A COUNTEREXAMPLE RELATED TO TOPOLOGICAL SUMS

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Abstract. In this paper we construct compact metric spaces $X$, $Y$ which are topologically distinct but whose topological sums $X \sqcup X$ and $Y \sqcup Y$ are homeomorphic.

1. Introduction and the main result

For topological spaces $X$, $Y$, we denote by $X \sqcup Y$ the topological sum of $X$ and $Y$. Let $X$ and $Y$ be topological spaces and assume that $X \sqcup X$ and $Y \sqcup Y$ are homeomorphic (we denote this by $X \sqcup X \approx Y \sqcup Y$). Then, one may ask whether $X \approx Y$ holds.

If $X$ is a connected space, obviously, this is always true. It seems that even for “most” disconnected spaces $X$, $Y$ such that $X \sqcup X \approx Y \sqcup Y$, $X$ and $Y$ are homeomorphic. But this is not true in general; we will describe a counterexample.

Theorem 1.1. There exist compact metric spaces $X$, $Y$ which satisfy the following property: $X$ and $Y$ are nonhomeomorphic, but the topological sums $X \sqcup X$ and $Y \sqcup Y$ are homeomorphic. Moreover, such spaces $X$, $Y$ can be constructed as subspaces of the Euclidean plane $\mathbb{R}^2$.

In this paper, for notational convenience, we will construct an example of the pair $X$, $Y$ as subspaces of $\mathbb{R}^4$. However, it will be easily observed that one can construct $X$, $Y$ with the same property as (compact) subspaces of $\mathbb{R}^2$ in a similar way.

For $n \geq 0$, we denote by

\[\bigcup_{n} X\]

the topological sum of $n$ copies of $X$. If $n = 0$, this means the empty set $\emptyset$. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ be the set of all positive integers, all integers, all rational numbers and all real numbers, respectively. Let $I = [0, 1] = \{x \in \mathbb{R} ; 0 \leq x \leq 1\}$ be the closed interval between 0 and 1. For points $x$, $y$ in the Euclidean space $\mathbb{R}^n$, put

\[\langle x, y \rangle = \{(1 - t)x + ty ; t \in I\} \subset \mathbb{R}^n.\]
Figure 1. The space $Z$

2. Construction of the example

The main idea is to construct spaces $X$ and $Z$ which have the following property:

**Property A.** The spaces $X$ and $X \sqcup Z$ are nonhomeomorphic, but $X$ and $X \sqcup Y \sqcup Z$ are homeomorphic.

If we can construct such a pair of spaces $X$ and $Z$, the spaces $X$ and $Y = X \sqcup Z$ clearly form a counterexample. Now, the problem is to construct $X$ and $Z$ satisfying Property A.

The space $Z$ is constructed as follows. Arrange the countable set $I \cap \mathbb{Q}$ to get a sequence $a_1, a_2, a_3, \ldots$. Then let $Z$ be the closed interval $I$ with $n$ “legs” at each rational point $a_n$ (see Figure 1). Precisely, let $v_i = (1, 1/i) \in \mathbb{R}^2$ for $i \in \mathbb{N}$ and let

$$F_n = \bigcup_{i=1}^{n} \left\{ 0, \frac{1}{n}v_i \right\} \subset \mathbb{R}^2$$

for $n \in \mathbb{N}$. Then we put

$$Z = (I \times \{(0,0)\}) \cup \bigcup_{n \in \mathbb{N}} \left( \{a_n\} \times F_n \right) \subset \mathbb{R}^3.$$

Observe that every open cover of $Z$ has a finite subcover, whence $Z$ is compact.

The following lemma is easily seen by counting the path components of the complements of each point of $Z$ (especially, the “node” $(a_n, 0, 0)$ of $Z$ for each $n \in \mathbb{N}$) and observing that the set $\{a_n; n \in \mathbb{N}\} = I \cap \mathbb{Q}$ is dense in $I$.

**Lemma 2.1.** Every homeomorphism $\varphi: Z \rightarrow Z$ fixes the points of $I \times \{(0,0)\}$. □

The other space $X$ has countably many copies of $Z$ as subspaces. Precisely, we construct $X$ as follows. Let $f: I \rightarrow I$ be the homeomorphism defined by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{3x}{2} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

(See Figure 2.) Clearly, $f$ induces a homeomorphism from $I \cap \mathbb{Q}$ to itself. Let $f^0 = \text{id}_I$, the identity map of $I$, $f^1 = f$ and $f^n = f \circ f^{n-1}$ for integers $n \geq 2$. Define $f^{-n} = (f^n)^{-1}$ for each $n \in \mathbb{N}$. Each $f^n$ is a homeomorphism from $I$ to itself and induces a homeomorphism from $I \cap \mathbb{Q}$ to itself.

Let

$$X_m = (I \times \{(0,0)\}) \cup \bigcup_{n \in \mathbb{N}} \left( \{f^{m-1}(a_n)\} \times \frac{1}{m}F_n \right) \subset \mathbb{R}^3$$

for each $m \in \mathbb{N}$ and let

$$X = \{0\} \times I \times \{(0,0)\} \cup \bigcup_{m \in \mathbb{N} \setminus \{0\}} \left( \frac{1}{m} \times X_{|m|} \right) \subset \mathbb{R}^4.$$
Then, clearly, $I_0 = \{0\} \times I \times \{(0,0)\}$ and

$$Z_m = \left\{ \frac{1}{m} \right\} \times X_{|m|} \quad (m \in \mathbb{Z} \setminus \{0\})$$

are path components of $X$. We easily see that $X$ is closed in $\mathbb{R}^4$. Since $X$ is a bounded subset of $\mathbb{R}^4$, $X$ is compact (see Figure 3).

Note that the map $Z \to Z_m$ given by

$$(x, y, z) \mapsto \left( \frac{1}{m}, f^{[m]-1}(x), y/|m|, z/|m| \right)$$

is a homeomorphism for each $m \in \mathbb{Z} \setminus \{0\}$. By this fact and Lemma 2.1, we have

**Lemma 2.2.** If $m, m' \in \mathbb{Z} \setminus \{0\}$, every homeomorphism $Z_m \to Z_{m'}$ induces the homeomorphism $\{1/m\} \times I \times \{(0,0)\} \to \{1/m'\} \times I \times \{(0,0)\}$ given by

$$\left( \frac{1}{m}, x, 0, 0 \right) \mapsto \left( \frac{1}{m'}, f^{[m']-|m|}(x), 0, 0 \right). \quad \square$$

Observe that $\ldots, f^{-(2)}(x), f^{-1}(x), x, f^1(x), f^2(x), \ldots$ are distinct in $I$ for each $x \in I \setminus \{0,1\}$. The next proposition follows from this fact and Lemma 2.2.

**Proposition 2.3.** Assume that $m, m' \in \mathbb{Z} \setminus \{0\}$ and that $\psi: Z_m \to Z_{m'}$ is a homeomorphism. If there exists a point $x \in I \setminus \{0,1\}$ such that

$$\psi \left( \frac{1}{m}, x, 0, 0 \right) = \left( \frac{1}{m'}, f^n(x), 0, 0 \right),$$

then $|m'| - |m| = n$. \quad \square

Now, we can prove the following essential proposition.

**Proposition 2.4.** The space $X \sqcup \bigsqcup_{N} Z$ is homeomorphic to $X$ if and only if $N$ is even. In particular, $X$ and $Z$ satisfy Property A.
Proof. First, assume that $N = 2n$ is an even positive integer. Define a map $\varphi: X \rightarrow X$ by $\varphi(0, x, 0, 0) = (0, f^n(x), 0, 0)$ for $x \in I$ and

$$\varphi\left(\frac{1}{m}, x, y, z\right) = \begin{cases} \left(\frac{1}{m + n}, f^n(x), \frac{m}{m + n} y, \frac{m}{m + n} z\right) & \text{if } m > 0, \\ \left(\frac{1}{m - n}, f^n(x), \frac{m}{m - n} y, \frac{m}{m - n} z\right) & \text{if } m < 0. \end{cases}$$

This $\varphi$ is clearly injective and satisfies

$$\varphi(X) = X \setminus \bigcup_{0 < |m| < n} Z_m.$$ 

Clearly, in order to prove the “if” part, it suffices to show that $\varphi$ is a homeomorphism onto its image. This is trivial because $\varphi$ is continuous and $X$ is compact.

To see the “only if” part, suppose that $\bigcup_N Z \sqcup X \approx X$.

Let $\varphi$ be a homeomorphism

$$\varphi: \bigcup_N Z \sqcup X \rightarrow X.$$
Since homeomorphisms take each path component onto another one and \( I_0 \approx I \neq Z \), each \( Z_m \) is mapped onto some \( Z_{m'} \), and \( I_0 \) onto \( I_0 \) by \( \varphi \). Take \( x \in I \) so that \( \varphi^{-1}(0, 1/2, 0, 0) = (x, 0, 0, 0) \). Then \( x \notin \{0, 1\} \).

Define a sequence \( \{p_j\}_{j \in \mathbb{N}} \) in \( X \) as follows: \( p_{2k} = (1/k, x, 0, 0) \) and \( p_{2k-1} = (-1/k, x, 0, 0) \). Obviously, \( \{p_j\} \) converges to \((0, x, 0, 0) \in I_0 \). Because \( \varphi \) is continuous, \( \{\varphi(p_j)\} \) must converge to \( \varphi(0, x, 0, 0) = (0, 1/2, 0, 0) \). Use Lemma 2.2 to observe that each \( \text{pr}_2(\varphi(p_j)) \) must belong to the set \( A = \{ f^n(x) ; n \in \mathbb{Z} \} \subset I \), where \( \text{pr}_2 \) denotes the second projection \( \mathbb{R}^3 \to \mathbb{R} \). Since the closure of \( A \) in \( I \) is \( A \cup \{0, 1\} \), the point \( 1/2 \) belongs to \( A \cup \{0, 1\} \), which means that \( 1/2 \in A \). Hence, there is an integer \( n \) such that \( 1/2 = f^n(x) \). Since \( 1/2 \) is an isolated point of \( A \) and \( \text{pr}_2(\varphi(p_j)) \to 1/2 \) as \( j \to \infty \), \( \text{pr}_2(\varphi(p_j)) = 1/2 \) for \( j \geq 2M - 1 \), where \( M \) is a sufficiently large number.

Take any integer \( m \) such that \(|m| \geq M\). Then there exists a number \( j \) such that \( p_j \in Z_m \). This \( j \) satisfies \( j \geq 2M - 1 \) and \( \text{pr}_2(\varphi(p_j)) = 1/2 \). Pick \( m' \in \mathbb{Z} \) so that \( \varphi(p_j) \in Z_{m'} \). Then, \( \varphi \) maps \( Z_m \) onto \( Z_{m'} \) homeomorphically. Notice that \( \text{pr}_2(p_j) = x \) and \( \text{pr}_2(\varphi(p_j)) = 1/2 = f^n(x) \). By Proposition 2.3 we have \( |m'| \neq |m| \). This means that \( \varphi(Z_m \cup Z_{-m}) = Z_{m+n} \cup Z_{-(m+n)} \) if \( m \geq M \).

Then, we have
\[
\varphi\left(\bigcup_{|m| \geq M} Z_m\right) = \bigcup_{|m| \geq M+n} Z_m,
\]
and hence,
\[
(\sharp) \quad \varphi\left(\bigcup_{N} Z \cup \bigcup_{0<|m|<M} Z_m\right) = \bigcup_{0<|m|<M+n} Z_m.
\]

In (\(\sharp\)), since \( \varphi \) is a homeomorphism, the numbers of path components within the bracket in the left-hand side and in the right-hand side must be the same. This means
\[
N + 2M - 2 = 2M + 2n - 2,
\]
that is, \( N = 2n \). Therefore, \( N \) is even.

Letting \( n = 1 \) or \( 2 \), we see from Proposition 2.4 that \( X \sqcup Z \sqcup Z \cong X \), and that \( X \sqcup Z \not\cong X \). This means that \( X \) and \( Z \) satisfy Property A. Hence, the spaces \( X \) and \( Y = X \sqcup Z \) give an example of the pair \( X, Y \) in Theorem 1.1.

Finally we note that, by a slight modification, we have an example of compact metric spaces \( X_n, Y_n \) such that \( \bigsqcup_k X_n \not\cong \bigsqcup_k Y_n \) for \( k < n \) but \( \bigsqcup_n X_n \cong \bigsqcup_n Y_n \).

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