DENSELY ALGEBRAIC BOUNDS
FOR THE EXPONENTIAL FUNCTION

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Abstract. An upper bound for $e^x$ that implies the inequality between the arithmetic and geometric means is generalized with the introduction of a new parameter $n$. The new upper bound is smoothly and densely algebraic in $n$, and valid for $-b < x < 1$ for arbitrarily large positive $b$ provided that $n (>1)$ is sufficiently close to 1. The range of its validity for negative $x$ is investigated through the study of a certain family of quadrinomials.

1. Introduction

In §4.2 of the classical treatise [1] the inequality between the arithmetic and geometric means is deduced from

$$1 + x \leq e^x.$$ 

This is the proof of “Pólya’s dream” [5]. With a change of variable this can be rewritten as

$$(1.1) \quad e^x \leq \frac{1}{1 - x}$$

for $x < 1$. In this paper, we shall establish the following generalization of which (1.1) is the case $n = 1$. For convenience, we let

$$U(n, x) = 1 - \frac{1}{n} + \frac{1}{n} \left( 1 + \frac{1 - \frac{1}{n}}{1 - \frac{x}{n}} \right)^n.$$ 

Theorem 1.1. For real $n \geq 1$ and

$$-\frac{n}{n-1} < x < n,$$

we have

$$e^x \leq U(n, x)$$

with equality if and only if $x = 0$. Moreover, for $0 \leq x < 1$ and $1 \leq n \leq 2$ we have

$$e^x \leq U(n, x) \leq \frac{1}{1 - x},$$
and for $x < 0$ and $0 < n \leq 1$ we have

$$e^x \leq \frac{1}{1 - x} \leq U(n, x).$$

Here $U(n, x)$ is smooth and densely algebraic in $n$ in the sense that it is an algebraic function of $x$ whenever $n$ is rational and this algebraic function changes by arbitrarily small amounts on compact sets for sufficiently small rational changes in $n$.

Another upper bound for $e^x$ that generalizes (1.1) is Karamata’s [2]

$$e^x \leq \sum_{k=0}^{n-1} \frac{x^k}{k!} + \frac{x^n}{n!} \frac{n}{n - x},$$

provided $n$ is a positive integer and $0 \leq x < n$. This is tighter for $0 < x < n$ but often fails for $x < 0$. Also, since $n - 1$ is the upper limit of the summation here, it is not smoothly algebraic in $n$. Another tighter bound for $e^x$ is Sewell’s [4]

$$e^x \leq \left(1 + \frac{x}{n}\right)^{n+1}$$

for $n$ a positive integer and $x \geq 0$. This is not algebraic in $x$, and can fail for $x < 0$.

The change of variable given by replacing $x$ with

$$\frac{n(x-1)}{n + x - 1}$$

plays an important role here. In fact, it is immediate that the first part of Theorem 1.1 is equivalent to

**Theorem 1.2.** For real $n \geq 1$ and $x > 0$ we have

$$\exp\left(\frac{n(x-1)}{n + x - 1}\right) \leq \frac{n-1 + x^n}{n}$$

with equality if and only if $x = 1$.

Our proof for Theorem 1.2 will be in the spirit of §2.15 of [1] where some “fundamental inequalities” that also lead to the inequality between the arithmetic and geometric means, including

$$x^r - 1 > r(x - 1), \quad r > 1, \quad x > 0, \quad x \neq 1,$$

are proved for all real $r > 1$ by first establishing them for rational numbers and then taking limits. Also the proof of Theorem 1.2 will rely on the polynomial

$$K(p, q, x) := q^2(x^{p+q} - x^p) + q(p-q)(x^p - x^{p-q}) + p(p-q)(x^{p-q} - 1),$$

where both $p$ and $q$ are integers and $p > q$. For our convenience, we write $K(x) = K(p, q, x)$.

For a survey of rational bounds for $e^x$ see pp. 266-270 of [3]. The examination of the inequalities between $(1 - x)^{-1}$ and $U(n, x)$ from the point of view of their power series expansions leads to questions about a certain sequence of polynomials; see §3. We remark here that for $0 \leq x < 1$ the power series of $U(n, x) - (1 - x)^{-1}$ about $x = 0$ is

$$\frac{(n-1)(n-2)}{2n}x^2 + \frac{1}{6}\left(2 + \frac{6}{n^2} - \frac{6}{n} - 3n + n^2\right)x^3 + \cdots.$$
So for \( n > 2 \) and small \( x > 0 \) we have \( U(n, x) > (1 - x)^{-1} \). For \( 0 \leq x < 1 \) and \( n \) large some simple asymptotics (details omitted) show that \((1 - x)^{-1} < U(n, x)\) for \( x \leq 1 - c^n \) for a fixed \( c > 1/2 \), while \( U(n, x) < (1 - x)^{-1} \) for \( x \geq 1 - c^n \) for a fixed \( c \leq 1/2 \).

2. Proofs

We begin with a lemma that leads to the inequalities between \((1 - x)^{-1}\) and \( U(n, x) \), and then proceed to the inequalities between \( e^x \) and \( U(n, x) \). This latter inequality is of course immediate when \( x < 1 \) and \((1 - x)^{-1} \leq U(n, x)\).

**Lemma 2.1.** (a) Let \( c = 1 - 1/n \), where \( 1 \leq n \leq 2 \) and \( 0 \leq x < 1 \). Then
\[
  l_1 := \frac{nc}{1 + cx} + \frac{1}{1 - \frac{c}{n}} \leq \frac{nc}{1 + ncx} + \frac{1}{1 - x} =: r_1.
\]
(b) Let \( 0 < a \leq b, 1 \leq b, \) and \( x \geq 0 \). Then
\[
  l_2 := \frac{a}{1 + \frac{2}{b}x} + \frac{b}{1 + bx} \leq \frac{a}{1 + ax} + \frac{b}{1 + x} =: r_2.
\]

**Proof.** For (a) we have
\[
  r_1 - l_1 = \frac{(n-1)(1+cn)x(2-n+cx(1+n))}{(x-1)(x-n)(1+cx)(1+nx)} \geq 0,
\]
while for (b) we have
\[
  r_2 - l_2 = \frac{(b-1)(b-a)x(a+b+a(1+b)x)}{(1+x)(1+ax)(b+ax)(bx+1)} \geq 0.
\]

\[\square\]

We now proceed to the right side of the second part of Theorem 1.1. Observe that
\[
  n \log (1 + cx) - n \log \left(1 - \frac{x}{n}\right) \leq \log (1 + ncx) - \log (1 - x)
\]
since there is equality when \( x = 0 \), and the corresponding inequality between the derivatives of each side follows from (a) of Lemma 2.1. Hence
\[
  \left(\frac{1+cx}{1-\frac{c}{n}}\right)^n \leq \frac{1+ncx}{1-x}
\]
and we obtain \( U(n, x) \leq 1/(1-x) \). For the right side of the third part of Theorem 1.1 a similar argument using (b) of Lemma 2.1 yields
\[
  \left(\frac{1+\frac{2}{b}x}{1+x}\right)^b \leq \frac{1+ax}{1+bx}.
\]
Here we may take \( b = \frac{1}{n} \) and \( a = \frac{1}{n} - 1 \) for \( 0 < n \leq 1 \), so
\[
  \frac{1-(n-1)x}{1+x} \leq \left(\frac{1-(1-\frac{1}{n})x}{1+\frac{c}{n}}\right)^n.
\]
Upon replacing \( x \) by \(-x\) (so that \( x \leq 0 \)), we obtain
\[
  \frac{1}{1-x} \leq U(n, x).
\]
For the proof of Theorem 1.2 (and hence of the remaining first part of Theorem 1.1) we observe that (1.2) is equivalent to

\[(2.1) \quad g(x) := \frac{n(x-1)}{n+x-1} \leq \log \left( \frac{n-1+x^n}{n} \right) =: f(x). \]

Since both sides of (2.1) are zero when \(x = 1\) we may apply the following lemma (proof omitted) to reduce it to an inequality not involving transcendental functions.

**Lemma 2.2.** Let \(f(x)\) and \(g(x)\) be differentiable functions on a finite or infinite interval \(I\) containing 1 such that \(f(1) = g(1)\), and such that \(g'(x) \geq f'(x)\) for \(x < 1\) and \(g'(x) \leq f'(x)\) for \(x > 1\). Then \(g(x) \leq f(x)\).

Now

\[ g'(x) = \frac{n^2}{(n - 1 + x)^2} \quad \text{and} \quad f'(x) = \frac{n x^{n-1}}{n - 1 + x}. \]

Replace \(n\) by \(p/q\) where both \(p\) and \(q\) are positive integers, \(p > q\). The change of variable \(x\) by \(x^s\) takes 1 to 1 and \((0, \infty)\) to \((0, \infty)\). To verify the hypothesis of Lemma 2.2 we need to show that \(H(x)\) has the same sign as \((x-1)\), where

\[ H(x) := H(p, q, x) := f'(x) - g'(x) \]
\[ = \frac{p x^{p-q}}{p + q(x^p - 1)} - \frac{p^2}{(p + q(x^p - 1))^2} \]
\[ = \frac{p K(x)}{(p - q + q x^p)(p - q + q x^q)^2} \]

and

\[ K(x) = q^2(x^{p+q} - x^p) + q(p - q)(x^p - x^{p-q}) + p(p - q)(x^{p-q} - 1). \]

Using the identity

\[(2.2) \quad x^s - x^t = \left( \frac{x^s - 1}{s - 1} - \frac{x^t - 1}{t - 1} \right) (x - 1) \]

for \(s \geq t \geq 0\) and the expansion of the terms in (2.2) into geometric series, we see that \(K(x)\) is the product of \(x-1\) with polynomials in \(x\), all of whose coefficients are nonnegative. Hence \(K(x)\) has the same sign as \(x-1\). The inequality of the theorem for rational \(n\) now follows from Lemma 2.2. For real \(n \geq 1\), it follows by letting \(p/q \to n\) where \(p\) and \(q\) run through sequences of integers such that \(p > q \geq 1\). The strict inequality for \(x \neq 1\) follows from the fact that the functions in \(x\)

\[ \exp \left( \frac{n(x-1)}{n+x-1} \right) \quad \text{and} \quad \frac{n-1+x^n}{n} \]

are strictly increasing for \(x > 0\).

3. A SEQUENCE OF POLYNOMIALS

It is possible that the \(U(n, x) \leq (1 - x)^{-1}\) inequality for \(1 \leq n \leq 2\) can be strengthened to an inequality between the corresponding power series coefficients. In fact, we can make a stronger conjecture. Write

\[ \frac{1}{n-1} \left( U(n, x) - (1 - x)^{-1} \right) = \sum_{k=2}^{\infty} \frac{P_k(n) x^k}{k! n^{k-1}}. \]
Here $P_2(n) = n - 2$, $P_3(n) = n^3 - 2n^2 - 6$ and
\[ P_4(n) = n^5 - 5n^4 + 18n^3 - 48n^2 + 12n - 24. \]

The unique real zeros of $P_2$, $P_3$ and $P_4$ are $n = 2$, $n = 2.7776 \cdots$ and $n = 3.5934 \cdots$, respectively. We conjecture that each $P_k(n)$ for $k \geq 4$ is a monic polynomial of degree $2k - 3$ whose coefficients alternate in sign, and has a unique real root $r_k$ that exceeds the real part of every other root of $P_k(n)$. Moreover,
\[ 0 < r_{k+1} - r_k < 1 \quad \text{and} \quad \lim_{k \to \infty} (r_{k+1} - r_k) = 1. \]

It also seems that $k!$ divides $P_k(2)$.

To describe qualitatively the conjectural zero distribution of $P_k(n)$ we employ polar coordinates $r$ and $\phi$ to describe a certain curve $\theta$. It is the cardioid $H$ given by $r = 1 + \cos \phi$ together with that part of the circle $C$ defined by $r = 1/4$ that lies outside of $H$. The left vertical tangent to $C$, call it $T$, is tangent to the cardioid at two points, and (much more crudely) the $\theta$-curve is topologically equivalent to the Greek letter $\theta$. Note that the cusp of $H$ is inside of the circle $C$. The conjecture is that the zeros of $P_k(n)$ for $k$ large lie on or very close to a curve similar (in the non-technical sense) to $\theta$ that has the imaginary axis as a line of triple tangency analogous to $T$. Also, about $1/3$ of the zeros lie on or inside the part of the $\theta$-curve that corresponds to the union of its $C$ part with that part of $H$ that lies inside of the full circle $C$.

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