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ON SUBMULTIPLICATIVITY OF SPECTRAL RADIUS AND TRANSITIVITY OF SEMIGROUPS

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Abstract. It is shown that a transitive, closed, homogeneous semigroup of linear transformations on a finite-dimensional space either has zero divisors or is simultaneously similar to a group consisting of scalar multiples of unitary transformations. The proof begins with the result that for each closed homogeneous semigroup with no zero divisors there is a k such that the spectral radius satisfies $r(AB) \leq kr(A)r(B)$ for all A and B in the semigroup.

It is also shown that the spectral radius is not k-submultiplicative on any transitive semigroup of compact operators.

1. Introduction

We consider semigroups of linear transformations or operators, by which we simply mean collections closed under multiplication. A semigroup $\mathcal{S}$ is said to be *transitive* if $\{Ax : A \in \mathcal{S}\}$ is dense in the space for every non-zero vector x (this is sometimes called 'topological transitivity', to distinguish it from the case where $\{Ax : A \in \mathcal{S}\}$ is the entire space for each $x \neq 0$). A semigroup is said to be *irreducible* if it has no non-trivial invariant subspaces; a transitive semigroup is irreducible (but an irreducible semigroup need not be transitive).

The spectral radius r is said to be *k-submultiplicative* on a semigroup if $r(AB) \leq kr(A)r(B)$ for all A and B in the semigroup. This paper was partially stimulated by [2], which contains results relating k-submultiplicativity of spectral radius to $\frac{1}{k}$-supermultiplicativity (i.e., $r(AB) \geq \frac{1}{k}r(A)r(B)$).

We begin with several results on finite-dimensional spaces and then establish a theorem for semigroups of compact operators.

2. Semigroups of linear transformations on finite-dimensional spaces

In this section we restrict our attention to linear operators on finite-dimensional complex vector spaces.

Definition 2.1. A semigroup of linear transformations is said to be *homogeneous* if it is closed under multiplication by complex scalars. By a *closed semigroup* we mean one that is closed in the norm topology.

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Theorem 2.2. *If a closed, homogeneous semigroup has no (non-zero) zero divisors, then there exists a $k \geq 1$ such that*

$$\frac{1}{k} r(A)r(B) \leq r(AB) \leq kr(A)r(B)$$

for all A and B in the semigroup.

Proof. Suppose there is no k satisfying $r(AB) \leq kr(A)r(B)$ for all A and B. For each n, choose a pair $\{A_n, B_n\}$ in the semigroup such that

$$r(A_n B_n) > nr(A_n)r(B_n).$$

Dividing by $r(A_n)r(B_n)$ yields operators C_n and D_n with $r(C_n) = r(D_n) = 1$ and

$$r(C_n D_n) > n.$$

Thus, at least one of the sequences $\{\|C_n\|\}$ and $\{\|D_n\|\}$ is unbounded. Assume, after replacing sequences by subsequences, that

$$\|C_n\| \to \infty \text{ and } \frac{C_n}{\|C_n\|} \to R.$$

Now, $R \neq 0$ (because $\|R\| = 1$) and, by continuity of the spectral radius, $r(R) = 0$. Hence R is nilpotent, so the semigroup has zero divisors, which is a contradiction. Thus there is a k such that

$$r(AB) \leq kr(A)r(B)$$

for all A and B in the semigroup. Since the semigroup contains members A with spectral radius one, and since $r(A^2) = r(A)^2$, it follows that $k \geq 1$. The fact that

$$r(AB) \geq \frac{1}{k} r(A)r(B)$$

then follows from Theorem 2 of [2]. □

Theorem 2.3. *A transitive, homogeneous, closed semigroup that has no zero divisors is simultaneously similar to a group consisting of scalar multiples of unitary transformations.*

Proof. Let $\mathcal{S}$ be such a semigroup.

By Lemma 3.1.6 of [3], since the semigroup is irreducible, there is an idempotent E of minimal non-zero rank in $\mathcal{S}$ such that the restriction of $E\mathcal{S}E \setminus \{0\}$ to the range of E is simultaneously similar to a group of multiples of unitaries. If E is the identity operator, there is nothing more to prove. Suppose otherwise; it must then be shown that $\mathcal{S}$ contains zero divisors.

Perform a similarity so that the restriction of $E\mathcal{S}E$ to the range of E consists of multiples of unitaries. Decompose members of $\mathcal{S}$ as block matrices with respect to the ranges of E and $I - E$; each $(1, 1)$ entry in this block decomposition is a multiple of a unitary operator.

Suppose there are no zero divisors in $\mathcal{S}$. By Theorem 2.2 above, there is then a k such that

$$r(AB) \leq kr(A)r(B)$$

for all A and B in $\mathcal{S}$. We will show that the transitivity of $\mathcal{S}$ implies there is no such k.

Since $\mathcal{S}$ is transitive, there is a T other than 0 in $\mathcal{S}$ whose (1,2) block entry T_{12} is not zero. Then T's (1,1) block T_{11} is also different from zero, since otherwise ET

would be a non-trivial nilpotent member of $\mathcal{S}$. Multiply T by an appropriate scalar so that T_{11} is unitary, and call the resulting operator R. The operator ER has the block form

$$\begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix}$$

with R_{11} unitary and $R_{12} \neq 0$. Choose a vector g in the range of $I - E$ such that $R_{12}g \neq 0$.

Let s and t be positive numbers, to be specified later. Fix any unit vector f in the range of E. By the transitivity of $\mathcal{S}$, we can find S in $\mathcal{S}$ so that Sf is within t of sg. Then

$$\|S_{11}f\| < t \text{ and } \|S_{21}f - sg\| < t.$$

Since f is a unit vector and S_{11} is a multiple of a unitary operator, it follows that $\|S_{11}\| < t$.

Let $A = SE$ and $B = ER$. Then $r(A) = r(S_{11}) < t$ and $r(B) = r(U_{11}) = 1$. Also,

$$BA = ERSE = \begin{pmatrix} R_{11}S_{11} + R_{12}S_{21} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$BAf = (R_{11}S_{11} + R_{12}S_{21})f.$$

By choosing t sufficiently small, BAf can be made as close to $sR_{12}g$ as desired (by the above inequalities). Thus the norm of BA may be made arbitrarily large by choosing s sufficiently large.

This leads to a contradiction, as follows: By the above, $r(A) < t$ and $r(B) = 1$. Note that $r(BA) = \|BA\|$, since the $(1,1)$ block of BA is a multiple of a unitary operator and the other entries are 0. Thus choosing s sufficiently large yields

$$r(BA) > kr(B)r(A). \qquad \square$$

Theorem 2.4. *If $\mathcal{S}$ is a transitive semigroup of linear transformations on which the spectral radius is k-submultiplicative for some k, then there is an invertible transformation A such that $\{A^{-1}SA : S \in \mathcal{S}\}$ consists of multiples of unitary transformations.*

Proof. Let $\mathcal{J}$ be the closure of $\mathbb{C}\mathcal{S}$. Note that continuity of the spectral radius implies that $\mathcal{J}$ also satisfies the hypothesis. Assume that the spectral radius is not multiplicative. Then Theorem 2.3 together with this assumption implies that there are non-zero A and B in $\mathcal{S}$ with $AB = 0$. Thus the set $B\mathcal{S}A$ consists of operators X with $X^2 = 0$. Since $B\mathcal{S}A \neq \{0\}$ (by the irreducibility of $\mathcal{S}$), the set $\mathcal{N}$ of nilpotent members of $\mathcal{S}$ is different from $\{0\}$. Now the k-submultiplicativity of spectral radius on $\mathcal{S}$ implies that $\mathcal{N}$ is a semigroup ideal of $\mathcal{S}$. Since $\mathcal{N}$ is triangularizable by Levitzki's Theorem ([1], Theorem 2.1.7 of [3]), and since every non-trivial ideal of an irreducible semigroup is irreducible (see Lemma 2.1.10 of [3]), this is the desired contradiction. $\square$

Corollary 2.5. *If the spectral radius is k-submultiplicative on a transitive semigroup of linear transformations, then the spectral radius is multiplicative on the semigroup.*

Proof. This follows immediately from Theorem 2.4. $\square$

As shown in [2], for a closed homogeneous semigroup $\mathcal{S}$ with no zero divisors and a number $k \geq 1$, the following statements are equivalent:

(1) $r(AB) \leq kr(A)r(B)$ for all A and B in $\mathcal{S}$,
(2) $r(AB) \geq \frac{1}{k}r(A)r(B)$ for all A and B in $\mathcal{S}$.

The following result shows that for every k, there is a semigroup 'sharply' satisfying (1) and (2). (An example with $k = 4$ is contained in [2].)

Proposition 2.6. *Let $m \geq 1$ be given. There is an irreducible semigroup $\mathcal{S}$ with no zero divisors that satisfies* (1) *and* (2) *above for $k = m$ but does not satisfy those inequalities for any $k < m$.*

Proof. For a non-zero column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \text{ and a positive number } v$$

we say that x is of *variation* v if

$$|\frac{x_i}{x_j}| \leq v$$

for all i and j. (Note that this implies $v \geq 1$ and $x_i \neq 0$ for all i.)

Let χ_v be the set of all column vectors of variation v. It is easily seen that χ_v is a closed set in $\mathbb{C}^n$ and $\alpha x \in \chi_v$ if $x \in \chi_v$ and $0 \neq \alpha \in \mathbb{C}$. Now define $\mathcal{S}_v$ to be the set of all matrices with exactly one column from χ_v whose other columns are all zero.

Each member of $\mathcal{S}_v$ is of the form xe_j^*, where $x \in \chi_v$ and where $\{e_j\}$ are the basis unit columns (with the j-th component of e_j equal to 1). If x and y are members of χ_v and i, j are integers in $[1, n]$, then

$$(xe_i^*)(ye_j^*) = (e_i^* y)xe_j^*.$$

This shows that $\mathcal{S}_v$ is a semigroup. Since

$$r(xe_i^*) = |x_i| = |e_i^* x|,$$

where x_i is the i-th component of x, we have

$$\begin{aligned} r((xe_i^*)(ye_j^*)) &= |e_i^* y| \cdot |x_j| = |y_i| \cdot |x_j| \\ &= |\frac{y_i}{y_j}| \cdot |\frac{x_j}{x_i}| \cdot |x_i| \cdot |y_j| \\ &\leq v^2 r(xe_i^*)r(ye_j^*), \end{aligned}$$

which shows that $\mathcal{S}$ satisfies (1) above with $k = v^2$. Thus (2) is also satisfied since $\mathcal{S}$ contains no zero divisors. It is easy to see that no $k < v^2$ would do: just consider

$$A = \begin{pmatrix} 1 & 0 & 0 & . & . & 0 \\ v & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & 0 & . & . & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & v & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 1 & 0 & . & . & 0 \end{pmatrix}. \tag{1}$$

Then $r(A) = r(B) = 1$, and $r(AB) = v^2$. □

3. Semigroups of compact operators

One of the above theorems can be extended to semigroups of compact operators on Hilbert spaces.

Theorem 3.1. *If $\mathcal{S}$ is a transitive semigroup of bounded linear operators on a Hilbert space, and if $\mathcal{S}$ contains a compact operator other than 0, then the spectral radius is not k-submultiplicative on $\mathcal{S}$ for any k.*

Proof. Suppose there is a k such that

$$r(AB) \leq kr(A)r(B)$$

for all A and B in $\mathcal{S}$. Since a non-zero semigroup ideal of a transitive semigroup is transitive, we can and do assume that $\mathcal{S}$ consists of compact operators. Turovski's Theorem ([4], [3], Theorem 8.1.11) states that a semigroup of quasinilpotent compact operators has a non-trivial invariant subspace, so $\mathcal{S}$ must contain operators that are not quasinilpotent.

Enlarge $\mathcal{S}$ by including all scalar multiples of the members of $\mathcal{S}$ and then taking the uniform closure; let $\mathcal{J}$ be the enlarged semigroup. Since spectral radius is continuous on the collection of compact operators,

$$r(AB) \leq kr(A)r(B)$$

for $A, B \in \mathcal{J}$. By Corollary 8.1.12 of [3], $\mathcal{J}$ contains finite-rank operators other than 0, and Lemma 8.1.15 of [3] yields a finite-rank idempotent E in $\mathcal{J}$ of minimal non-zero rank.

It follows from the finite-dimensional case that the restriction of $E\mathcal{J}E \setminus \{0\}$ to the range of E is simultaneously similar to a subgroup of multiples of unitary operators.

Now the proof proceeds along the lines of the last part of the proof of Theorem 2.3 above. Since E is of finite rank, and thus is not the identity operator, the transitivity of $\mathcal{J}$ contradicts the k-submultiplicativity of the spectral radius. □

Corollary 3.2. *If $\mathcal{S}$ is a semigroup of bounded linear operators on a Hilbert space that contains a compact operator other than 0, and if the spectral radius is k-submultiplicative on $\mathcal{S}$ for some k, then there is a closed subset of the Hilbert space that is invariant under all the operators in $\mathcal{S}$ and is different from $\{0\}$ and the entire space.*

Proof. If $\mathcal{S}$ did not have such a proper, closed, invariant set, then $\mathcal{S}$ would be transitive, contradicting the previous theorem. □

Proposition 3.3. *Let $k \geq 1$. There is an irreducible semigroup of finite-rank operators on l^2 which has no zero divisors and satisfies*

$$\frac{1}{k}r(A)r(B) \leq r(AB) \leq kr(A)r(B)$$

for all A and B in the semigroup.

Proof. The example $\mathcal{S}_v$ of Proposition 2.6 with $v = \sqrt{k}$ does not occur in infinite dimensions, since l^2 contains no vectors of any fixed variation. There are known semigroups of finite-rank operators, with no zero divisors on l^2, with multiplicative spectral radius. Let $\mathcal{J}$ be such a semigroup (e.g. see Example 8.6.6 of [3]). Let $\mathcal{S}_v$

be the example of Proposition 2.6, with $v = \sqrt{k}$ on a space of dimension 2. Now let $\mathcal{S} = \mathcal{S}_v \otimes \mathcal{J}$ (on $l^2 \oplus l^2$, identified with l^2).

The irreducibility of $\mathcal{S}$ and its lack of zero divisors follow from the same properties of $\mathcal{S}_v$ and $\mathcal{J}$. All that is left to prove is that if $\mathcal{S}_i$, $i = 1, 2$ are semigroups satisfying

$$\frac{1}{k_i} r(A)r(B) \leq r(AB) \leq k_i r(A)r(B)$$

for all A and B in $\mathcal{S}_i$, $i = 1, 2$, then the members of $\mathcal{S}_1 \otimes \mathcal{S}_2$ satisfy

$$\frac{1}{k_1 k_2} r(A)r(B) \leq r(AB) \leq k_1 k_2 r(A)r(B).$$

But since $r(S \otimes T) = r(S)r(T)$ for all S and T, we have, for A_1, B_1 in $\mathcal{S}_1$ and $A_2, B_2 \in \mathcal{S}_2$, that

$$\begin{aligned} r((A_1 \otimes A_2)(B_1 \otimes B_2)) &= r(A_1B_1 \otimes A_2B_2) = r(A_1B_1) \cdot r(A_2B_2) \\ &\leq k_1 r(A_1)r(B_1) \cdot k_2 r(A_2)r(B_2) \\ &= k_1 k_2 r(A_1)r(A_2) \cdot r(B_1)r(B_2) \\ &= k_1 k_2 r(A_1 \otimes A_2) \cdot r(B_1 \otimes B_2). \end{aligned}$$

The inequality involving $\frac{1}{k_1 k_2}$ can be similarly established. □

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