

GIBBS' PHENOMENON AND SURFACE AREA

L. DE MICHELE AND D. ROUX

(Communicated by Michael T. Lacey)

ABSTRACT. If a function f is of bounded variation on T^N ($N \geq 1$) and $\{\varphi_n\}$ is a positive approximate identity, we prove that the area of the graph of $f * \varphi_n$ converges from below to the relaxed area of the graph of f . Moreover we give asymptotic estimates for the area of the graph of the square partial sums of multiple Fourier series of functions with suitable discontinuities.

1. INTRODUCTION

Let T be the one-dimensional torus. In a recent paper [5], R. S. Strichartz discussed the asymptotic behaviour of the length l of the graph of the convolution product $f * \varphi_n$, where $f \in L^1(T)$ and $\{\varphi_n\}$ is a sequence of kernels.

For partial sums $s_n(f)$ of Fourier series he proved that if f is a continuous piecewise \mathcal{C}^1 function

$$\lim_{n \rightarrow +\infty} l(s_n(f)) = l(f)$$

and if f is a piecewise \mathcal{C}^1 function with a finite number of jump discontinuities,

$$l(s_n(f)) = \mathcal{O}(\log n).$$

Moreover, if $\{\varphi_n\}$ is a positive approximate identity in L^1 and f is a function of bounded variation, Strichartz proved that the length of the graph of $f * \varphi_n$ converges from below to the length of the graph of f , defined as the sum of the length of the graph of the continuous part of f and the sum of the essential jumps of f .

In this paper, we study the same problems on the N -dimensional torus (for $N \geq 1$).

Although summation methods are very relevant for convergence problems, it is easy to see that (Proposition 1) if f is a Lipschitz function, then for every summation method the area \mathcal{A} of the graph of the partial sums of the Fourier series of f converges to the area $\mathcal{A}(f)$ of the graph of f , as in the one-dimensional case.

On the other hand, if f is the characteristic function of a set E in R^n with sufficiently smooth boundary and \mathcal{D}_n^N is the N -dimensional Dirichlet kernel, we prove that, as in the one-dimensional case,

$$\mathcal{A}(\chi_E * \mathcal{D}_n^N) = \mathcal{O}(\log n).$$

Finally we prove that if $\{\varphi_n\}$ is a positive approximate identity and f is a function of bounded variation (see e.g. [1]), the Lebesgue area $A(f * \varphi_n)$ of the

Received by the editors June 21, 2005.

2000 *Mathematics Subject Classification*. Primary 42B99.

Key words and phrases. Gibbs phenomenon, Fourier series, approximate identity.

©2006 American Mathematical Society
Reverts to public domain 28 years from publication

graph of the continuous function $f * \varphi_n$ converges from below to the relaxed area (see e.g. [3]) $\overline{A}(f)$ of the graph of f .

Moreover if f is a nonbounded variation L^1 -function, then $\overline{A}(f * \varphi_n)$ diverges as $n \rightarrow +\infty$. Therefore f is of bounded variation if and only if $\overline{A}(f * \varphi_n)$ is uniformly bounded.

These results extend some old results in the 2-dimensional case due to C. Goffman [4] to the case of a general positive approximate identity. As far as we know, no other result was proved in the N -dimensional case.

The techniques in the proof of Goffman are strictly connected to the particular form of φ_n and to the various notions of area used in the fifties for L^1 functions. These techniques do not seem to work in the general case.

2. THE RESULTS

Let $x = (x_1, \dots, x_N) \in T^N$ (the N -dimensional torus) and let $\{I_n\}$ be a sequence of finite subset of Z^N such that $I_n \subset I_{n+1}$ and $\bigcup_n I_n = Z^N$. Set

$$\mathcal{D}_{I_n}(x) = \sum_{k \in I_n} e^{ikt}, \quad t \in T^N, \quad k = (k_1, \dots, k_N) \text{ and } kt = \sum_{i=1}^N k_i t_i.$$

If $f : T^N \mapsto R$ is a Lipschitz function, we denote by $\mathcal{A}(f)$ the area of the graph of f (see e.g. [2]). We have the following proposition.

Proposition 1. *If $f : T^N \mapsto R$ is a Lipschitz function, then*

$$(2.1) \quad \lim_{n \rightarrow +\infty} \mathcal{A}(f * \mathcal{D}_{I_n}) = \mathcal{A}(f).$$

If f has some jump discontinuities we have the Gibbs' phenomenon; the oscillations of $f * \mathcal{D}_{I_n}$ are uniformly bounded, but this is no longer true for the area of the graph.

Indeed, let

$$E_1 = [a, b), \quad 0 \leq a < b \leq 2\pi,$$

and for $i = 2, \dots, n$

$$E_i = \left\{ (x_1, \dots, x_i) : (x_1, \dots, x_{i-1}) \in E_{i-1}, \right. \\ \left. g_{i-1}(x_1, \dots, x_{i-1}) \leq x_i < h_{i-1}(x_1, \dots, x_{i-1}) \right\},$$

where $g_i, h_i : E_{i-1} \mapsto [0, 2\pi]$ belong to $C^{(1)}(E_{i-1})$, and $g_i(x_1, \dots, x_{s-1}, \bullet, x_{s+1}, \dots, x_{i-1})$, $h_i(x_1, \dots, x_{s-1}, \bullet, x_{s+1}, \dots, x_{i-1})$ are monotone functions with respect to x_s for $s = 1, 2, \dots, i - 1$. Moreover, for convenience, let us set $E_n = E$. Then, if

$$\mathcal{D}_n^N(x) = \mathcal{D}_n^N(x_1, x_2, \dots, x_N) = \prod_{j=1}^N \mathcal{D}_n^1(x_j) = \prod_{j=1}^N \sum_{p=-n}^n e^{ipx_j},$$

the following theorem holds true.

Theorem 1. *If χ_E is the characteristic function of E , then*

$$(2.2) \quad \mathcal{A}(\chi_E * \mathcal{D}_n^N) = \mathcal{O}(\log n).$$

Remark. This estimate depends heavily on the summation method used. Indeed from the proof of the theorem it is easy to see that, if for example $N = 2$ and $\tilde{\mathcal{D}}_n^2(x_1, x_2) = \mathcal{D}_n(x_1)\mathcal{D}_{e^n}(x_2)$, we have

$$\mathcal{A}(\chi_E * \tilde{\mathcal{D}}_n^2) = \mathcal{O}(n).$$

For convenience, we recall that $\{\varphi_n\}_{n=1}^\infty$ is a positive approximate identity in $L^1(T^N)$ if for every n

$$\varphi_n(x) \geq 0 \text{ a.e., } \int_{T^N} \varphi_n(x) \, dx = 1,$$

and moreover for every open set E containing the point zero

$$\lim_{n \rightarrow \infty} \int_{E^c} \varphi_n(x) \, dx = 0.$$

We also recall that $f \in L^1(T^N)$ is said to be of *bounded variation in T^N* if the distributional derivative $\nabla f = (D_1 f, \dots, D_N f)$ is represented by a finite Radon measure in T^N , i.e. if

$$\int_{T^N} f \frac{\partial \phi}{\partial x_i} \, dx = - \int_{T^N} \phi \, dD_i f \quad \forall \phi \in C^\infty(T^N), \quad i = 1, \dots, N,$$

and the relaxed area of f is given by

$$\overline{A}(f) = \int_{T^N} \left(1 + |\nabla_a f|^2\right)^{1/2} \, dx + \|\nabla_s f\|_{M(T^N)}$$

where ∇_a and ∇_s are, respectively, the absolutely continuous and the singular part of the distributional derivative of f and $\|\cdot\|_{M(T^N)}$ is the norm of the vector-valued finite measures on T^N .

Then we have the following result.

Theorem 2. *If $f : T^N \rightarrow R$ is a bounded variation function and $\{\varphi_n\}$ is a positive approximate unit, then for every n*

$$(2.3) \quad A(f * \varphi_n) \leq \overline{A}(f)$$

and

$$(2.4) \quad \lim_{n \rightarrow \infty} A(f * \varphi_n) = \overline{A}(f).$$

If $f \in L^1(T^N)$ is not bounded variation we have the following.

Proposition 2. *If $f \in L^1(T^N)$ is not bounded variation, then for every subsequence $\{f * \varphi_{n_h}\}$ of bounded variation functions we have*

$$(2.5) \quad \lim_{h \rightarrow \infty} \overline{A}(f * \varphi_{n_h}) = +\infty.$$

Indeed, the functional relaxed area is lower semicontinuous in L^1 and $\overline{A}(f) = +\infty$.

3. PROOFS

Proof of Proposition 1. We have

$$|\mathcal{A}(f * \mathcal{D}_{I_n}) - \mathcal{A}(f)| = \left| \int_{T^N} \left(1 + |\nabla(f * \mathcal{D}_{I_n})|^2\right)^{1/2} - \left(1 + |\nabla f|^2\right)^{1/2} \, dx \right|,$$

where $|\cdot|$ is the euclidean norm. Since $y \mapsto (1 + y^2)^{1/2}$ is a Lipschitz function

$$\begin{aligned} |\mathcal{A}(f * \mathcal{D}_{I_n}) - \mathcal{A}(f)| &\leq \int_{T^N} |\nabla(f * \mathcal{D}_{I_n}) - \nabla f| \, dx \\ &= \|\nabla(f * \mathcal{D}_{I_n}) - \nabla f\|_1 \leq (2\pi)^N \|\nabla(f * \mathcal{D}_{I_n}) - \nabla f\|_2 \\ &= (2\pi)^N \|\nabla f * \mathcal{D}_{I_n} - \nabla f\|_2, \end{aligned}$$

where $\| \cdot \|_1$ and $\| \cdot \|_2$ are respectively the L^1 and the L^2 norm of vector-valued functions. Because the last term by Plancherel goes to zero, Proposition 1 follows. \square

Proof of Theorem 1. We have

$$\begin{aligned} H^{(n)}(x_1, \dots, x_N) &= \chi_E * \mathcal{D}_n^N(x) \\ &= \int_E \mathcal{D}_n(x_1 - u_1) \cdots \mathcal{D}_n(x_N - u_N) du_1 \dots du_N \\ &= \int_a^b \mathcal{D}_n(x_1 - u_1) du_1 \cdot \int_{g_1(x_1)}^{h_1(x_1)} \mathcal{D}_n(x_2 - u_2) du_2 \\ &\quad \cdots \int_{g_{i-1}(x_1, \dots, x_{i-1})}^{h_{i-1}(x_1, \dots, x_{i-1})} \mathcal{D}_n(x_i - u_i) du_i \cdots \int_{g_{N-1}(x_1, \dots, x_{N-1})}^{h_{N-1}(x_1, \dots, x_{N-1})} \mathcal{D}_n(x_N - u_N) du_N \\ &= H_1^{(n)}(x_1) \cdot H_2^{(n)}(x_1, x_2) \cdots H_i^{(n)}(x_1, \dots, x_i) \cdots H_N^{(n)}(x_1, \dots, x_N). \end{aligned}$$

Then

$$(3.1) \quad \frac{\partial}{\partial x_i} H^{(n)} = \sum_{k=i}^N \frac{\partial}{\partial x_i} H_k^{(n)} \prod_{j=1, j \neq k}^N H_j^{(n)}.$$

Since for every $c, d, 0 \leq c < d \leq 2\pi$,

$$\left| \int_c^d \mathcal{D}_n(y) dy \right| < K$$

uniformly with respect to c, d and n , we have

$$\left\| \frac{\partial}{\partial x_i} H^{(n)} \right\|_1 = \mathcal{O} \left(\left\| \sum_{k=1}^N \left| \frac{\partial}{\partial x_i} H_k^{(n)} \right| \right\|_1 \right).$$

Now we give an estimate of $\frac{\partial}{\partial x_i} H_k^{(n)}$ ($i \leq k$). We have

$$\begin{aligned} (3.2) \quad \frac{\partial}{\partial x_i} H_k^{(n)}(x_1, \dots, x_k) &= \mathcal{D}_n(x_k - h_{k-1}(x_1, \dots, x_{k-1})) \cdot \frac{\partial}{\partial x_i} h_{k-1}(x_1, \dots, x_{k-1}) \\ &\quad - \mathcal{D}_n(x_k - g_{k-1}(x_1, \dots, x_{k-1})) \cdot \frac{\partial}{\partial x_i} g_{k-1}(x_1, \dots, x_{k-1}) \\ &\quad - \delta_{i,k} \{ \mathcal{D}_n(x_k - h_{k-1}(x_1, \dots, x_{k-1})) - \mathcal{D}_n(x_k - g_{k-1}(x_1, \dots, x_{k-1})) \}, \end{aligned}$$

where $\delta_{i,k}$ is the Kronecker symbol.

The L^1 norm of every term of (3.2) is $\mathcal{O}(\log n)$. Indeed

$$\begin{aligned} &\int_{T^N} |\mathcal{D}_n(x_k - h_{k-1}(x_1, \dots, x_{k-1}))| \cdot \left| \frac{\partial}{\partial x_i} h_{k-1}(x_1, \dots, x_{k-1}) \right| dx_1 \dots dx_N \\ &\leq \int_{T^N} |\mathcal{D}_n(t_k - t_i)| dt_1 \dots dt_N \\ &= \int_{T^{N-1}} dt_1 \dots dt_{k-1} \cdot dt_{k+1} \dots dt_N \int_T |\mathcal{D}_n(t_k - t_i)| dt_k \\ &= \mathcal{O}(\log n). \end{aligned}$$

Trivially a similar estimate holds for the last term in (3.2). Therefore, by (3.1) and (3.2) we obtain

$$\left\| \frac{\partial}{\partial x_i} H^{(n)}(x_1, \dots, x_N) \right\|_1 = \mathcal{O}(\log n)$$

for every $i = 1, \dots, N$. Since

$$\mathcal{A}(\chi_E * \mathcal{D}_n^N) = \int_{T^N} \left\{ 1 + |\nabla(f * \mathcal{D}_n^N)|^2 \right\}^{1/2} dx_1 \dots dx_N,$$

Theorem 1 easily follows. □

Proof of Theorem 2. Since f is bounded and the translation in L^1 is continuous, we have

$$f * \varphi_n \in \mathcal{C}^{(0)}(T^N).$$

Moreover, the distributional derivative $\nabla(f * \varphi_n)$ is a vector-valued L^1 -function, because $\nabla(f * \varphi_n) = \nabla f * \varphi_n$ and ∇f is a vector-valued finite measure.

Then $f * \varphi_n \in W^{1,1}$ and its graph has finite Lebesgue area given by

$$A(f * \varphi_n) = \int_{T^N} \left(1 + |\nabla f * \varphi_n|^2 \right)^{1/2} dx,$$

where $|\cdot|$ is the euclidean norm in R^N .

We have

$$A(f * \varphi_n) = \int_{T^N} \left(1 + |(\nabla_a f + \nabla_s f) * \varphi_n|^2 \right)^{1/2} dx.$$

From the trivial inequality

$$(1 + (a + b)^2)^{1/2} \leq (1 + a^2)^{1/2} + |b|, \quad a, b \in R,$$

we have

$$\begin{aligned} A(f * \varphi_n) &\leq \int_{T^N} (1 + |\nabla_a f * \varphi_n|^2)^{1/2} dx + \int_{T^N} |\nabla_s f * \varphi_n| dx \\ (3.3) \quad &\leq \int_{T^N} (1 + |\nabla_a f * \varphi_n|^2)^{1/2} dx + \|\nabla_s f\|_{M(T^N)} \\ &= I + \|\nabla_s f\|_{M(T^N)}. \end{aligned}$$

We now prove that

$$(3.4) \quad I \leq \int_{T^N} (1 + |\nabla_a f|^2)^{1/2} dx.$$

Minkowski's inequality implies that

$$I \leq \int_{T^N} (1 + (|\nabla_a f| * \varphi_n)^2)^{1/2} dx.$$

Moreover, by the Jensen inequality we have a.e.

$$(3.5) \quad (1 + (|\nabla_a f| * \varphi_n)^2)^{1/2}(x) \leq \int_{T^N} (1 + |\nabla_a f(t - x)|^2)^{1/2} \varphi_n(t) dt.$$

By integration of both terms of (3.5) and using Fubini's theorem, we obtain

$$\begin{aligned} I &\leq \int_{T^N} dt \int_{T^N} (1 + |\nabla_a f(t-x)|^2)^{1/2} \varphi_n(t) dx \\ &\leq \int_{T^N} \varphi_n(t) dt \int_{T^N} (1 + |\nabla_a f(x)|^2)^{1/2} dx \\ &= \int_{T^N} (1 + |\nabla_a f|^2)^{1/2} dx \end{aligned}$$

and we have (3.4). Then (2.3) follows by (3.3) and (3.4).

Since the functional relaxed area is lower semicontinuous in L^1 , (2.4) follows immediately by (2.3). \square

REFERENCES

- [1] L. Ambrosio, N. Fusco, D. Pallara: Functions of bounded variation and free discontinuity problems, Clarendon Press, Oxford (2000). MR1857292 (2003a:49002)
- [2] L. C. Evans, R. F. Gariepy: Measure theory and fine property of functions, CRC Press Inc., Boca Raton (1992). MR1158660 (93f:28001)
- [3] M. Giaquinta, G. Modica, J. Souček: Cartesian currents in the calculus of variations II, Springer-Verlag, Berlin (1998). MR1645082 (2000b:49001b)
- [4] C. Goffman: Convergence in area of integrals means, Amer. J. Math. 77 (1955) 563-574. MR0070698 (17:22b)
- [5] R. S. Strichartz: Gibbs' phenomenon and arclength, J. Fourier Anal. Appl. 6 (2000) 533-536. MR1781092 (2002f:42005)

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA,
VIA R. COZZI 53, 20126 MILANO, ITALIA

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA,
VIA R. COZZI 53, 20126 MILANO, ITALIA