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# A COUNTEREXAMPLE RELATED TO TOPOLOGICAL SUMS

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Abstract. In this paper we construct compact metric spaces $X$, $Y$ which are topologically distinct but whose topological sums $X \sqcup X$ and $Y \sqcup Y$ are homeomorphic.

## 1. Introduction and the main result

For topological spaces $X$, $Y$, we denote by $X \sqcup Y$ the topological sum of $X$ and $Y$. Let $X$ and $Y$ be topological spaces and assume that $X \sqcup X$ and $Y \sqcup Y$ are homeomorphic (we denote this by $X \sqcup X \approx Y \sqcup Y$). Then, one may ask whether $X \approx Y$ holds.

If $X$ is a connected space, obviously, this is always true. It seems that even for "most" disconnected spaces $X, Y$ such that $X \sqcup X \approx Y \sqcup Y$, $X$ and $Y$ are homeomorphic. But this is not true in general; we will describe a counterexample.

**Theorem 1.1.** *There exist compact metric spaces $X$, $Y$ which satisfy the following property: $X$ and $Y$ are nonhomeomorphic, but the topological sums $X \sqcup X$ and $Y \sqcup Y$ are homeomorphic. Moreover, such spaces $X$, $Y$ can be constructed as subspaces of the Euclidean plane $\mathbb{R}^2$.*

In this paper, for notational convenience, we will construct an example of the pair $X$, $Y$ as subspaces of $\mathbb{R}^4$. However, it will be easily observed that one can construct $X$, $Y$ with the same property as (compact) subspaces of $\mathbb{R}^2$ in a similar way.

For $n \geqq 0$, we denote by

$$\bigsqcup_n X$$

the topological sum of $n$ copies of $X$. If $n = 0$, this means the empty set $\emptyset$. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ be the set of all positive integers, all integers, all rational numbers and all real numbers, respectively. Let $I = [0,1] = \{x \in \mathbb{R}\,;\, 0 \leqq x \leqq 1\}$ be the closed interval between 0 and 1. For points $x, y$ in the Euclidean space $\mathbb{R}^n$, put

$$\langle x, y\rangle = \{(1-t)x + ty\,;\, t \in I\} \subset \mathbb{R}^n.$$

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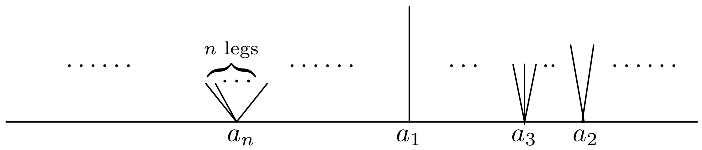


FIGURE 1. The space $Z$

## 2. Construction of the example

The main idea is to construct spaces $X$ and $Z$ which have the following property:

**Property A.** *The spaces $X$ and $X \sqcup Z$ are nonhomeomorphic, but $X$ and $X \sqcup Z \sqcup Z$ are homeomorphic.*

If we can construct such a pair of spaces $X$ and $Z$, the spaces $X$ and $Y = X \sqcup Z$ clearly form a counterexample. Now, the problem is to construct $X$ and $Z$ satisfying Property A.

The space $Z$ is constructed as follows. Arrange the countable set $I \cap \mathbb{Q}$ to get a sequence $a_1, a_2, a_3, \ldots$. Then let $Z$ be the closed interval $I$ with $n$ "legs" at each rational point $a_n$ (see Figure 1). Precisely, let $v_i = (1, 1/i) \in \mathbb{R}^2$ for $i \in \mathbb{N}$ and let

$$F_n = \bigcup_{i=1}^{n} \left\langle 0, \frac{1}{n} v_i \right\rangle \subset \mathbb{R}^2$$

for $n \in \mathbb{N}$. Then we put

$$Z = (I \times \{(0,0)\}) \cup \bigcup_{n \in \mathbb{N}} (\{a_n\} \times F_n) \subset \mathbb{R}^3.$$

Observe that every open cover of $Z$ has a finite subcover, whence $Z$ is compact.

The following lemma is easily seen by counting the path components of the complements of each point of $Z$ (especially, the "node" $(a_n, 0, 0)$ of $Z$ for each $n \in \mathbb{N}$) and observing that the set $\{a_n \,; n \in \mathbb{N}\} = I \cap \mathbb{Q}$ is dense in $I$.

**Lemma 2.1.** *Every homeomorphism $\varphi \colon Z \longrightarrow Z$ fixes the points of $I \times \{(0,0)\}$.* $\square$

The other space $X$ has countably many copies of $Z$ as subspaces. Precisely, we construct $X$ as follows. Let $f \colon I \longrightarrow I$ be the homeomorphism defined by

$$f(x) = \begin{cases} x/2 & \text{if } 0 \leqq x \leqq 1/2, \\ (3x-1)/2 & \text{if } 1/2 \leqq x \leqq 1. \end{cases}$$

(See Figure 2.) Clearly, $f$ induces a homeomorphism from $I \cap \mathbb{Q}$ to itself. Let $f^0 = \mathrm{id}_I$, the identity map of $I$, $f^1 = f$ and $f^n = f \circ f^{n-1}$ for integers $n \geqq 2$. Define $f^{-n} = (f^n)^{-1}$ for each $n \in \mathbb{N}$. Each $f^n$ is a homeomorphism from $I$ to itself and induces a homeomorphism from $I \cap \mathbb{Q}$ to itself.

Let

$$X_m = (I \times \{(0,0)\}) \cup \bigcup_{n \in \mathbb{N}} \left( \{f^{m-1}(a_n)\} \times \frac{1}{m} F_n \right) \subset \mathbb{R}^3$$

for each $m \in \mathbb{N}$ and let

$$X = (\{0\} \times I \times \{(0,0)\}) \cup \bigcup_{m \in \mathbb{Z} \setminus \{0\}} \left( \left\{ \frac{1}{m} \right\} \times X_{|m|} \right) \subset \mathbb{R}^4.$$

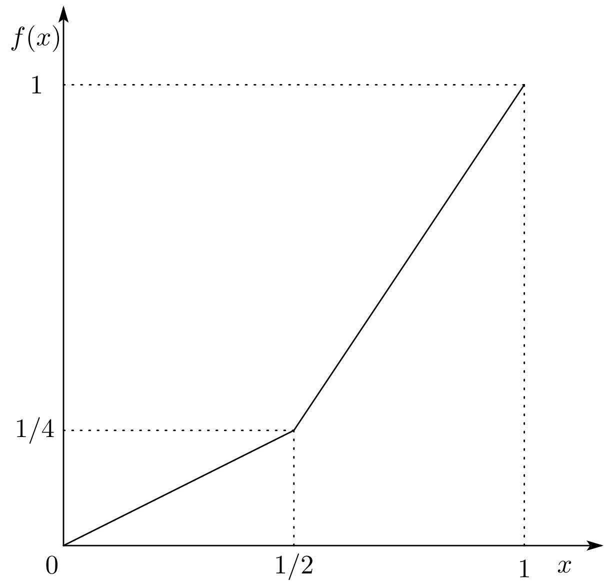


FIGURE 2. The homeomorphism $f$

Then, clearly, $I_0 = \{0\} \times I \times \{(0,0)\}$ and

$$Z_m = \left\{ \frac{1}{m} \right\} \times X_{|m|} \quad (m \in \mathbb{Z} \setminus \{0\})$$

are path components of $X$. We easily see that $X$ is closed in $\mathbb{R}^4$. Since $X$ is a bounded subset of $\mathbb{R}^4$, $X$ is compact (see Figure 3).

Note that the map $Z \longrightarrow Z_m$ given by

$$(x, y, z) \longmapsto (1/m, f^{|m|-1}(x), y/|m|, z/|m|)$$

is a homeomorphism for each $m \in \mathbb{Z} \setminus \{0\}$. By this fact and Lemma 2.1, we have

**Lemma 2.2.** *If $m, m' \in \mathbb{Z} \setminus \{0\}$, every homeomorphism $Z_m \longrightarrow Z_{m'}$ induces the homeomorphism $\{1/m\} \times I \times \{(0,0)\} \longrightarrow \{1/m'\} \times I \times \{(0,0)\}$ given by*

$$\left( \frac{1}{m}, x, 0, 0 \right) \longmapsto \left( \frac{1}{m'}, f^{|m'|-|m|}(x), 0, 0 \right). \quad \square$$

Observe that $\ldots, f^{-2}(x), f^{-1}(x), x, f^1(x), f^2(x), \ldots$ are distinct in $I$ for each $x \in I \setminus \{0, 1\}$. The next proposition follows from this fact and Lemma 2.2.

**Proposition 2.3.** *Assume that $m, m' \in \mathbb{Z} \setminus \{0\}$ and that $\psi \colon Z_m \longrightarrow Z_{m'}$ is a homeomorphism. If there exists a point $x \in I \setminus \{0, 1\}$ such that*

$$\psi \left( \frac{1}{m}, x, 0, 0 \right) = \left( \frac{1}{m'}, f^n(x), 0, 0 \right),$$

*then $|m'| - |m| = n$.* $\square$

Now, we can prove the following essential proposition.

**Proposition 2.4.** *The space $X \sqcup \bigsqcup_N Z$ is homeomorphic to $X$ if and only if $N$ is even. In particular, $X$ and $Z$ satisfy Property A.*

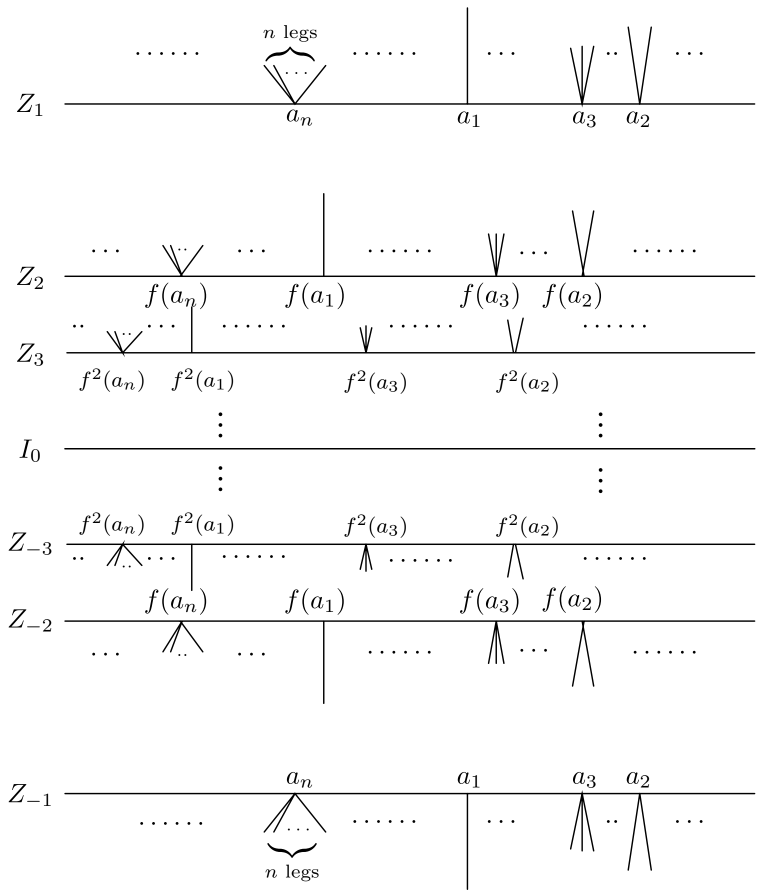


FIGURE 3. The space $X$

*Proof.* First, assume that $N = 2n$ is an even positive integer. Define a map $\varphi\colon X \longrightarrow X$ by $\varphi(0,x,0,0) = (0, f^n(x), 0, 0)$ for $x \in I$ and

$$\varphi\left(\frac{1}{m}, x, y, z\right) = \begin{cases} \left(\frac{1}{m+n}, f^n(x), \frac{m}{m+n}y, \frac{m}{m+n}z\right) & \text{if } m > 0, \\ \left(\frac{1}{m-n}, f^n(x), \frac{m}{m-n}y, \frac{m}{m-n}z\right) & \text{if } m < 0. \end{cases}$$

This $\varphi$ is clearly injective and satisfies

$$\varphi(X) = X \setminus \bigcup_{0<|m|<n} Z_m.$$

Clearly, in order to prove the "if" part, it suffices to show that $\varphi$ is a homeomorphism onto its image. This is trivial because $\varphi$ is continuous and $X$ is compact.

To see the "only if" part, suppose that $\bigsqcup_N Z \sqcup X \approx X$.

Let $\varphi$ be a homeomorphism

$$\varphi\colon \bigsqcup_N Z \sqcup X \longrightarrow X.$$

Since homeomorphisms take each path component onto another one and $I_0 \approx I \not\approx Z$, each $Z_m$ is mapped onto some $Z_{m'}$, and $I_0$ onto $I_0$ by $\varphi$. Take $x \in I$ so that $\varphi^{-1}(0, 1/2, 0, 0) = (0, x, 0, 0)$. Then $x \notin \{0, 1\}$.

Define a sequence $\{p_j\}_{j\in\mathbb{N}}$ in $X$ as follows: $p_{2k} = (1/k, x, 0, 0)$ and $p_{2k-1} = (-1/k, x, 0, 0)$. Obviously, $\{p_j\}$ converges to $(0, x, 0, 0) \in I_0$. Because $\varphi$ is continuous, $\{\varphi(p_j)\}$ must converge to $\varphi(0, x, 0, 0) = (0, 1/2, 0, 0)$. Use Lemma 2.2 to observe that each $\mathrm{pr}_2(\varphi(p_j))$ must belong to the set $A = \{f^n(x)\,;\, n \in \mathbb{Z}\} \subset I$, where $\mathrm{pr}_2$ denotes the second projection $\mathbb{R}^4 \longrightarrow \mathbb{R}$. Since the closure of $A$ in $I$ is $A \cup \{0, 1\}$, the point $1/2$ belongs to $A \cup \{0, 1\}$, which means that $1/2 \in A$. Hence, there is an integer $n$ such that $1/2 = f^n(x)$. Since $1/2$ is an isolated point of $A$ and $\mathrm{pr}_2(\varphi(p_j)) \to 1/2$ as $j \to \infty$, $\mathrm{pr}_2(\varphi(p_j)) = 1/2$ for $j \geqq 2M - 1$, where $M$ is a sufficiently large number.

Take any integer $m$ such that $|m| \geqq M$. Then there exists a number $j$ such that $p_j \in Z_m$. This $j$ satisfies $j \geqq 2M - 1$ and $\mathrm{pr}_2(\varphi(p_j)) = 1/2$. Pick $m' \in \mathbb{Z}$ so that $\varphi(p_j)$ belongs to $Z_{m'}$. Then, $\varphi$ maps $Z_m$ onto $Z_{m'}$ homeomorphically. Notice that $\mathrm{pr}_2(p_j) = x$ and $\mathrm{pr}_2(\varphi(p_j)) = 1/2 = f^n(x)$. By Proposition 2.3, we have $|m'| - |m| = n$. This means that $\varphi(Z_m \cup Z_{-m}) = Z_{m+n} \cup Z_{-(m+n)}$ if $m \geqq M$.

Then, we have

$$\varphi\left(\bigcup_{|m|\geqq M} Z_m\right) = \bigcup_{|m|\geqq M+n} Z_m,$$

and hence,

$$(\sharp) \qquad \varphi\left(\bigsqcup_N Z \cup \bigcup_{0<|m|<M} Z_m\right) = \bigcup_{0<|m|<M+n} Z_m.$$

In ($\sharp$), since $\varphi$ is a homeomorphism, the numbers of path components within the bracket in the left-hand side and in the right-hand side must be the same. This means

$$N + 2M - 2 = 2M + 2n - 2,$$

that is, $N = 2n$. Therefore, $N$ is even. □

Letting $n = 1$ or $2$, we see from Proposition 2.4 that $X \sqcup Z \sqcup Z \approx X$, and that $X \sqcup Z \not\approx X$. This means that $X$ and $Z$ satisfy Property A. Hence, the spaces $X$ and $Y = X \sqcup Z$ give an example of the pair $X$, $Y$ in Theorem 1.1.

Finally we note that, by a slight modification, we have an example of compact metric spaces $X_n, Y_n\,(n \in \mathbb{N})$ such that $\bigsqcup_k X_n \not\approx \bigsqcup_k Y_n$ for $k < n$ but $\bigsqcup_n X_n \approx \bigsqcup_n Y_n$.

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