  $S_2 = -1$ . Consequently,

$$R = \sqrt{\frac{98p^6 + 196p^4 + 235p^2 + 176}{2(49p^4 + 44)}},$$

and (6) becomes

$$\frac{1089(768 + 637p^4)}{4(49p^4 + 44)^2} = 0,$$

which is impossible as well, since  $p$  is real.

All in all, we must therefore have that  $S_1 = 0$ . Again, (7) reduces to

$$(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)(S_2 + 1) = 0,$$

so if  $S_2$  does *not* equal  $-1$ , we have

$$(9) \quad -8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4 = 0.$$

Now, the eighth equation of system (\*) is

$$(1 + S_2)[ -7(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)p^2S_2 + (132 + 378p^2 - 252p^2R^2 + 486R^2 + 63p^2R^4 - 189p^4R^2 - 414R^4 + 378p^4 + 57p^6 + 69R^6)] = 0,$$

and so assuming (9),

$$132 + 378p^2 - 252p^2R^2 + 486R^2 + 63p^2R^4 - 189p^4R^2 - 414R^4 + 378p^4 + 57p^6 + 69R^6 = 0.$$

Taking this times 49, adding  $-483R^2 + 996 + 249p^2$  times (9), and dividing the result by 90, this reduces to

$$(16p^4 + 7)R^2 - (16p^6 + 32p^4 + 31p^2 + 14) = 0,$$

i.e.,

$$R = \sqrt{\frac{16p^6 + 32p^4 + 31p^2 + 14}{16p^2 + 7}}.$$

Substituting this into (9), we get

$$-\frac{1764}{(16p^4 + 7)^2} = 0.$$

This is of course absurd, so we conclude that  $S_2 = -1$ .

Thus,  $S_1 = 0$  and  $S_2 = -1$  is the only solution, and the proposition is proved in the generic case.

Turning to the remaining special cases, we note that if  $z_k = \infty$ , then the corresponding term  $a$ ,  $b$ ,  $c$ , or  $d$  in the left-hand side of system (\*) is absent, which does not effect further computation.

Next, since  $f(z)$  is analytic at  $z = 0$ , it follows from (4) that  $z_k = 0$  for some  $k$  if and only if  $w_1 = 0$  or  $w_2 = 0$ . Assume  $w_1 = 0$ . Then also  $z_1 = 0$  since  $z_1$  denotes the preimage of  $w_1$ . Now we can of course find equations similar to those

of system (\*), where the left-hand side of the equations are Newton power sums in  $b = 1/z_2$ ,  $c = 1/z_3$ , and  $d = 1/z_4$ , and the right-hand side are expressions in  $r$ ,  $p$  and  $B = 1/w_2$ . In this case, having six variables, we need the first six equations. Again, the Newton power sums are rewritten in terms of the elementary symmetric symbols  $s_1 = b + c + d$ ,  $s_2 = bc + bd + cd$ , and  $s_3 = bcd$ , and the first three equations are used to solve for  $s_1$ ,  $s_2$ , and  $s_3$  in terms of  $r$ ,  $p$ , and  $B$ . As a further simplification, we replace  $r + 1/r$  by  $R$ , to get the remaining three equations

$$\begin{aligned} L_4 &:= 11R^4 + (50p^2 - 44)R^2 - (61p^4 + 100p^2 + 64) = 0, \\ L_5 &:= (24R^4 - (70p^2 + 92)R^2 + (47p^4 + 140p^2 + 128))B = 0, \\ L_6 &:= -7p^2(24R^4 - (70p^2 + 92)R^2 + (47p^4 + 140p^2 + 128))B^2 \\ &\quad + 269R^6 + (1085p^2 - 1614)R^4 + (936 - 4340p^2 - 1729p^4)R^2 \\ &\quad + (375p^6 + 3458p^4 + 4760p^2 + 2432) = 0. \end{aligned}$$

Here, we note an obvious simplification:  $L_6 + 7p^2BL_5$  does not depend on  $B$ , but is simply a polynomial in  $R$  and  $p$ . Furthermore,

$$\begin{aligned} &\frac{1}{480}(121(L_6 + 7p^2BL_5) + (5918 + 1515p^2 - 2959R^2)L_4) \\ &= 2(49p^2 + 44)R^2 - (98p^6 + 196p^4 + 235p^2 + 176) = 0, \end{aligned}$$

so we must have

$$R = \sqrt{\frac{98p^6 + 196p^4 + 235p^2 + 176}{2(49p^2 + 44)}}.$$

But then

$$L_4 = -\frac{1089(637p^4 + 768)}{4(49p^2 + 44)^2},$$

which cannot be zero, since  $p$  is real.

Therefore, there are no solutions in this case. This finishes the proof of Proposition 1.

*Remarks.* (1) Actually our proof gives a stronger result: Let  $f$  admit analytic continuation along any path in a domain (not necessarily simply connected)  $\Omega \subset \overline{\mathbb{C}}$ . Suppose there are two branches  $f_1$  and  $f_2$  of  $f$  and two discs  $D_1$  and  $D_2$ ,  $D_1 \neq D_2$ , such that  $f_k$  maps  $D_k$  one-to-one onto an interior of an ellipse. Then  $f$  is a Möbius transformation.

(2) Equating the right-hand sides of (4) and (5) we can express  $w$  as an algebraic function of  $z$ . Then one may finish the proof of Proposition 1 by invoking the fact that the elliptic integral in (1) cannot be reduced to an algebraic form for any value of  $r$  under consideration. But we could not find a reference for such a result in the accessible literature.

(3) It is worth mentioning that the existence of a common solution  $w = f(z)$  to two equations  $w'^2 = R_1(z, w)$  and  $w'^2 = R_2(z, w)$  with rational  $R_1$  and  $R_2$  does not guarantee in general that  $R_1 \equiv R_2$ . Indeed, the Koebe function  $w = z/(1-z)^2$ , being extremal in numerous problems of geometric function theory, satisfies a variety of such equations; see [2, Ch. VI].

## REFERENCES

1. M. Chuaqui, P. Duren, and B. Osgood, Ellipses, near ellipses, and harmonic Möbius transformations. *Proc. Amer. Math. Soc.*, **133** (2005), no. 9, 2705–2710. MR2146217 (2005m:31002)
2. J. A. Jenkins, *Univalent functions and conformal mapping*, 2nd ed. Springer, Berlin, 1965. MR0096806 (20:3288)
3. B. L. van der Waerden, *Algebra*. Vol. I. Springer-Verlag, New York, 1991. MR1080172 (91h:00009a)

DEPARTMENT OF MATHEMATICS AND STATISTICS, TEXAS TECH UNIVERSITY, BOX 41042, LUBBOCK, TEXAS 79409

*E-mail address:* `arne.ledet@ttu.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, TEXAS TECH UNIVERSITY, BOX 41042, LUBBOCK, TEXAS 79409

*E-mail address:* `alex.solynin@ttu.edu`