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A VARIATIONAL METHOD IN FIXED POINT RESULTS WITH INWARDNESS CONDITIONS

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Abstract. We generalize, in a metric space setting, the result due to Lim (2000), that a weakly inward multivalued contraction, defined on a nonempty closed subset of a Banach space, has a fixed point. The simple proof of this generalization, avoiding the use of a transfinite induction as in Lim's paper, is based on Ekeland's variational principle (1974), along the lines of Hamel (1994) and Takahashi (1991). Moreover, we give a sharp estimate for the distance from any point to the fixed point set.

1. Introduction and notation

In this note, we consider a metric space (X,d), a nonempty subset D of X, and a multifunction $T : D \to 2^X$ with nonempty values, and we let $\mathcal{F}_T := \{x \in D : x \in T(x)\}$ denote the (possibly empty) set of fixed points of T. We shall assume that T has a closed graph $\mathcal{G}_T := \{(x,y) \in D \times X : \; y \in T(x)\}$. Given $0 \le \kappa < 1$, for $0 < \varepsilon < (1-\kappa)(1+\kappa)^{-1}$, we let

$$\sigma_\varepsilon := \frac{1-\varepsilon}{1+\varepsilon} - \kappa > 0\,,$$

and we consider the following property, hereafter denoted by $(P)_{\sigma_\varepsilon}$: for every $(x,y) \in \mathcal{G}_T$ with $y \neq x$, there exist $u \in X$ and $z \in D$ such that

$$d(x,y) = d(x,u) + d(u,y)\,, \qquad d(u,z) < \varepsilon\, d(u,x)\,, \tag{1}$$

and

$$d(y, T(z)) \le \kappa\, d(x,z)\,. \tag{2}$$

(Note that since $\varepsilon \le 1$, both u and z differ from x.) We say that T *satisfies property* $(P)_{1-\kappa}$ if T satisfies property $(P)_{\sigma_\varepsilon}$ for all $0 < \varepsilon < (1-\kappa)(1+\kappa)^{-1}$.

Let us first compare these properties with more standard notions. Recall that T is a *contraction of modulus* κ $(0 \le \kappa < 1)$ if

$$e(T(x),T(z)) := \sup_{y \in T(x)} d(y, T(z)) \le \kappa\, d(x,z) \qquad \text{for all } x,z \in D \tag{3}$$

(where $d(y,T(z)) := \inf_{w \in T(z)} d(y,w)$). Of course, this is equivalent to

$$\mathcal{H}(T(x),T(z)) \le \kappa\, d(x,z) \qquad \text{for all } x,z \in D\,,$$

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where $\mathcal{H}(T(x),T(z)) := \max\{e(T(x),T(z)), e(T(z),T(x))\}$ is the classical Hausdorff distance between $T(x)$ and $T(z)$. If T is a contraction of modulus κ, and if for every $(x,y) \in \mathcal{G}_T$ with $y \neq x$ there exists $u \in X$ such that

$$d(x,y) = d(x,u) + d(u,x) \qquad \text{and} \qquad d(u,D) < \varepsilon\, d(u,x)\,, \tag{4}$$

then (1) and (2) are satisfied, for some $z \in D$. Note further that if D is closed and if T is a contraction with (nonempty) closed values, then $\mathcal{G}_T$ is closed.

On the other hand, if $(X, \|\cdot\|)$ is a normed space (and $d(x,y) := \|x-y\|$), and if for some $\varepsilon > 0$ we have

$$\inf_{t\geq 1,\, z\in D} \|y-x-t(z-x)\| < \varepsilon\|y-x\| \qquad \text{for all } (x,y) \in \mathcal{G}_T\,,\ y \neq x\,, \tag{5}$$

then T satisfies (4) (indeed, if $(x,y) \in \mathcal{G}_T$ with $y \neq x$ and if $z \in D$ and $t \geq 1$ are such that $\|y-x-t(z-x)\| < \varepsilon\|y-x\|$, then (4) holds with $u := x + t^{-1}(y-x)$ since $\|u-z\| < \varepsilon\|u-x\|$ — of course, (4) and (5) are not equivalent, for the set $\{u \in X:\ \|x-y\| = \|x-u\| + \|u-y\|\}$ may in general differ from the geometrical segment joining x and y). Now, (5) clearly holds for every $\varepsilon > 0$ if and only if

$$T(x) - x \subset \overline{[1,+\infty[(D-x)} \qquad \text{for all } x \in D\,. \tag{6}$$

Summing up this discussion, we see that if D is a (nonempty) closed subset of a normed space X, and if $T : D \to 2^X \setminus \{\emptyset\}$ is a closed-valued contraction of modulus κ verifying (6), then T satisfies property $(P)_{1-\kappa}$.

Assume now that $D = X$. Given $0 \leq \kappa < 1$, we say that the multifunction $T : X \to 2^X \setminus \{\emptyset\}$ defined on the metric space (X,d) is a *directional contraction of modulus* κ if $\mathcal{G}_T$ is closed and if for every $(x,y) \in \mathcal{G}_T$ with $y \neq x$, there exists $z \in X \setminus \{x\}$ such that

$$d(x,y) = d(x,z) + d(z,y) \qquad \text{and} \qquad d(y,T(z)) \leq \kappa\, d(x,z)\,. \tag{7}$$

Observe that a closed-valued contraction is a directional contraction: $\mathcal{G}_T$ is closed, while if $(x,y) \in \mathcal{G}_T$ with $y \neq x$ and if T satisfies (3), then (7) is satisfied with $z := y$. Moreover, if T satisfies (7), then T satisfies property $(P)_{\sigma_\varepsilon}$ (with $u := z$) for all $0 < \varepsilon < (1-\kappa)(1+\kappa)^{-1}$. Thus, if $T : X \to 2^X \setminus \{\emptyset\}$ is a directional contraction of modulus κ, then T satisfies property $(P)_{1-\kappa}$.

In [7], Lim proved the following nice result: if D is a closed subset of a Banach space X and $T : D \to 2^X \setminus \{\emptyset\}$ is a closed-valued contraction satisfying (6) — that is to say, T is a *weakly inward* contraction — then T has a fixed point. Here, we give an elementary proof of a generalization of this result, using property $(P)_{\sigma_\varepsilon}$ and a variational method directly derived from the Ekeland variational principle, thus avoiding the use of transfinite induction as in the quoted reference. Moreover, we also provide an estimate for the distance $d(x, \mathcal{F}_T)$. The use of variational methods in fixed point theory dates back to the celebrated paper of Caristi [3]. However, our point of view is quite different, more in line with the method of Hamel [6] and Takahashi [12], and is based upon an immediate application of the basic Ekeland's principle in the spirit of Penot [10, 11].

Our main result also provides an extension of a result in Clarke [4], dealing with continuous single-valued directional contractions, as well as containing the classical result of Nadler [9].

In the sequel, we shall use the following notation: given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, if $-\infty < \alpha < \beta \leq +\infty$, we set

$$[f{\leq}\beta] := \{x \in X :\ f(x) \leq \beta\}$$

(so that $[f{\le}{+\infty}] = X$), and we define in a similar way the sets $[f{<}\beta]$, $[f{>}\alpha]$, and $[\alpha{<}f{<}\beta] := [f{<}\beta] \setminus [f{\le}\alpha]$. We say that f is *proper* if $\mathrm{dom} f := [f{<}{+\infty}] \neq \emptyset$.

2. Fixed point results

Let X be a metric space endowed with the metric d, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. A point $x \in X$ is said to be a *d-point* of f if

$$f(x) < f(z) + d(z,x) \qquad \text{for all } z \in X\,, z \neq x\,.$$

(Observe that d-points are in $\mathrm{dom} f$, and that global minima are d-points.) With our notation, the basic form of Ekeland's variational principle is the following (see [11, Theorem B]; see also [2, Theorem 2.1]):

Theorem 2.1. *If (X,d) is complete and $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous, and bounded from below, then f has a d-point.*

This result is proved through a simple iterative procedure involving a sequence of sets of the type $M_x := \{z \in X : \ f(z) + d(z,x) \le f(x)\} \subset [f{\le}f(x)]$, $x \in X$. It is readily seen that x is a d-point of f if and only if $M_x = \{x\}$, and that, using the triangle inequality we have

$$\bar{x} \in M_x \text{ is a } d\text{-point of the restriction of } f \text{ to } M_x \implies \bar{x} \text{ is a } d\text{-point of } f\,.$$

Thus, applying Theorem 2.1 to the restriction of f to M_x, we immediately obtain the following.

Corollary 2.1. *Let (X,d) be a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and bounded from below. Assume that for every $x \in \mathrm{dom} f$, the set $[f{\le}f(x)]$ is complete. Then, for every $x \in X$, f has a d-point in M_x.*

Note that Corollary 2.1 applies to $([f{<}\beta], d)$, $\beta \in \mathbb{R} \cup \{+\infty\}$, whenever (X,d) is complete. The result with (X,d) complete is given in [11, Theorem B]. We finally state the following immediate consequence of Corollary 2.1, which we shall use for the proof of our main result.

Corollary 2.2. *Let (X,d) be a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Assume that for every $x \in \mathrm{dom} f$ the set $[f{\le}f(x)]$ is complete, and that $\mu := \inf_X f \in \mathbb{R}$. Set $S := [f{\le}\mu]$. Then, the following are equivalent:*

(a) *f has no d-point in $X \setminus S$;*

(b) *for every $x \in X$, there exists $\bar{x} \in M_x \cap S$ (i.e., $f(x) - \mu \ge d(x,\bar{x})$).*

(Indeed, a d-point of f in M_x, given by Corollary 2.1, must be in S if (a) holds, and conversely, $M_x \neq \{x\}$ for $x \in X \setminus S$ if (b) holds.) Corollary 2.2 slightly extends [6, Theorem 2], which was itself an extension of the results of [12].

Remark 2.1. Let (X,d) be complete, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, and let $-\infty < \alpha < \beta \le +\infty$. Then, applying Corollary 2.2 to the metric space $([f{<}\beta], d)$ and to the function $g : [f{<}\beta] \to \mathbb{R} \cup \{+\infty\}$ defined by $g(x) := (f(x) - \alpha)^+$, yields that the following are equivalent:

(a) for every $x \in [\alpha{<}f{<}\beta]$ there exists $y \in [f{<}\beta] \setminus \{x\}$ such that $f(x) - \alpha \ge (f(y) - \alpha)^+ + d(y,x)$;

(b) for every $x \in [\alpha{<}f{<}\beta]$ there exists $\bar{x} \in [f{\le}\alpha]$ such that $f(x) - \alpha \ge d(x,\bar{x})$.

See [2] and the references therein for related results.

The following is the main result of this note.

Theorem 2.2. *Let D be a nonempty subset of a complete metric space (X,d), and let $T : D \to 2^X$ be a multifunction with nonempty values and closed graph. Let $0 \leq \kappa < 1$, let $\varepsilon > 0$ be such that*

$$\sigma_\varepsilon := \frac{1-\varepsilon}{1+\varepsilon} - \kappa > 0\,,$$

and assume that T satisfies property $(P)_{\sigma_\varepsilon}$ (recall (1)-(2)). Then, $\mathcal{F}_T \neq \emptyset$ and

$$d(x, T(x)) \geq \sigma_\varepsilon\, d(x, \mathcal{F}_T) \qquad \textit{for all } x \in D\,.$$

Proof. Assume first that $\kappa > 0$. Let $(x,y) \in \mathcal{G}_T$, $y \neq x$, be fixed. According to property $(P)_{\sigma_\varepsilon}$, we find $u \in X$ and $z \in D$ such that $d(x,y) = d(x,u) + d(u,y)$, $d(u,z) < \varepsilon\, d(u,x)$, and $d(y, T(z)) \leq \kappa\, d(x,z)$. Then

$$d(x,z) \leq d(x,u) + d(u,z) < (1+\varepsilon) d(x,u) \tag{8}$$

and

$$d(y, T(z)) \leq \kappa d(x,z) < \kappa(1+\varepsilon) d(x,u)\,,$$

so that we find $w \in T(z)$ with

$$d(y,w) \leq \kappa(1+\varepsilon) d(x,u)\,. \tag{9}$$

On the other hand, we also have

$$\begin{aligned} d(z,w) &\leq d(z,u) + d(u,y) + d(y,w) \\ &= d(z,u) + d(x,y) - d(x,u) + d(y,w) \\ &< \varepsilon d(u,x) + d(x,y) - d(x,u) + \kappa(1+\varepsilon) d(x,u)\,, \end{aligned}$$

so that

$$d(x,y) > d(z,w) + (1 - \varepsilon - \kappa(1+\varepsilon)) d(x,u) = d(z,w) + \sigma_\varepsilon (1+\varepsilon) d(x,u)\,. \tag{10}$$

Now, consider $X \times X$ as endowed with the metric

$$\tilde{d}((x',y'),(x'',y'')) := \sigma_\varepsilon \max\left\{ d(x',x''), \frac{1}{\kappa} d(y',y'') \right\}\,.$$

Since $\mathcal{G}_T$ is closed in $(X \times X, \tilde{d})$, $(\mathcal{G}_T, \tilde{d})$ is complete. Consider the continuous, nonnegative function $f := d_{|\mathcal{G}_T}$ (the restriction of d to $\mathcal{G}_T$). Then, combining (8), (9), and (10), we see that for every $(x,y) \in \mathcal{G}_T$ with $y \neq x$, we find $(z,w) \in \mathcal{G}_T$ such that

$$f(x,y) > f(z,w) + \tilde{d}((x,y),(z,w))\,.$$

Thus, f has no $\tilde{d}$-point in $[f{>}0]$, and we deduce from Corollary 2.2 that $\inf f = 0$, that $[f{\leq}0] = \{(x,x) :\ x \in \mathcal{F}_T\} \neq \emptyset$, and that for every $(x,y) \in \mathcal{G}_T$ there exists $\bar{x} \in \mathcal{F}_T$ such that

$$d(x,y) \geq \tilde{d}((x,y),(\bar{x},\bar{x})) \geq \sigma_\varepsilon d(x,\bar{x}) \geq \sigma_\varepsilon d(x, \mathcal{F}_T)\,,$$

so that $d(x,T(x)) \geq \sigma d(x, \mathcal{F}_T)$, since y is arbitrary in $T(x)$.

If $\kappa = 0$, considering $\tilde{\kappa} > 0$ such that $\tilde{\sigma}_\varepsilon := (1-\varepsilon)(1+\varepsilon)^{-1} - \tilde{\kappa} > 0$, we see that T satisfies property $(P)_{\tilde{\sigma}_\varepsilon}$, so that, arguing as above and then letting $\tilde{\kappa} \to 0$ yields the conclusion. □

Corollary 2.3. *Let D be a nonempty subset of a complete metric space (X,d), let $T : D \to 2^X$ be a multifunction with nonempty values and closed graph, and let $0 \leq \kappa < 1$. Assume that T satisfies property $(P)_{1-\kappa}$. Then, $\mathcal{F}_T \neq \emptyset$ and $d(x,T(x)) \geq (1-\kappa)d(x,\mathcal{F}_T)$ for all $x \in D$.*

Proof. Recall that T satisfying property $(P)_{1-\kappa}$ means that it satisfies property $(P)_{\sigma_\varepsilon}$ for any $0 < \varepsilon < (1-\kappa)(1+\kappa)^{-1}$. Thus, the conclusion follows from Theorem 2.2. $\square$

Remark 2.2. Taking into account what we already said in the introduction, Corollary 2.3 is an extension of Lim's result [7, Theorem 1], where D is a nonempty closed subset of a Banach space X, and T is a contraction with nonempty, closed values satisfying the inwardness condition (6). This result was previously established for single-valued maps by Martinez-Yanez [8], and by Xu [13, Theorem 3.3] under the additional assumption that each $x \in D$ has a nearest point in $T(x)$. Corollary 2.3 also extends Clarke's Theorem 7.6.2 in [4], where T is a continuous single-valued, directional contraction from the complete metric space X into itself, as well as it contains Nadler's theorem: every multi-valued contraction with nonempty, closed values, from a complete metric space into itself, has a fixed point (indeed, Nadler considers multifunctions with bounded values — see Remark 2.3 below). A simple example of a (single-valued) directional contraction which is not a contraction is given in [4, Remark 7.6.3].

We note that a variational proof of Nadler's theorem was already given by Takahashi [12], while Clarke's proof of his result also relies on Ekeland's principle, applied to the function $f(x) := d(x,T(x))$, which is continuous if so is (the single-valued map) T. Our proof of Theorem 2.2 suggests that it is more appropriate to work in $\mathcal{G}_T$ — that is, in the product space $X \times X$ — when possible (in the same spirit, see [1, Section 5]). Still, we now give a variant of Theorem 2.2, where we apply Corollary 2.2 to the function $x \mapsto d(x,T(x))$; see Remark 2.3 below for comments.

Theorem 2.3. *Let D be a nonempty closed subset of a complete metric space (X,d), and let $T : D \to 2^X$ be a multifunction with nonempty, closed values. Assume that every $x \in D$ has a nearest point in $T(x)$, and that the function $x \mapsto d(x,T(x))$ is lower semicontinuous on D. Let further $0 \leq \kappa < 1$ and $\varepsilon > 0$ be such that*

$$\sigma := \frac{1-\varepsilon}{1+\varepsilon} - \kappa > 0\,,$$

and assume that for every $(x,y) \in \mathcal{G}_T$ with $y \neq x$, there exist $u \in X$ and $z \in D$ such that

$$d(x,y) = d(x,u) + d(u,y)\,, \qquad d(u,z) < \varepsilon\, d(u,x)\,, \tag{11}$$

and

$$d(z,T(z)) \leq d(z,y) + \kappa\, d(z,x)\,. \tag{12}$$

Then, $\mathcal{F}_T \neq \emptyset$ and

$$d(x,T(x)) \geq \sigma\, d(x,\mathcal{F}_T) \qquad \text{for all } x \in D\,.$$

Proof. We consider D as endowed with the metric $\tilde{d}(x,y) := \sigma\, d(x,y)$ and the lower semicontinuous function $f : D \to \mathbb{R}_+$ defined by $f(x) := d(x,T(x))$. Then, $(D,\tilde{d})$ is complete and $[f\leq 0] = \mathcal{F}_T$. Let $x \in D$ be such that $f(x) > 0$, and let $y \in T(x)$

be such that $f(x) = d(x,y)$ — so that $y \neq x$. Then let $u \in X$ and $z \in D$ be as in (11) and (12). We obtain from (11) that $d(z,x) < (1+\varepsilon)d(u,x)$ and

$$\begin{aligned} d(z,y) + \tfrac{1-\varepsilon}{1+\varepsilon} d(z,x) &< d(z,y) + (1-\varepsilon)d(u,x) \\ &= d(z,y) + d(x,y) - d(u,y) - \varepsilon d(u,x) \\ &< d(z,y) - (d(u,y) + d(u,z)) + d(x,y) \ \leq \ d(x,y)\,. \end{aligned}$$

Then, using also (12), we have

$$d(z,T(z)) + \sigma\, d(z,x) \leq d(z,y) + \frac{1-\varepsilon}{1+\varepsilon}\, d(z,x)\,,$$

so that, combining the previous inequalities, we have $f(z) + \tilde{d}(z,x) < f(x)$. Thus, f has no $\tilde{d}$-point in $[f{>}0]$, and we deduce from Corollary 2.2 that $[f{\leq}0] = \mathcal{F}_T \neq \emptyset$, and that for every $x \in D$ there exists $\bar{x} \in \mathcal{F}_T$ such that $d(x,T(x)) \geq \sigma\, d(x,\bar{x}) \geq \sigma\, d(x,\mathcal{F}_T)$. □

Remark 2.3. The assumption that $x \mapsto d(x,T(x))$ is lower semicontinuous, in Theorem 2.3, is satisfied, for example, if T is *$\mathcal{H}$-upper semicontinuous* ("$\mathcal{H}$" is for Hausdorff), that is, if $\lim_{z\to x} e(T(z),T(x)) = 0$ for all $x \in D$.

Note that (2) implies (12) since $d(z,T(z)) \leq d(z,y) + d(y,T(z))$. Similarly, if T is a contraction of modulus κ, then T satisfies

$$d(z,T(z)) \leq d(z,T(x)) + \kappa\, d(z,x) \qquad \text{for all } z,x \in D\,. \tag{13}$$

When the multifunction T has unbounded values, the requirement that it be a contraction is difficult to satisfy, in general. We show in the following simple example that condition (13) is strictly weaker than the contractiveness of T.

Example 2.1. Let $X := \mathbb{R}$, $D := [0,1]$, and $T : D \to 2^X$ be defined by

$$T(x) := \begin{cases} \left(-\infty, \frac{x}{2} + \frac{1}{4}\right] & \text{if } x > 0\,, \\ \left[0, \frac{1}{4}\right] & \text{if } x = 0\,. \end{cases}$$

Then, for any $x,z \in [0,1]$ we have $d(z,T(x)) = \left(z - \frac{x}{2} - \frac{1}{4}\right)^+$, whence an easy computation yields $d(z,T(z)) - d(z,T(x)) \leq \frac{1}{2}|z-x|$, so that (13) is verified with $\kappa := \frac{1}{2}$. But T is not a contraction since $\mathcal{H}(T(0),T(x)) = +\infty$ for any $x > 0$. Observe that T indeed satisfies property $(P)_{1-\kappa}$, so that Theorem 2.2 applies, as well as Theorem 2.3. Since $\mathcal{F}_T = \left[0, \frac{1}{2}\right]$, we see in this example that the estimate $d(x,T(x)) \geq (1-\kappa)d(x,\mathcal{F}_T)$ is exact.

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