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CONVOLUTION CONGRUENCES FOR THE PARTITION FUNCTION

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Abstract. Ahlgren and Boylan recently proved the uniqueness of the Ramanujan congruences for the primes 5, 7, and 11 by using the modularity of a certain partition function. Here we use their result to find universal congruences, of a different type, which hold for the partition function modulo all primes $\ell \geq 5$.

1. Introduction and statement of results

Let $p(n)$ denote the number of partitions of the positive integer n, that is, the number of representations of n as the sum of a sequence of non-increasing positive integers. For more on partitions see [And98] or [AO01]. For convenience, we let $p(0) = 1$, and $p(\alpha) = 0$ if $\alpha < 0$. Euler introduced the generating function

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}.$$

If we denote the pentagonal numbers

$$\omega(k) := \frac{1}{2}k(3k+1), \quad k \in \mathbb{Z}, \tag{1.1}$$

then we have

$$\sum_{k\in\mathbb{Z}} (-1)^k q^{\omega(k)} \cdot \sum_{n\geq 0} p(n)q^n = 1.$$

This gives rise to the recursive formula

$$p(n) = \sum_{k\in\mathbb{Z}-\{0\}} (-1)^{k+1} p(n-\omega(k)),$$

which holds for all positive integers n. While this formula is useful for quickly computing values of $p(n)$, it has not offered much information about the arithmetic of the partition function.

The famous Ramanujan congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

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may be reformulated in the following peculiar way. For primes $\ell \geq 5$, define

$$\delta_\ell := \frac{\ell^2 - 1}{24},$$

and let

$$\mathcal{P}_\ell(z) := \sum_{k \in \mathbb{Z}} (-1)^k q^{\ell\omega(k)} \sum_{n \geq 0} p(\ell n - \delta_\ell) q^n. \tag{1.2}$$

The Ramanujan congruences are equivalent to the assertion that $\mathcal{P}_\ell(z) \equiv 0 \pmod{\ell}$ for $\ell \in \{5, 7, 11\}$. For example, we have

$$\begin{aligned} \mathcal{P}_5(z) &= 5q + 30q^2 + 135q^3 + 490q^4 + 1575q^5 + \cdots \equiv 0 \pmod{5}, \\ \mathcal{P}_7(z) &= 7q + 77q^2 + 490q^3 + 2436q^4 + 10143q^5 + \cdots \equiv 0 \pmod{7}, \\ \mathcal{P}_{11}(z) &= 11q + 297q^2 + 3718q^3 + 31185q^4 + 204226q^5 + \cdots \equiv 0 \pmod{11}. \end{aligned} \tag{1.3}$$

Ahlgren and Boylan [AB03] have proven Ramanujan's famous conjecture that congruences of the form $p(\ell n - \beta) \equiv 0 \pmod{\ell}$ exist only for primes $5, 7$, and 11. Therefore, it is natural to ask whether some refinement of (1.3) continues to hold for all primes $\ell \geq 5$. Here we show that this is indeed the case.

To make this precise, let M_k denote the space of weight k holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$. For more on modular forms see [Kob93]. We shall denote a modular form $f(z) \in M_k$ by its Fourier expansion

$$f(z) := \sum_{n=0}^{\infty} a_f(n) q^n, \qquad q := e^{2\pi i z}.$$

For each prime ℓ we have the $U(\ell)$-operator defined by

$$f(z)|U(\ell) = \sum_{n=0}^{\infty} a_f(\ell n) q^n. \tag{1.4}$$

Furthermore, for even $k \geq 4$, let $E_k(z) \in M_k$ denote the normalized Eisenstein series of weight k,

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.5}$$

where the rational numbers B_k are the Bernoulli numbers defined by

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 + \cdots$$

and

$$\sigma_{k-1}(n) = \sum_{1 \leq d | n} d^{k-1}.$$

Our refinement of (1.3) is the following.

Theorem 1.1. *Suppose $\ell \geq 5$ is prime and $m \in \{4, 6, 8, 10, 14\}$. If $\mathcal{P}_\ell^{(m)}(z) := E_m(z) P_\ell(z)$, then*

$$\mathcal{P}_\ell^{(m)}(z)|U(\ell) \equiv 0 \pmod{\ell}.$$

Remark. Note that $E_4(z) = 1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n \equiv 1 \pmod 5$. Similarly, $E_6(z) \equiv 1 \pmod 7$ and $E_{10}(z) \equiv 1 \pmod{11}$. While this does not yield the full Ramanujan congruences, it does show

$$p(\ell^2 n - \delta_\ell) \equiv 0 \pmod \ell \quad \text{for } \ell = 5, 7, 11.$$

Example. If $m = 4$ and $\ell = 13$, then

$$\mathcal{P}_{13}^{(4)}(z) = 11q + 3130q^2 + 149709q^3 + 3225214q^4 + 43626740q^5 + \cdots,$$

$$\mathcal{P}_{13}^{(4)}(z)|U(13) = 64580467571997q + 16669309558732899621 8q^2 + \cdots \equiv 0 \pmod{13}.$$

If we let $b(4) := 240, b(6) := -540, b(8) := 480, b(10) := -264, b(14) := -24$, then Theorem 1.1 implies the following recursive congruence.

Theorem 1.2. *For each prime $\ell \geq 5$ and each $m \in \{4, 6, 8, 10, 14\}$ we have*

$$\begin{aligned} p(\ell^2 N - \delta_\ell) \equiv & \sum_{k\in\mathbb{Z}-\{0\}} (-1)^{k+1} p(\ell^2(N - \omega(k)) - \delta_\ell) \\ & + b(m) \sum_{\substack{n\geq 1\\ k\in\mathbb{Z}}} \sigma_{m-1}(n)(-1)^{k+1} p(\ell^2(N - \omega(k) - n/\ell) - \delta_\ell) \pmod \ell \end{aligned}$$

for all positive integers N.

Theorem 1.1 is a consequence of a generic theorem for cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients. Let S_k denote the space of cusp forms of weight k, that is, the subspace of modular forms $f(z) \in M_k$ such that $a_f(0) = 0$. For example,

$$\Delta(z) := q\prod_{n=1}^{\infty}(1 - q^n)^{24} = q - 24q^2 + \cdots \in S_{12} \cap \mathbb{Z}[[q]].$$

Theorem 1.3. *Fix a prime $\ell \geq 5$. Suppose $k = a(\ell - 1) + b$ for some $1 \leq a \leq 12$ and $b \in \{4, 6, 8, 10, 14\}$ with $b \geq a + 2$. If $f(z) \in S_k \cap \mathbb{Z}[[q]]$, then $f(z)|U(\ell) \equiv 0 \pmod \ell$.*

We prove all three of the above theorems in the next section. We then conclude with a discussion of an analogous function to $\mathcal{P}_\ell$ for the small primes $\ell = 2, 3$ in the final section.

2. Proof of the theorems

We begin with some preliminary information about modular forms. Recall that for even k we know M_k is a finite-dimensional $\mathbb{C}$-vector space of dimension

$$d(k) := \dim_{\mathbb{C}}(M_k) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases} \tag{2.1}$$

For even k define the function $\delta(k) \in \{0, 4, 6, 8, 10, 14\}$ by the congruence

$$\delta(k) \equiv k \pmod{12}. \tag{2.2}$$

The above function $\delta(k)$ gives information about the zeroes of $f(z) \in M_k$ by the valence formula below. Let $\omega := e^{2\pi i/3}$. If $f(z) \in M_k$ and $v_\tau(f)$ is the order vanishing of f at τ, then

$$\frac{k}{12} = v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{\substack{\tau\in\mathcal{H}/\Gamma\\ \tau\notin\{i,\omega\}}} v_\tau(f), \tag{2.3}$$

where $\mathcal{H}$ denotes the upper half place and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The valence formula tells us, for example, that $E_{\ell-1}(z)/E_{\delta(\ell-1)}(z)$ and $E_8(z)/E_4(z)$ are holomorphic modular forms. In addition, we have some well-known facts about the Eisenstein series $E_k(z)$. For convenience we define $E_0(z) := 1 \in M_0$.

(1) $E_k(z) = 1 + \cdots \in M_k \cap \mathbb{Z}[[q]]$ for each $k \in \{0, 4, 6, 8, 10, 14\}$.
(2) $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$ for $\ell \geq 5$ prime.

We will also need the following theorem of Choie, Kohnen, and Ono [CKO05].

Theorem 2.1 ([CKO05, Theorem 3.1])**.** *If $f(z) \in M_k$ and $g(z) \in M_{12N}$, then*

$$\mathrm{const}\left(\frac{E_{14-\delta(k)}(z)g(z)}{\Delta(z)^{d(k)+N}} \cdot f(z)\right) = 0,$$

where $\mathrm{const}(h(z))$ *denotes the constant term of the power series $h(z)$.*

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Fix a prime $\ell \geq 5$. Let k be of the specified form. Fix a cusp form $f(z) \in S_k \cap \mathbb{Z}[[q]]$. It suffices to show that $a_f(\ell n) \equiv 0 \pmod{\ell}$ for each $n \geq 1$. This will follow by induction on n.

Recall, as f is a cusp form, $a_f(0) = 0$. Fix an integer n and suppose $a_f(\ell m) \equiv 0 \pmod{\ell}$ for each $0 \leq m \leq n-1$. Consider the quantity

$$c(a, b, \ell) := \delta(a(\ell-1)) - 12\left(d(b + \delta(a(\ell-1))) - 1\right).$$

Define the function

$$g_n(z) := E_{14-b}(z)^{\ell-1} E_{\ell-1}(z)^{b-a-2} E_4(z)^{3\ell(n-1)} \tilde{g}(z),$$

where

$$\tilde{g}(z) := \begin{cases} E_{\delta(a(\ell-1))}(z) & \text{if } c(a,b,\ell) > 2, \\ E_{14-b}/E_{12-b}(z) & \text{if } c(a,b,\ell) = 2, \\ E_0(z) & \text{if } c(a,b,\ell) = 0, \\ E_{14-b}/E_{16-b}(z) & \text{if } c(a,b,\ell) = -2, \\ 1/E_{-c(a,b,\ell)}(z) & \text{if } c(a,b,\ell) < -2. \end{cases}$$

We first show that the above function $g_n(z)$ is a holomorphic modular form by considering each of the three non-trivial possibilities for $c(a, b, \ell)$ separately.

Case i) $c(a, b, \ell) = 2$.

$$g_n(z) = E_{14-b}(z)^{\ell-1} E_{\ell-1}(z)^{b-a-2} E_4(z)^{3\ell(n-1)} E_{14-b}(z)/E_{12-b}(z).$$

If $c(a, b, \ell) = 2$, then $d(b+\delta(a(\ell-1))) = 2$ and $\delta(a(\ell-1)) = 14$. Hence, $\delta(\ell-1) = 10$. Furthermore, $b \in \{4, 6, 8\}$. Since $\delta(\ell-1) = 10$, it follows by the valence formula that $E_{\ell-1}(z)E_{14-b}(z)/E_{12-b}(z)$ is holomorphic for each $b \in \{4, 6, 8\}$. It suffices to show that if $c(a, b, \ell) = 2$, then $b - a - 2 > 0$. If $b - a - 2 = 0$, then a is even, which contradicts $\delta(a(\ell-1)) = 14$.

Case ii) $c(a, b, \ell) = -2$.

$$g_n(z) = E_{14-b}(z)^{\ell-1} E_{\ell-1}(z)^{b-a-2} E_4(z)^{3\ell(n-1)} E_{14-b}(z)/E_{16-b}(z).$$

If $c(a, b, \ell) = -2$, then $d(b + \delta(a(\ell-1))) = 2$ and $\delta(a(\ell-1)) = 10$, so $\delta(\ell-1) = 10$ and $b \in \{6, 8, 10\}$. Again, the valence formula implies that $E_{\ell-1}(z)E_{14-b}(z)/E_{16-b}(z)$ is holomorphic for $b \in \{6, 10\}$ if $b-a-2 > 0$ and for $b = 8$ if $b-a-2 > 1$. If $b - a - 2 = 0$, then a is even, which contradicts the fact that $\delta(a(\ell-1)) = 10$. If $b = 8$ and $b - a - 2 = 1$, then $a = 5$, but then $a(\ell-1) \equiv 5(10) \not\equiv 10 \pmod{12}$.

Case iii) $c(a,b,\ell) < -2$.

$$g_n(z) = E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}E_4(z)^{3\ell(n-1)}/E_{-c(a,b,\ell)}(z).$$

As $c(a,b,\ell) < -2$, we have $d(b+\delta(a(\ell-1))) \geq 2$. As $b \leq 14$ and $\delta(a(\ell-1)) \leq 14$, we have $d(b+\delta(a(\ell-1))) \leq 3$. Moreover, if $\delta(a(\ell-1)) \in \{10,14\}$, then $d(b+\delta(a(\ell-1))) = 3$, and if $\delta(a(\ell-1)) \in \{4,6,8\}$, then $d(b+\delta(a(\ell-1))) = 2$.

First suppose $\delta(a(\ell-1)) \in \{10,14\}$, so $\delta(\ell-1) = 10$. If $\delta(a(\ell-1)) = 10$, then $-c(a,b,\ell) = 14$, and if $\delta(a(\ell-1)) = 14$, then $-c(a,b,\ell) = 10$. By the valence formula, $E_{\ell-1}(z)^{b-a-2}/E_{-c(a,b,\ell)}$ is a holomorphic modular form if $\delta(a(\ell-1)) = 14$ and $b-a-2 > 0$ or $\delta(a(\ell-1)) = 10$ and $b-a-2 > 1$. As in the previous cases, $\delta(\ell-1) = 10$ implies $b-a-2 > 0$. Suppose $\delta(a(\ell-1)) = 10$ and $b-a-2 = 1$. As $d(b+\delta(a(\ell-1))) = 3$, this implies that $b = 14$. Thus, $a = 11$, which yields $11(\ell-1) \equiv 11(10) \not\equiv 10 \pmod{12}$, a contradiction.

Now suppose $\delta(a(\ell-1)) \in \{4,6,8\}$, so $E_{-c(a,b,\ell)}(z) \in \{E_8(z), E_6(z), E_4(z)\}$, respectively. Begin with $\delta(a(\ell-1)) = 4$; then $\delta(\ell-1) = 4$ or 10. By the valence formula, $E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}/E_8(z)$ is a holomorphic modular form if $b-a-2 > 1$. As $d(b+a(\ell-1)) = 2$, this implies that $b = 8, 14$, and so we need only check that $a \neq 5,6,11,12$. Note that in each of these cases $a(\ell-1) \not\equiv 4 \pmod{12}$.

Next suppose $\delta(a(\ell-1)) = 6$, so $\delta(\ell-1) = 6$ or 10. It again follows by the valence formula that $E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}/E_6(z)$ is a holomorphic modular form if $b-a-2 > 0$. If $\delta(a(\ell-1)) = 6$ and $b-a-2 = 0$, then a is even, which contradicts $\delta(a(\ell-1)) = 6$.

Finally, consider $\delta(a(\ell-1)) = 8$, so $\delta(\ell-1) = 4$ or 10. If $b-a-2 > 0$ or $b \neq 8, 14$, then $E_{14-b}(z)^{\ell-1}E_{\ell-1}(z)^{b-a-2}/E_4(z)$ is a holomorphic modular form. If $b = 8, 14$, then $b-a-2 = 0$ implies $a = 6, 12$, for which $\delta(a(\ell-1)) \not\equiv 8 \pmod{12}$.

We can apply Theorem 2.1 to show $a_f(\ell n) \equiv 0 \pmod{\ell}$. Take $f(z)$ as above and $g(z) = g_n(z)$. We have $k = a(\ell-1) - \delta(a(\ell-1)) + \delta(a(\ell-1)) + b$; it follows that

$$d(k) = \frac{a(\ell-1) - \delta(a(\ell-1))}{12} + d(b+\delta(a(\ell-1))).$$

Similarly,

$$\begin{aligned}\delta(k) &= \delta(b+\delta(a(\ell-1))) \\ &= b + \delta(a(\ell-1)) - 12(d(b+\delta(a(\ell-1))) - 1) \\ &= b + c(a,b,\ell).\end{aligned}$$

Moreover, $\tilde{g}(z)$ has weight $c(a,b,\ell)$; hence, $g_n(z)$ has weight

$$(14-b)(\ell-1) + (\ell-1)(b-a-2) + 12\ell(n-1) + c(a,b,\ell) = 12N,$$

where

$$\begin{aligned}N &= \ell n - \frac{a(\ell-1) - \delta(a(\ell-1))}{12} - d(b+\delta(a(\ell-1))) \\ &= \ell n - d(k).\end{aligned}$$

Thus, by Theorem 2.1,

$$\text{const}\left(\frac{E_{14-b-c(a,b,\ell)}(z)g_n(z)}{\Delta(z)^{\ell n}} \cdot f(z)\right) = 0.$$

Each possibility for $g_n(z)$ allows us to conclude that

$$\text{const}\left(\frac{E_{14-b}(z)^{\ell}E_{\ell-1}(z)^{b-a-2}E_4(z)^{3\ell(n-1)}}{\Delta(z)^{\ell n}}\cdot f(z)\right)=0,$$

where we use the observation that

$$E_{14-b-c(a,b,\ell)}(z)\tilde{g}(z)=E_{14-b}(z).$$

Now consider the above function modulo ℓ, recalling $E_{\ell-1}(z)\equiv 1 \pmod{\ell}$. The constant term of

$$\begin{aligned}&\frac{E_{14-b}(z)^{\ell}E_{\ell-1}(z)^{b-a-2}E_4(z)^{3\ell(n-1)}}{\Delta(z)^{\ell n}}\cdot f(z)\\ &\equiv \frac{E_{14-b}(\ell z)E_4(\ell z)^{3(n-1)}}{\Delta(\ell z)^{n}}\cdot f(z) \pmod{\ell}\\ &\equiv E_{14-b}(\ell z)E_4(\ell z)^{3(n-1)}\left(q^{-\ell}+24+324q^{\ell}+\cdots\right)^{n}\sum_{n=0}^{\infty}a_f(n)q^n \pmod{\ell} \qquad (2.4)\end{aligned}$$

is congruent to zero modulo ℓ. Recall that $E_{14-b}(\ell z)$ and $E_4(\ell z)^{3(n-1)}$ have leading coefficient 1, so by multiplying out the above expression, we find for each $0\le m\le n-1$ integers $\alpha_{b,n}(m)$ such that

$$a_f(\ell n)+\sum_{m=0}^{n-1}\alpha_{b,n}(m)a_f(\ell m)\equiv 0 \pmod{\ell}. \qquad (2.5)$$

In particular, $a_f(\ell m)\equiv 0 \pmod{\ell}$ for each $0\le m\le n-1$ by the inductive hypothesis, and hence $a_f(\ell n)\equiv 0 \pmod{\ell}$ by induction. □

Remark. The above proof fails for the small primes $\ell=2,3$; for example, $E_{\ell-1}(z)$ is not a holomorphic modular form for these primes. We can, however, modify the proof using the fact that $E_k(z)\equiv 1 \pmod{2,3}$ for $k\in\{0,4,6,8,10,14\}$. It suffices to show that for each $n\ge 1$ we can find a non-negative integer N such that $d(k)+N=\ell n$, as we can then conclude by Theorem 2.1 that

$$\text{const}\left(\frac{E_{14-\delta(k)}(z)E_4(z)^{3N}}{\Delta(z)^{d(k)+N}}\cdot f(z)\right)\equiv \text{const}\left(\Delta(\ell z)^{-n}\cdot f(z)\right) \pmod{\ell}.$$

This yields equations analogous to (2.4) and (2.5). For $\ell=3$ we have $2a+b\le 38$, so for each $n\ge 1$ we can take $N=3n-d(k)\ge 0$, with the exception of $k=36$, where $d(36)=4$. Similarly, for $\ell=2$ we have $a+b\le 26$, so again we can take $N=2n-d(k)\ge 0$ with the exception of $k=24$, where $d(24)=3$. For these exceptions the conclusion of Theorem 1.3 does not hold as $\Delta(z)^{\ell}=q^{\ell}+\cdots\in S_{12\ell}\cap\mathbb{Z}[[q]]$.

We are now ready to prove the main theorem of the paper, which follows as a consequence of Theorem 1.3 above and the following theorem of Ahlgren and Boylan [AB03].

Theorem 2.2 ([AB03, Theorem 3])**.** *If $\ell\ge 5$ is prime and $j\ge 1\in\mathbb{Z}$, then let $1\le\beta_{\ell,j}\le\ell^j$ be the unique integer for which $24\beta_{\ell,j}\equiv 1 \pmod{\ell^j}$. Define $k_{\ell,j}$ by*

$$k_{\ell,j}:=\begin{cases}\frac{\ell^{j-1}(\ell-1)}{2}-\frac{1}{2}\left(\frac{24\beta_{\ell,j}-1}{\ell^j}+1\right) & \text{if } j \text{ is odd,}\\ \ell^{j-1}(\ell-1)-\frac{1}{2}\left(\frac{24\beta_{\ell,j}-1}{\ell^j}+1\right) & \text{if } j \text{ is even.}\end{cases}$$

Then there exists a modular form $F_{\ell,j} \in M_{k_{\ell,j}} \cap \mathbb{Z}[[q]]$ *such that*

$$\sum_{n=0}^{\infty} p(\ell^j n + \beta_{\ell,j})q^n \equiv \prod_{n=1}^{\infty}(1-q^n)^{\frac{24\beta_{\ell,j}-1}{\ell^j}} \cdot F_{\ell,j}(z) \pmod{\ell^j}.$$

Proof of Theorem 1.1. We now apply Theorem 2.2 in the case where ℓ is any prime at least 5 and $j = 1$. In this case

$$k_\ell := k_{\ell,1} = (\ell - 1) - 12\left(\lfloor \ell/24 \rfloor + 1\right),$$
$$\beta_\ell := \beta_{\ell,1} = -\frac{(\ell^2-1)}{24} + \ell\left(\lfloor \ell/24 \rfloor + 1\right) = -\delta_\ell + \ell\left(\lfloor \ell/24 \rfloor + 1\right).$$

Thus, we have

$$\sum_{n=0}^{\infty} p(\ell n+\beta_\ell)q^n \equiv \prod_{n=1}^{\infty}(1-q^n)^{\frac{24\beta_\ell-1}{\ell}} \cdot F_{\ell,1} \equiv \prod_{n=1}^{\infty}(1-q^n)^{24(\lfloor \ell/24 \rfloor+1)-\ell} \cdot F_{\ell,1} \pmod{\ell}.$$

Multiplication by $q^{\lfloor \ell/24 \rfloor+1}\prod_{n=1}^{\infty}(1-q^n)^\ell$ yields

$$q^{\lfloor \ell/24 \rfloor+1}\prod_{n=1}^{\infty}(1-q^n)^\ell \sum_{n=0}^{\infty} p(\ell n+\beta_\ell)q^n \equiv \Delta(z)^{\lfloor \ell/24 \rfloor+1} \cdot F_{\ell,1} \pmod{\ell},$$
$$\prod_{n=1}^{\infty}(1-q^{\ell n}) \sum_{n=0}^{\infty} p(\ell n-\delta_\ell)q^n \equiv \Delta(z)^{\lfloor \ell/24 \rfloor+1} \cdot F_{\ell,1} \pmod{\ell}.$$

Note that $\Delta(z)^{\lfloor \ell/24 \rfloor+1} \cdot F_{\ell,1} \pmod{\ell} \in S_{\ell-1} \cap \mathbb{Z}[[q]]$, and so, by Theorem 1.3, we can multiply both sides of the equation above by any Eisenstein series $E_m(z)$ for $m \in \{4, 6, 8, 10, 14\}$ to find

$$\left(E_m(z)\prod_{n=1}^{\infty}(1-q^{\ell n}) \sum_{n=0}^{\infty} p(\ell n-\delta_\ell)\right)|U(\ell) \equiv 0 \pmod{\ell}. \tag{2.6}$$

Applying Euler's Pentagonal Number Theorem, which states

$$\prod_{n=1}^{\infty}(1-q^{\ell n}) = \sum_{k\in\mathbb{Z}}(-1)^k q^{\ell\omega(k)},$$

completes the proof. □

Equation (2.6) brings us to Theorem 1.2.

Proof of Theorem 1.2. Suppose $\ell \geq 5$ is prime. For each $m \in \{4, 6, 8, 10, 14\}$ define the numbers $A_m(n)$ by

$$\sum_{n\geq 0} A_m(n)q^n := \left(1 + b(m)\sum_{n\geq 1}\sigma_{m-1}(n)q^n\right)\left(\sum_{k\in\mathbb{Z}}(-1)^k q^{\ell\omega(k)}\right)\left(\sum_{n\geq 0} p(\ell n-\delta_\ell)q^n\right). \tag{2.7}$$

In particular,

$$A_m(N) = p(\ell N-\delta_\ell) + \sum_{k\in\mathbb{Z}-\{0\}}(-1)^k p(\ell(N-\ell\omega(k))-\delta_\ell)$$
$$+ b(m)\sum_{\substack{n\geq 1\\ k\in\mathbb{Z}}}\sigma_{m-1}(n)(-1)^k p(\ell(N-\ell\omega(k)-n)-\delta_\ell).$$

The result now follows by Theorem 1.1 as

$$\sum_{n\geq 0} A_m(n)q^n|U(\ell) \equiv 0 \pmod{\ell}. \qquad \square$$

3. The primes 2 and 3

It is difficult to prove anything about the distribution of the partition function $p(n)$ modulo 2 and 3. For these primes $\ell = 2, 3$, we cannot define $\mathcal{P}_\ell(z)$ since δ_ℓ is no longer an integer. We can, however, define the following analog for these small primes. For $\ell = 2, 3$ define

$$\mathcal{P}_\ell^*(z) := \sum_{k\in\mathbb{Z}} (-1)^k q^{\ell\omega(k)} \sum_{n\geq 0} p(n)q^n. \tag{3.1}$$

Note that for these small primes

$$\mathcal{P}_\ell^*(z) \equiv \prod_{n\geq 0} (1-q^n)^{\ell-1} \pmod{\ell}.$$

For $\ell = 2, 3$ define the numbers $a_\ell(n)$ by

$$\sum_{n\geq 0} a_\ell(n)q^n = \mathcal{P}_\ell^*(z).$$

It follows that

$$\mathcal{P}_2^*(z) \equiv \sum_{k\in\mathbb{Z}} (-1)^k q^{\omega(k)} \pmod{2},$$

and so $a_2(N) \equiv 0 \pmod{2}$ precisely when N is not a pentagonal number. Similarly,

$$\mathcal{P}_3^*(z) \equiv \sum_{m,n\in\mathbb{Z}} (-1)^{m+n} q^{\omega(m)+\omega(n)} \pmod{3}.$$

In particular, $a_3(N) \equiv 0 \pmod{3}$ for N that do not satisfy $24N + 2 = (6m+1)^2 + (6n+1)^2$, where $m, n \in \mathbb{Z}$. Note that while it is difficult to prove anything about $p(n) \pmod{2, 3}$, in sharp contrast we have that $a_\ell(n) \equiv 0 \pmod{\ell}$ with arithmetic density one.

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