REMARK ON “A PROBLEM OF PRESCRIBING GAUSSIAN CURVATURE ON $S^2$”
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Abstract. In this note, we remark on a 2001 paper of S. Goyal and V. Goyal. The main result of this work is that they used some elementary method to find a class of functions $K(x) = K(x_1, x_2, x_3)$ for which the solutions to
\[ \Delta u + K(x)e^{2u} = 1 \]
on $S^2$ can be obtained. We observe that this class of functions that they studied is actually the trivial one, i.e. the class of positive constant functions.

1. Statement of the main result in [1]

For clarity, we first state the main result of above-mentioned paper.

Theorem 1.0.1. On the 2-sphere $(S^2, g_0)$ with standard metric, if the function $K(x) = K(x_1, x_2, x_3)$ is positive and $\ln(K(x))$ is harmonic, then $u = \frac{1}{2}\ln\left(\frac{1}{K(x)}\right)$ is a solution of
\[ \Delta_{g_0} u + K(x)e^{2u} = 1. \]

2. Remark

Given an $n$-dimensional Riemannian manifold $(M^n, g)$, the Laplace operator $\Delta$ on functions is well known. Taking an orthonormal frame \{\(E_i\)\} on $(M^n, g)$, we can define the Laplace operator acting on functions $f$ as
\[ \Delta f = \sum_{i=1}^{n} \nabla^2_{E_i, E_i} f, \]
where $\nabla^2 f$ is the Hessian of the function $f$.

In local coordinates \(\{x^i\}\),
\[ \Delta = \frac{1}{\sqrt{\text{det}(g)}} \frac{\partial}{\partial x^i} \left( \sqrt{\text{det}(g)} g^{ij} \frac{\partial}{\partial x^j} \right). \]

It is immediate from the above definition that $\Delta$ is an elliptic operator. This is the analyst’s definition. On the other hand, geometers usually consider the
Laplace operator on functions \( \tilde{\Delta} \) as a positive operator, i.e. they have non-negative eigenvalues. This is often called Hodge Laplacian on functions:

\[
\tilde{\Delta} = \delta d,
\]
where \( d \) is the exterior differentiation and \( \delta \) is the Hodge dual of \( d \).

It can easily be shown that these two definitions are only different by a sign, i.e. \( \Delta = -\tilde{\Delta} \). In [1] and this note, the Laplace operator is considered as an elliptic operator.

We observe that if \( K(x) > 0 \) and \( \Delta \ln(K(x)) = 0 \), they \( K \) must be constant functions.

**Proof.**

\[
\Delta \ln K = \frac{\Delta K}{K} - \frac{|\nabla K|^2}{K^2}.
\]

\( \ln K \) is harmonic implies that

\[
\frac{\Delta K}{K} = \frac{|\nabla K|^2}{K^2}.
\]

Since \( K > 0 \), multiplying both sides by \( K^2 \), we get

\[
K \Delta K = |\nabla K|^2.
\]

Integrating both sides on \( S^2 \) using standard volume form, we get

\[
\int_{S^2} K \Delta K d\text{vol}(g_0) = \int_{S^2} |\nabla K|^2 d\text{vol}(g_0).
\]

Now integration by parts shows that

\[
- \int_{S^2} |\nabla K|^2 d\text{vol}(g_0) = \int_{S^2} |\nabla K|^2 d\text{vol}(g_0).
\]

This in turn shows that

\[
|\nabla K| \equiv 0.
\]

We conclude that \( K(x) \) must be a constant function. This shows that the class of functions for which to prescribe Gaussian curvature in [1] is trivial. □

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**REFERENCES**


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