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DENSELY ALGEBRAIC BOUNDS FOR THE EXPONENTIAL FUNCTION

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Abstract. An upper bound for e^x that implies the inequality between the arithmetic and geometric means is generalized with the introduction of a new parameter n. The new upper bound is smoothly and densely algebraic in n, and valid for $-b < x < 1$ for arbitrarily large positive b provided that n (> 1) is sufficiently close to 1. The range of its validity for negative x is investigated through the study of a certain family of quadrinomials.

1. Introduction

In §4.2 of the classical treatise [1] the inequality between the arithmetic and geometric means is deduced from

$$1 + x \leq e^x.$$

This is the proof of "Pólya's dream" [5]. With a change of variable this can be rewritten as

$$e^x \leq \frac{1}{1-x} \tag{1.1}$$

for $x < 1$. In this paper, we shall establish the following generalization of which (1.1) is the case $n = 1$. For convenience, we let

$$U(n,x) = 1 - \frac{1}{n} + \frac{1}{n}\left(\frac{1 + \left(1 - \frac{1}{n}\right)x}{1 - \frac{x}{n}}\right)^n.$$

Theorem 1.1. *For real $n \geq 1$ and*

$$-\frac{n}{n-1} < x < n,$$

we have

$$e^x \leq U(n,x)$$

with equality if and only if $x = 0$. Moreover, for $0 \leq x < 1$ and $1 \leq n \leq 2$ we have

$$e^x \leq U(n,x) \leq \frac{1}{1-x},$$

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and for $x < 0$ *and* $0 < n \leq 1$ *we have*

$$e^x \leq \frac{1}{1-x} \leq U(n,x).$$

Here $U(n,x)$ is smoothly and densely algebraic in n in the sense that it is an algebraic function of x whenever n is rational and this algebraic function changes by arbitrarily small amounts on compact sets for sufficiently small rational changes in n.

Another upper bound for e^x that generalizes (1.1) is Karamata's [2]

$$e^x \leq \sum_{k=0}^{n-1} \frac{x^k}{k!} + \frac{x^n}{n!}\frac{n}{n-x},$$

provided n is a positive integer and $0 \leq x < n$. This is tighter for $0 < x < n$ but often fails for $x < 0$. Also, since $n-1$ is the upper limit of the summation here, it is not smoothly algebraic in n. Another tighter bound for e^x is Sewell's [4]

$$e^x \leq \left(1 + \frac{x}{n}\right)^{n+(x/2)}$$

for n a positive integer and $x \geq 0$. This is not algebraic in x, and can fail for $x < 0$.

The change of variable given by replacing x with

$$\frac{n(x-1)}{n+x-1}$$

plays an important role here. In fact, it is immediate that the first part of Theorem 1.1 is equivalent to

Theorem 1.2. *For real* $n \geq 1$ *and* $x > 0$ *we have*

$$\exp\left(\frac{n(x-1)}{n+x-1}\right) \leq \frac{n-1+x^n}{n} \tag{1.2}$$

with equality if and only if $x = 1$.

Our proof for Theorem 1.2 will be in the spirit of §2.15 of [1] where some "fundamental inequalities" that also lead to the inequality between the arithmetic and geometric means, including

$$x^r - 1 > r(x-1), \quad r > 1, \ x > 0, \ x \neq 1,$$

are proved for all real $r > 1$ by first establishing them for rational numbers and then taking limits. Also the proof of Theorem 1.2 will rely on the polynomial

$$K(p,q,x) := q^2(x^{p+q} - x^p) + q(p-q)(x^p - x^{p-q}) + p(p-q)(x^{p-q} - 1),$$

where both p and q are integers and $p > q$. For our convenience, we write $K(x) = K(p,q,x)$.

For a survey of rational bounds for e^x see pp. 266-270 of [3]. The examination of the inequalities between $(1-x)^{-1}$ and $U(n,x)$ from the point of view of their power series expansions leads to questions about a certain sequence of polynomials; see §3. We remark here that for $0 \leq x < 1$ the power series of $U(n,x) - (1-x)^{-1}$ about $x = 0$ is

$$\frac{(n-1)(n-2)}{2n}x^2 + \frac{1}{6}\left(2 + \frac{6}{n^2} - \frac{6}{n} - 3n + n^2\right)x^3 + \cdots.$$

So for $n > 2$ and small $x > 0$ we have $U(n,x) > (1-x)^{-1}$. For $0 \le x < 1$ and n large some simple asymptotics (details omitted) show that $(1-x)^{-1} < U(n,x)$ for $x \le 1 - c^n$ for a fixed $c > 1/2$, while $U(n,x) < (1-x)^{-1}$ for $x \ge 1 - c^n$ for a fixed $c \le 1/2$.

2. Proofs

We begin with a lemma that leads to the inequalities between $(1-x)^{-1}$ and $U(n,x)$, and then proceed to the inequalities between e^x and $U(n,x)$. This latter inequality is of course immediate when $x < 1$ and $(1-x)^{-1} \le U(n,x)$.

Lemma 2.1. (a) *Let $c = 1 - 1/n$, where $1 \le n \le 2$ and $0 \le x < 1$. Then*

$$l_1 := \frac{nc}{1+cx} + \frac{1}{1-\frac{x}{n}} \le \frac{nc}{1+ncx} + \frac{1}{1-x} =: r_1.$$

(b) *Let $0 < a \le b$, $1 \le b$, and $x \ge 0$. Then*

$$l_2 := \frac{a}{1+\frac{a}{b}x} + \frac{b}{1+bx} \le \frac{a}{1+ax} + \frac{b}{1+x} =: r_2.$$

Proof. For (a) we have

$$r_1 - l_1 = \frac{(n-1)(1+cn)x(2-n+cx(1+n))}{(x-1)(x-n)(1+cx)(1+cnx)} \ge 0,$$

while for (b) we have

$$r_2 - l_2 = \frac{(b-1)(b-a)x(a+b+a(1+b)x)}{(1+x)(1+ax)(b+ax)(bx+1)} \ge 0.$$

□

We now proceed to the right side of the second part of Theorem 1.1. Observe that

$$n\log(1+cx) - n\log\left(1-\frac{x}{n}\right) \le \log(1+ncx) - \log(1-x)$$

since there is equality when $x = 0$, and the corresponding inequality between the derivatives of each side follows from (a) of Lemma 2.1. Hence

$$\left(\frac{1+cx}{1-\frac{x}{n}}\right)^n \le \frac{1+ncx}{1-x}$$

and we obtain $U(n,x) \le 1/(1-x)$. For the right side of the third part of Theorem 1.1 a similar argument using (b) of Lemma 2.1 yields

$$\left(\frac{1+\frac{a}{b}x}{1+x}\right)^b \le \frac{1+ax}{1+bx}.$$

Here we may take $b = \frac{1}{n}$ and $a = \frac{1}{n} - 1$ for $0 < n \le 1$, so

$$\frac{1-(n-1)x}{1+x} \le \left(\frac{1-\left(1-\frac{1}{n}\right)x}{1+\frac{x}{n}}\right)^n.$$

Upon replacing x by $-x$ (so that $x \le 0$), we obtain

$$\frac{1}{1-x} \le U(n,x).$$

For the proof of Theorem 1.2 (and hence of the remaining first part of Theorem 1.1) we observe that (1.2) is equivalent to

$$g(x) := \frac{n(x-1)}{n+x-1} \le \log\left(\frac{n-1+x^n}{n}\right) =: f(x). \tag{2.1}$$

Since both sides of (2.1) are zero when $x = 1$ we may apply the following lemma (proof omitted) to reduce it to an inequality not involving transcendental functions.

Lemma 2.2. *Let $f(x)$ and $g(x)$ be differentiable functions on a finite or infinite interval I containing 1 such that $f(1) = g(1)$, and such that $g'(x) \ge f'(x)$ for $x < 1$ and $g'(x) \le f'(x)$ for $x > 1$. Then $g(x) \le f(x)$.*

Now

$$g'(x) = \frac{n^2}{(n-1+x)^2} \qquad \text{and} \qquad f'(x) = \frac{nx^{n-1}}{n-1+x^n}.$$

Replace n by p/q where both p and q are positive integers, $p > q$. The change of variable x by x^q takes 1 to 1 and $(0,\infty)$ to $(0,\infty)$. To verify the hypothesis of Lemma 2.2 we need to show that $H(x)$ has the same sign as $(x-1)$, where

$$\begin{aligned} H(x) := H(p,q,x) &:= f'(x) - g'(x) \\ &= \frac{px^{p-q}}{p+q(x^p-1)} - \frac{p^2}{(p+q(x^q-1))^2} \\ &= \frac{pK(x)}{(p-q+qx^p)(p-q+qx^q)^2} \end{aligned}$$

and

$$K(x) = q^2(x^{p+q} - x^p) + q(p-q)(x^p - x^{p-q}) + p(p-q)(x^{p-q} - 1).$$

Using the identity

$$x^s - x^t = \left(\frac{x^s-1}{x-1} - \frac{x^t-1}{x-1}\right)(x-1) \tag{2.2}$$

for $s \ge t \ge 0$ and the expansion of the terms in (2.2) into geometric series, we see that $K(x)$ is the product of $x-1$ with polynomials in x, all of whose coefficients are nonnegative. Hence $K(x)$ has the same sign as $x-1$. The inequality of the theorem for rational n now follows from Lemma 2.2. For real $n \ge 1$, it follows by letting $p/q \to n$ where p and q run through sequences of integers such that $p > q \ge 1$. The strict inequality for $x \ne 1$ follows from the fact that the functions in x

$$\exp\left(\frac{n(x-1)}{n+x-1}\right) \quad \text{and} \quad \frac{n-1+x^n}{n}$$

are strictly increasing for $x > 0$.

3. A sequence of polynomials

It is possible that the $U(n,x) \le (1-x)^{-1}$ inequality for $1 \le n \le 2$ can be strengthened to an inequality between the corresponding power series coefficients. In fact, we can make a stronger conjecture. Write

$$\frac{1}{n-1}\left(U(n,x) - (1-x)^{-1}\right) = \sum_{k=2}^{\infty} \frac{P_k(n)x^k}{k!n^{k-1}}.$$

Here $P_2(n) = n - 2$, $P_3(n) = n^3 - 2n^2 - 6$ and

$$P_4(n) = n^5 - 5n^4 + 18n^3 - 48n^2 + 12n - 24.$$

The unique real zeros of P_2, P_3 and P_4 are $n = 2$, $n = 2.7776\cdots$ and $n = 3.5934\cdots$, respectively. We conjecture that each $P_k(n)$ for $k \geq 4$ is a monic polynomial of degree $2k-3$ whose coefficients alternate in sign, and has a unique real root r_k that exceeds the real part of every other root of $P_k(n)$. Moreover,

$$0 < r_{k+1} - r_k < 1 \qquad \text{and} \qquad \lim_{k\to\infty} (r_{k+1} - r_k) = 1.$$

It also seems that $k!$ divides $P_k(2)$.

To describe qualitatively the conjectural zero distribution of $P_k(n)$ we employ polar coordinates r and ϕ to describe a certain curve θ. It is the cardioid H given by $r = 1 + \cos\phi$ together with that part of the circle C defined by $r = 1/4$ that lies outside of H. The left vertical tangent to C, call it T, is tangent to the cardioid at two points, and (much more crudely) the θ-curve is topologically equivalent to the Greek letter θ. Note that the cusp of H is inside of the circle C. The conjecture is that the zeros of $P_k(n)$ for k large lie on or very close to a curve similar (in the non-technical sense) to θ that has the imaginary axis as a line of triple tangency analogous to T. Also, about $1/3$ of the zeros lie on or inside the part of the θ-curve that corresponds to the union of its C part with that part of H that lies inside of the full circle C.

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