GIBBS’ PHENOMENON AND SURFACE AREA

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Abstract. If a function $f$ is of bounded variation on $T^N$ ($N \geq 1$) and $\{\varphi_n\}$ is a positive approximate identity, we prove that the area of the graph of $f \ast \varphi_n$ converges from below to the relaxed area of the graph of $f$. Moreover we give asymptotic estimates for the area of the graph of the square partial sums of multiple Fourier series of functions with suitable discontinuities.

1. Introduction

Let $T$ be the one-dimensional torus. In a recent paper \cite{5}, R. S. Strichartz discussed the asymptotic behaviour of the length $l$ of the graph of the convolution product $f \ast \varphi_n$, where $f \in L^1(T)$ and $\{\varphi_n\}$ is a sequence of kernels.

For partial sums $s_n(f)$ of Fourier series he proved that if $f$ is a continuous piecewise $C^1$ function

$$\lim_{n \to +\infty} l(s_n(f)) = l(f)$$

and if $f$ is a piecewise $C^1$ function with a finite number of jump discontinuities,

$$l(s_n(f)) = O(\log n).$$

Moreover, if $\{\varphi_n\}$ is a positive approximate identity in $L^1$ and $f$ is a function of bounded variation, Strichartz proved that the length of the graph of $f \ast \varphi_n$ converges from below to the length of the graph of $f$, defined as the sum of the length of the graph of the continuous part of $f$ and the sum of the essential jumps of $f$.

In this paper, we study the same problems on the $N$-dimensional torus (for $N \geq 1$).

Although summation methods are very relevant for convergence problems, it is easy to see that (Proposition 1) if $f$ is a Lipschitz function, then for every summation method the area $A$ of the graph of the partial sums of the Fourier series of $f$ converges to the area $A(f)$ of the graph of $f$, as in the one-dimensional case.

On the other hand, if $f$ is the characteristic function of a set $E$ in $R^n$ with sufficiently smooth boundary and $D^N_n$ is the $N$-dimensional Dirichlet kernel, we prove that, as in the one-dimensional case,

$$A(\chi_E \ast D^N_n) = O(\log n).$$

Finally we prove that if $\{\varphi_n\}$ is a positive approximate identity and $f$ is a function of bounded variation (see e.g. \cite{1}), the Lebesgue area $A(f \ast \varphi_n)$ of the
graph of the continuous function $f \ast \varphi_n$ converges from below to the relaxed area (see e.g. [3]) $\overline{A}(f)$ of the graph of $f$.

Moreover if $f$ is a nonbounded variation $L^1$-function, then $\overline{A}(f \ast \varphi_n)$ diverges as $n \to +\infty$. Therefore $f$ is of bounded variation if and only if $\overline{A}(f \ast \varphi_n)$ is uniformly bounded.

These results extend some old results in the 2-dimensional case due to C. Goffman [4] to the case of a general positive approximate identity. As far as we know, no other result was proved in the $N$-dimensional case.

The techniques in the proof of Goffman are strictly connected to the particular form of $\varphi_n$ and to the various notions of area used in the fifties for $L^1$ functions. These techniques do not seem to work in the general case.

2. The Results

Let $x = (x_1, \ldots, x_N) \in T^N$ (the $N$-dimensional torus) and let $\{I_n\}$ be a sequence of finite subset of $Z^N$ such that $I_n \subseteq I_{n+1}$ and $\bigcup_n I_n = \mathbb{Z}^N$. Set

$$D_{I_n}(x) = \sum_{k \in I_n} e^{ikt}, \quad t \in \mathbb{T}^N, \ k = (k_1, \ldots, k_N) \text{ and } kt = \sum_{i=1}^N k_it_i.$$  

If $f : \mathbb{T}^N \to \mathbb{R}$ is a Lipschitz function, we denote by $A(f)$ the area of the graph of $f$ (see e.g. [2]). We have the following proposition.

**Proposition 1.** If $f : \mathbb{T}^N \to \mathbb{R}$ is a Lipschitz function, then

$$\lim_{n \to +\infty} A(f \ast D_{I_n}) = A(f). \quad (2.1)$$

If $f$ has some jump discontinuities we have the Gibbs’ phenomenon; the oscillations of $f \ast D_{I_n}$ are uniformly bounded, but this is no longer true for the area of the graph.

Indeed, let

$$E_1 = [a, b], \quad 0 \leq a < b \leq 2\pi,$$

and for $i = 2, \ldots, n$

$$E_i = \{(x_1, \ldots, x_i) : (x_1, \ldots, x_{i-1}) \in E_{i-1}, \quad g_{i-1}(x_1, \ldots, x_{i-1}) \leq x_i < h_{i-1}(x_1, \ldots, x_{i-1})\},$$

where $g_i, h_i : E_{i-1} \to [0, 2\pi]$ belong to $C^1(E_{i-1})$, and $g_i(x_1, \ldots, x_{s-1}, \bullet, x_{s+1}, \ldots, x_{i-1}), h_i(x_1, \ldots, x_{s-1}, \bullet, x_{s+1}, \ldots, x_{i-1})$ are monotone functions with respect to $x_s$ for $s = 1, 2, \ldots, i-1$. Moreover, for convenience, let us set $E_n = E$. Then, if

$$D_n^N(x) = D_n^N(x_1, x_2, \ldots, x_N) = \prod_{j=1}^N D_n^1(x_j) = \prod_{j=1}^N \prod_{p=-n}^n e^{ipx_j},$$

the following theorem holds true.

**Theorem 1.** If $\chi_E$ is the characteristic function of $E$, then

$$A(\chi_E \ast D_n^N) = O(\log n). \quad (2.2)$$

**Remark.** This estimate depends heavily on the summation method used. Indeed from the proof of the theorem it is easy to see that, if for example $N = 2$ and $D_2(x_1, x_2) = D_2(x_1)D_2(x_2)$, we have

$$A(\chi_E \ast D_n^2) = O(n).$$
For convenience, we recall that \( \{\varphi_n\}_{n=1}^{\infty} \) is a positive approximate identity in \( L^1(T^N) \) if for every \( n \)
\[
\varphi_n(x) \geq 0 \text{ a.e., } \quad \int_{T^N} \varphi_n(x) \, dx = 1,
\]
and moreover for every open set \( E \) containing the point zero
\[
\lim_{n \to \infty} \int_E \varphi_n(x) \, dx = 0.
\]
We also recall that \( f \in L^1(T^N) \) is said to be of bounded variation in \( T^N \) if the distributional derivative \( \nabla f = (D_1 f, \ldots, D_N f) \) is represented by a finite Radon measure in \( T^N \), i.e. if
\[
\int_{T^N} \frac{\partial \phi}{\partial x_i} \, dx = -\int_{T^N} \phi \, dD_i f \quad \forall \phi \in C^\infty(T^N), \quad i = 1, \ldots, N,
\]
and the relaxed area of \( f \) is given by
\[
\bar{A}(f) = \int_{T^N} \left( 1 + |\nabla_a f|^2 \right)^{1/2} \, dx + \|\nabla_s f\|_{M(T^N)}
\]
where \( \nabla_a \) and \( \nabla_s \) are, respectively, the absolutely continuous and the singular part of the distributional derivative of \( f \) and \( \|\cdot\|_{M(T^N)} \) is the norm of the vector-valued finite measures on \( T^N \).

Then we have the following result.

**Theorem 2.** If \( f : T^N \to \mathbb{R} \) is a bounded variation function and \( \{\varphi_n\} \) is a positive approximate unit, then for every \( n \)
\[
A(f * \varphi_n) \leq \bar{A}(f) \tag{2.3}
\]
and
\[
\lim_{n \to \infty} A(f * \varphi_n) = \bar{A}(f). \tag{2.4}
\]

If \( f \in L^1(T^N) \) is not bounded variation we have the following.

**Proposition 2.** If \( f \in L^1(T^N) \) is not bounded variation, then for every subsequence \( \{f * \varphi_{n_h}\} \) of bounded variation functions we have
\[
\lim_{h \to \infty} \bar{A}(f * \varphi_{n_h}) = +\infty. \tag{2.5}
\]

Indeed, the functional relaxed area is lower semicontinuous in \( L^1 \) and \( \bar{A}(f) = +\infty \).

### 3. Proofs

**Proof of Proposition 1.** We have
\[
|A(f * D_{I_n}) - A(f)| = \left| \int_{T^N} \left( 1 + |\nabla(f * D_{I_n})|^2 \right)^{1/2} - \left( 1 + |\nabla f|^2 \right)^{1/2} \, dx \right|
\]
where \( |\cdot| \) is the euclidean norm. Since \( y \mapsto (1 + y^2)^{1/2} \) is a Lipschitz function
\[
|A(f * D_{I_n}) - A(f)| \leq \int_{T^N} |\nabla(f * D_{I_n}) - \nabla f| \, dx
= \|\nabla(f * D_{I_n}) - \nabla f\|_1 \leq (2\pi)^N \|\nabla(f * D_{I_n}) - \nabla f\|_2
= (2\pi)^N \|\nabla f * D_{I_n} - \nabla f\|_2,
\]

where \( \| \|_1 \) and \( \| \|_2 \) are respectively the \( L^1 \) and the \( L^2 \) norm of vector-valued functions. Because the last term by Plancherel goes to zero, Proposition 1 follows.

**Proof of Theorem 1.** We have

\[
H^{(n)}(x_1, \ldots, x_N) = \chi_E * D_n^N(x)
\]

\[
= \int_E D_n(x_1 - u_1) \cdot \ldots \cdot D_n(x_N - u_N) du_1 \ldots du_N
\]

\[
= \int_a^b D_n(x_1 - u_1) du_1 \cdot \int_{g_1(x_1)}^{h_1(x_1)} D_n(x_2 - u_2) du_2
\]

\[
\ldots \cdot \int_{g_{i-1}(x_1, \ldots, x_{i-1})}^{h_{i-1}(x_1, \ldots, x_{i-1})} D_n(x_i - u_i) du_i \ldots \int_{g_{N-1}(x_1, \ldots, x_{N-1})}^{h_{N-1}(x_1, \ldots, x_{N-1})} D_n(x_N - u_N) du_N
\]

\[
= H_1^{(n)}(x_1) \cdot H_2^{(n)}(x_1, x_2) \ldots \cdot H_i^{(n)}(x_1, \ldots, x_i) \ldots \cdot H_N^{(n)}(x_1, \ldots, x_N). 
\]

Then

\[
\frac{\partial}{\partial x_i} H^{(n)} = \sum_{k=1}^N \frac{\partial}{\partial x_i} H_k^{(n)} \prod_{j=1, j \neq k}^N H_j^{(n)}.
\]

Since for every \( c, d, 0 \leq c < d \leq 2\pi \),

\[
\left| \int_c^d D_n(y) dy \right| < K
\]

uniformly with respect to \( c, d \) and \( n \), we have

\[
\left\| \frac{\partial}{\partial x_i} H^{(n)} \right\|_1 = O \left( \left\| \sum_{k=1}^N \frac{\partial}{\partial x_i} H_k^{(n)} \right\|_1 \right).
\]

Now we give an estimate of \( \frac{\partial}{\partial x_i} H_k^{(n)} (i \leq k) \). We have

\[
\frac{\partial}{\partial x_i} H_k^{(n)}(x_1, \ldots, x_k) = D_n(x_k - h_{k-1}(x_1, \ldots, x_{k-1})) \cdot \frac{\partial}{\partial x_i} h_{k-1}(x_1, \ldots, x_{k-1})
\]

\[
- D_n(x_k - g_{k-1}(x_1, \ldots, x_{k-1})) \cdot \frac{\partial}{\partial x_i} g_{k-1}(x_1, \ldots, x_{k-1})
\]

\[
- \delta_{i,k} \{ D_n(x_k - h_{k-1}(x_1, \ldots, x_{k-1})) - D_n(x_k - g_{k-1}(x_1, \ldots, x_{k-1})) \},
\]

where \( \delta_{i,k} \) is the Kronecker symbol.

The \( L^1 \) norm of every term of (3.2) is \( O(\log n) \). Indeed

\[
\int_{T^N} |D_n(x_k - h_{k-1}(x_1, \ldots, x_{k-1}))| \cdot \left| \frac{\partial}{\partial x_i} h_{k-1}(x_1, \ldots, x_{k-1}) \right| dx_1 \ldots dx_N
\]

\[
\leq \int_{T^N} |D_n(t_k - t_i)| dt_1 \ldots dt_N
\]

\[
= \int_{T^{N-1}} dt_1 \ldots dt_{k-1} \cdot dt_{k+1} \ldots dt_N \int_T |D_n(t_k - t_i)| dt_k
\]

\[
= O(\log n).
\]
Trivially a similar estimate holds for the last term in (3.2). Therefore, by (3.1) and (3.2) we obtain
\[
\left\| \frac{\partial}{\partial x_i} H^{(n)}(x_1, \ldots, x_N) \right\|_1 = O(\log n)
\]
for every \(i = 1, \ldots, N\). Since
\[
A(\chi_{\mathcal{E}} \ast D_n^N) = \int_{T^N} \left\{ 1 + |\nabla (f \ast D_n^N)|^2 \right\}^{1/2} \, dx_1 \ldots dx_N,
\]
Theorem 1 easily follows. \(\square\)

**Proof of Theorem 2.** Since \(f\) is bounded and the translation in \(L^1\) is continuous, we have
\[
f \ast \varphi_n \in C^0(T^N).
\]
Moreover, the distributional derivative \(\nabla (f \ast \varphi_n)\) is a vector-valued \(L^1\)-function, because \(\nabla (f \ast \varphi_n) = \nabla f \ast \varphi_n\) and \(\nabla f\) is a vector-valued finite measure.

Then \(f \ast \varphi_n \in W^{1,1}\) and its graph has finite Lebesgue area given by
\[
A(f \ast \varphi_n) = \int_{T^N} \left( 1 + |\nabla f \ast \varphi_n|^2 \right)^{1/2} \, dx,
\]
where \(\cdot\) is the euclidean norm in \(\mathbb{R}^N\).

We have
\[
A(f \ast \varphi_n) = \int_{T^N} \left( 1 + |(\nabla_a f + \nabla_s f) \ast \varphi_n|^2 \right)^{1/2} \, dx.
\]
From the trivial inequality
\[
(1 + (a + b)^2)^{1/2} \leq (1 + a^2)^{1/2} + |b|, \quad a, b \in \mathbb{R},
\]
we have
\[
A(f \ast \varphi_n) \leq \int_{T^N} \left( 1 + |\nabla_a f \ast \varphi_n|^2 \right)^{1/2} \, dx + \int_{T^N} |\nabla_s f \ast \varphi_n| \, dx
\]
\[
\leq \int_{T^N} \left( 1 + |\nabla_a f \ast \varphi_n|^2 \right)^{1/2} \, dx + \|\nabla_s f\|_{M(T^N)}
\]
\[
= I + \|\nabla_s f\|_{M(T^N)}.
\]

We now prove that
\[
I \leq \int_{T^N} \left( 1 + |\nabla_a f|^2 \right)^{1/2} \, dx.
\]
Minkowski’s inequality implies that
\[
I \leq \int_{T^N} \left( 1 + (|\nabla_a f| \ast \varphi_n)^2 \right)^{1/2} \, dx.
\]
Moreover, by the Jensen inequality we have a.e.
\[
(1 + (|\nabla_a f| \ast \varphi_n)^2)^{1/2}(x) \leq \int_{T^N} \left( 1 + |\nabla_a f(t - x)|^2 \right)^{1/2} \varphi_n(t) \, dt.
\]
By integration of both terms of (3.5) and using Fubini’s theorem, we obtain
\[
I \leq \int_{T_N} dt \int_{T_N} \left( 1 + |\nabla_a f(t-x)|^2 \right)^{1/2} \varphi_n(t) \, dx \\
\leq \int_{T_N} \varphi_n(t) \, dt \int_{T_N} \left( 1 + |\nabla_a f(x)|^2 \right)^{1/2} \, dx \\
= \int_{T_N} \left( 1 + |\nabla_a f|^2 \right)^{1/2} \, dx
\]
and we have (3.4). Then (2.3) follows by (3.3) and (3.4).

Since the functional relaxed area is lower semicontinuous in $L^1$, (2.4) follows immediately by (2.3). □

References


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