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# REMARK ON "A PROBLEM OF PRESCRIBING GAUSSIAN CURVATURE ON $S^2$" [PROC. AMER. MATH. SOC. 129 (2001), No. 12, 3757–3758]

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Abstract. In this note, we remark on a 2001 paper of S. Goyal and V. Goyal. The main result of this work is that they used some elementary method to find a class of functions $K(x) = K(x_1, x_2, x_3)$ for which the solutions to

$$\Delta u + K(x)e^{2u} = 1$$

on $S^2$ can be obtained. We observe that this class of functions that they studied is actually the trivial one, i.e. the class of positive constant functions.

## 1. Statement of the main result in [1]

For clarity, we first state the main result of above-mentioned paper.

**Theorem 1.0.1.** *On the 2-sphere $(S^2, g_0)$ with standard metric, if the function $K(x) = K(x_1, x_2, x_3)$ is positive and $ln(K(x))$ is harmonic, then $u = \frac{1}{2}ln(\frac{1}{K(x)})$ is a solution of*

$$\Delta_{g_0} u + K(x)e^{2u} = 1.$$

## 2. Remark

Given an $n$-dimensional Riemannian manifold $(M^n, g)$, the Laplace operator $\Delta$ on functions is well known. Taking an orthonormal frame $\{E_i\}$ on $(M^n, g)$, we can define the Laplace operator acting on functions $f$ as

$$\Delta f = \sum_{i=1}^{n} \nabla^2_{E_i, E_i} f,$$

where $\nabla^2 f$ is the Hessian of the function $f$.

In local coordinates $\{x^i\}$,

$$\Delta = \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} (\sqrt{det(g)} g^{ij} \frac{\partial}{\partial x^j}).$$

It is immediate from the above definition that $\Delta$ is an elliptic operator. This is the analyst's definition. On the other hand, geometers usually consider the

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Laplace operator on fuctiuons $\tilde{\Delta}$ as a positive operator, i.e. they have non-negative eigenvalues. This is often called Hodge Laplacian on functions:

$$\tilde{\Delta} = \delta d,$$

where $d$ is the exterior differentiation and $\delta$ is the Hodge dual of $d$.

It can easily be shown that these two definitions are only different by a sign, i.e. $\Delta = -\tilde{\Delta}$. In [1] and this note, the Laplace operator is considered as an elliptic operator.

We observe that if $K(x) > 0$ and $\Delta ln(K(x)) = 0$, they $K$ must be constant functions.

*Proof.*

$$\Delta lnK = \frac{\Delta K}{K} - \frac{|\nabla K|^2}{K^2};$$

$lnK$ is harmonic implies that

$$\frac{\Delta K}{K} = \frac{|\nabla K|^2}{K^2}.$$

Since $K > 0$, multiplying both sides by $K^2$, we get

$$K\Delta K = |\nabla K|^2.$$

Integrating both sides on $S^2$ using standard volume form, we get

$$\int_{S^2} K\Delta K dvol(g_0) = \int_{S^2} |\nabla K|^2 dvol(g_0).$$

Now integration by parts shows that

$$-\int_{S^2} |\nabla K|^2 dvol(g_0) = \int_{S^2} |\nabla K|^2 dvol(g_0).$$

This in turn shows that

$$|\nabla K| \equiv 0.$$

We conclude that $K(x)$ must be a constant function. This shows that the class of functions for which to prescribe Gaussian curvature in [1] is trivial. □

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## References

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