ON THE STRUCTURE
OF QUANTUM PERMUTATION GROUPS

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Abstract. The quantum permutation group of the set \(X_n = \{1, \ldots, n\}\) corresponds to the Hopf algebra \(A_{\text{aut}}(X_n)\). This is an algebra constructed with generators and relations, known to be isomorphic to \(C(S_n)\) for \(n \leq 3\), and to be infinite dimensional for \(n \geq 4\). In this paper we find an explicit representation of the algebra \(A_{\text{aut}}(X_n)\), related to Clifford algebras. For \(n = 4\) the representation is faithful in the discrete quantum group sense.

Introduction

A general theory of unital Hopf \(C^\ast\)-algebras was developed by Woronowicz in [11], [12], [13]. The main results are the existence of the Haar functional, an analogue of Peter-Weyl theory and of Tannaka-Krein duality, and explicit formulae for the square of the antipode. As for examples, these include algebras of continuous functions on compact groups, \(q\)-deformations of them with \(q > 0\), and \(C^\ast\)-algebras of discrete groups.

Of particular interest is the algebra \(A_{\text{aut}}(X_n)\) constructed by Wang in [9]. This is the universal Hopf \(C^\ast\)-algebra coacting on the set \(X_n = \{1, \ldots, n\}\). In other words, the compact quantum group associated to it is a kind of analogue of the symmetric group \(S_n\).

The algebra \(A_{\text{aut}}(X_n)\) is constructed with generators and relations. There are \(n^2\) generators, labeled \(u_{ij}\) with \(i, j = 1, \ldots, n\). The relations are those making \(u\) a magic biunitary matrix. This means that all coefficients \(u_{ij}\) are projections, and on each row and each column of \(u\) these projections are mutually orthogonal, and sum up to 1. The comultiplication is given by \(\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}\), and the fundamental coaction is given by \(\alpha(\delta_i) = \sum u_{ji} \otimes \delta_j\).

For \(n = 1, 2, 3\) the canonical quotient map \(A_{\text{aut}}(X_n) \to C(S_n)\) is an isomorphism. For \(n \geq 4\) the algebra \(A_{\text{aut}}(X_n)\) is infinite dimensional, and just a few things are known about it. Its irreducible corepresentations are classified in [3], with the conclusion that their fusion rules coincide with those for irreducible representations of \(SO(3)\), independently of \(n \geq 4\). In [10] Wang proves that the compact quantum
group associated to $A_{\text{aut}}(X_n)$ with $n \geq 4$ is simple. In [3] it is shown that the discrete quantum group associated to $A_{\text{aut}}(X_n)$ with $n \geq 5$ is not amenable. Various quotients of $A_{\text{aut}}(X_n)$, corresponding to quantum symmetry groups of polyhedra, colored graphs etc., are studied in [4] by using planar algebra techniques.

These results certainly bring some light on the structure of $A_{\text{aut}}(X_n)$. However, for $n \geq 4$ this remains an abstract algebra, constructed with generators and relations.

In this paper we find an explicit representation of $A_{\text{aut}}(X_n)$. The construction works when $n$ is a power of 2, and uses a magic biunitary matrix related to Clifford algebras. For $n = 4$ the representation is inner faithful, in the sense that the corresponding unitary representation of the discrete quantum group associated to $A_{\text{aut}}(X_4)$ is faithful.

As a conclusion, there might be a geometric interpretation of Hopf algebras of type $A_{\text{aut}}(X_n)$. We should mention here that for the algebra $A_{\text{aut}}(X)$ with an $X$ finite graph, such an interpretation would be of real help, for instance in computing fusion rules.

1. Magic biunitary matrices

Let $A$ be a unital C$^*$-algebra. That is, we have a unital algebra $A$ over the field of complex numbers $\mathbb{C}$, with an antilinear antimultiplicative map $a \mapsto a^*$ satisfying $a^{**} = a$, and with a Banach space norm satisfying $|a^* a| = |a|^2$.

A projection is an element $p \in A$ satisfying $p^2 = p^* = p$. Two projections $p, q$ are said to be orthogonal when $pq = 0$. A partition of unity in $A$ is a finite set of projections, which are mutually orthogonal, and sum up to 1.

**Definition 1.1.** A matrix $v \in M_n(A)$ is called magic biunitary if all its rows and columns are partitions of the unity of $A$.

A magic biunitary is indeed a biunitary, in the sense that both $v$ and its transpose $v^t$ are unitaries. The other word – magic – comes from a vague similarity with magic squares.

The basic example comes from the symmetric group $S_n$. Consider the sets of permutations $\{ \sigma \in S_n \mid \sigma(i) = j \}$. When $i$ is fixed and $j$ varies, or vice versa, these sets form partitions of $S_n$. Thus their characteristic functions $v_{ij} \in \mathbb{C}(S_n)$ form a magic biunitary.

Of particular interest is the “universal” magic biunitary matrix. This has coefficients in the universal algebra $A_{\text{aut}}(X_n)$ constructed by Wang in [9].

**Definition 1.2.** $A_{\text{aut}}(X_n)$ is the universal C$^*$-algebra generated by $n^2$ elements $u_{ij}$, subject to the magic biunitarity condition.

In other words, we have the following universal property. For any magic biunitary matrix $v \in M_n(A)$ there is a morphism of C$^*$-algebras $A_{\text{aut}}(X_n) \to A$ mapping $u_{ij} \mapsto v_{ij}$.

A more elaborate version of this property, to be discussed now, states that $A_{\text{aut}}(X_n)$ is a Hopf C$^*$-algebra, whose underlying quantum group is a kind of analogue of $S_n$.

The following definition is due to Woronowicz [13].

**Definition 1.3.** A unital Hopf C$^*$-algebra is a unital C$^*$-algebra $A$, together with a morphism of C$^*$-algebras $\Delta : A \to A \otimes A$, subject to the following conditions.
(i) Coassociativity condition: \((\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta\).

(ii) Cocancellation condition: \(\Delta(A)(1 \otimes A)\) and \(\Delta(A)(A \otimes 1)\) are dense in \(A \otimes A\).

The basic example is the algebra \(\mathbb{C}(G)\) of continuous functions on a compact group \(G\), with \(\Delta(\varphi) : (g, h) \to \varphi(gh)\). Here coassociativity of \(\Delta\) follows from associativity of the multiplication of \(G\), and cocancellation in \(\mathbb{C}(G)\) follows from cancellation in \(G\).

Another example is the group algebra \(\mathbb{C}^*\) of a discrete group \(\Gamma\). This is obtained from the usual group algebra \(\mathbb{C}[\Gamma]\) by a standard completion procedure. The comultiplication is defined on generators \(g \in \Gamma\) by the formula \(\Delta(g) = g \otimes g\).

In general, associated to a Hopf \(\mathbb{C}^*\)-algebra \(A\) are a compact quantum group \(G\) and a discrete quantum group \(\Gamma\), according to the heuristic formula \(A = \mathbb{C}(G) = \mathbb{C}^*(\Gamma)\).

**Definition 1.4.** A coaction of \(A\) on a finite set \(X\) is a morphism of \(\mathbb{C}^*\)-algebras \(\alpha : \mathbb{C}(X) \to \mathbb{C}(X) \otimes A\), subject to the following conditions:

(i) Coassociativity condition: \((\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta)\alpha\).

(ii) Natural condition: \((\Sigma \otimes \text{id})v = \Sigma(\varphi)\), where \(\Sigma(\varphi)\) is the sum of values of \(\varphi\).

The basic example is with a group \(G\) of permutations of \(X\). Consider the action map \(\alpha : X \times G \to X\), given by \(a(i, \sigma) = \sigma(i)\). The formula \(\alpha \varphi = \varphi \alpha\) defines a morphism of \(\mathbb{C}^*\)-algebras \(\alpha : \mathbb{C}(X) \to \mathbb{C}(X \times G)\). This can be regarded as a coaction of \(\mathbb{C}(G)\) on \(X\).

In general, coactions of \(A\) can be thought of as coming from actions of the underlying compact quantum group \(G\). With this interpretation, the natural condition says that the action of \(G\) must preserve the counting measure on \(X\). This assumption cannot be dropped.

The following fundamental result is due to Wang [9].

**Theorem 1.1.** (i) \(A_{\text{aut}}(X_n)\) is a Hopf \(\mathbb{C}^*\)-algebra, with comultiplication \(\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}\).

(ii) The linear map \(\alpha(\delta_j) = \sum \delta_i \otimes u_{ji}\) is a coaction of \(A_{\text{aut}}(X_n)\) on \(X_n = \{1, \ldots, n\}\).

(iii) \(A_{\text{aut}}(X_n)\) is the universal Hopf \(\mathbb{C}^*\)-algebra coacting on \(X_n\).

The idea for proving (i) is that we can define \(\Delta\) by using the universal property of \(A_{\text{aut}}(X_n)\). Coassociativity is clear, and cocancellation follows from a result of Woronowicz in [13], stating that this is automatic whenever there is a counit and an antipode. But these can be defined by \(\varepsilon(u_{ij}) = \delta_{ij}\) and \(S(u_{ij}) = u_{ji}\), once again by using universality of \(A_{\text{aut}}(X_n)\).

We know that the compact quantum group \(G_n\) associated to \(A_{\text{aut}}(X_n)\) is a kind of quantum analogue of the symmetric group \(S_n\). In particular there should be an inclusion \(S_n \subset G_n\). Here is the exact formulation of this observation; see Wang [9] for details.

**Proposition 1.1.** There is a Hopf \(\mathbb{C}^*\)-algebra morphism \(\pi_n : A_{\text{aut}}(X_n) \to \mathbb{C}(S_n)\), mapping the generators \(u_{ij}\) to the characteristic functions of the sets \(\{\sigma \in S_n \mid \sigma(j) = i\}\).

The question is now whether \(\pi_n\) is an isomorphism or not. For instance a \(2 \times 2\) magic biunitary must be of the following special form, where \(p\) is a projection:

\[
\begin{pmatrix}
p & 1-p \\
1-p & p
\end{pmatrix}.
\]
The algebra generated by $p$ is canonically isomorphic to $\mathbb{C}^2$ if $p \neq 0,1$, and to $\mathbb{C}$ if not. Thus the universal algebra $A_{\text{aut}}(X_2)$ is isomorphic to $\mathbb{C}^2$, and $\pi_2$ is an isomorphism.

The map $\pi_3$ is an isomorphism as well; see [4] for a proof.

At $n = 4$ we have the following example of a magic biunitary matrix:

$$\begin{pmatrix}
p & 1-p & 0 & 0 \\
1-p & p & 0 & 0 \\
0 & 0 & q & 1-q \\
0 & 0 & 1-q & q
\end{pmatrix}.$$  

We can choose the projections $p,q$ such that the algebra $\langle p,q \rangle$ they generate is infinite dimensional and not commutative. It follows that $A_{\text{aut}}(X_4)$ is infinite dimensional and not commutative, so $\pi_4$ cannot be an isomorphism.

**Proposition 1.2.** For $n \geq 4$ the algebra $A_{\text{aut}}(X_n)$ is infinite dimensional and not commutative. In particular $\pi_n$ is not an isomorphism.

This follows by gluing an identity matrix of size $n-4$ to the above $4 \times 4$ matrix.

There is a quantum group interpretation here. Consider the compact and discrete quantum groups defined by the formula $A_{\text{aut}}(X_4) = \mathbb{C}(G_4) = \mathbb{C}^*(\Gamma_4)$. When $p,q$ are free the surjective morphism of $\mathbb{C}^*$-algebras $A_{\text{aut}}(X_4) \to \langle p,q \rangle$ can be thought of as coming from a surjective morphism of discrete quantum groups $\Gamma_4 \to \mathbb{Z}_2 * \mathbb{Z}_2$. This makes it clear that $\Gamma_4$ is infinite. Now $G_4$ being the Pontrjagin dual of $\Gamma_4$, it must be infinite as well.

See Bichon [5], Wang [9], [10] and Banica [4] for further speculations on this subject.

## 2. Inner faithful representations

We would like to find an explicit representation of $A_{\text{aut}}(X_n)$. As with any Hopf $\mathbb{C}$*-algebra, there is a problem here, because there are two notions of faithfulness.

Consider for instance a discrete subgroup $\Gamma$ of the unitary group $U(n)$. The inclusion $\Gamma \subset U(n)$ can be regarded as a unitary group representation $\Gamma \to U(n)$, and we get a $\mathbb{C}$*-algebra representation $\mathbb{C}^*(\Gamma) \to M_n(\mathbb{C})$. This latter representation is far from being faithful: for instance its kernel is infinite dimensional, hence non-empty, when $\Gamma$ is an infinite group. However, the representation $\mathbb{C}^*(\Gamma) \to M_n(\mathbb{C})$ must be “inner faithful” in some Hopf $\mathbb{C}$*-algebra sense, because the representation $\Gamma \to U(n)$ it comes from is faithful.

So, we are led to the following question. Let $H$ be a unital Hopf $\mathbb{C}$*-algebra, and let $\pi : H \to A$ be a morphism of $\mathbb{C}$*-algebras. If $\Gamma$ is the discrete quantum group associated to $H$ we know that $\pi$ corresponds to a unitary representation $\pi_i : \Gamma \to U(A)$. The question is: when is $\pi$ inner faithful, meaning that $\pi_i$ is faithful?

A simple answer is obtained by using the formalism of Kustermans and Vaes [6]. Associated to $H$ is a von Neumann algebra $\hat{H}_{vN}$, obtained by a certain completion procedure. Now coefficients of $\pi$ belong to the dual algebra $\hat{H}_{vN}$, and we can say that $\pi$ is inner faithful if these coefficients generate $\hat{H}_{vN}$. This notion is used by Vaes in [7], and a version of it is used by Wang in [9].

In this paper we use an equivalent definition from [2].
Definition 2.1. Let $H$ be a unital Hopf $\mathbb{C}^*$-algebra. A $\mathbb{C}^*$-algebra representation $\pi : H \to A$ is called inner faithful if the $*$-algebra generated by its coefficients is dense in $H_{\text{alg}}^*$. Here $H_{\text{alg}}$ is the dense $*$-subalgebra of $H$ consisting of “representative functions” on the underlying compact quantum group, constructed by Woronowicz in [13]. This is a Hopf $*$-algebra in the usual sense. Its dual complex vector space $H_{\text{alg}}^*$ is a $*$-algebra, with multiplication $\Delta^*$ and involution $^*$. Finally, coefficients of $\pi$ are the linear forms $\phi_{\pi}$ with $\phi \in A^*$, and the density assumption is with respect to the weak topology on $H_{\text{alg}}^*$. See e.g. the book of Abe [1] for Hopf algebras and [2] for details regarding this definition.

The main example is with a discrete group $\Gamma$. As expected, a representation $\mathbb{C}^*(\Gamma) \to A$ is inner faithful if and only if the corresponding unitary group representation $\Gamma \to U(A)$ is faithful. Some other examples are discussed in [2]. For how to use inner faithfulness see Vaes [7].

Definition 2.2. The character of a magic biunitary matrix $v \in M_n(A)$ is the sum of its diagonal entries $\chi(v) = v_{11} + v_{22} + \ldots + v_{nn}$.

The terminology comes from the case where $v = u$ is the universal magic biunitary matrix, with coefficients in $A = A_{\text{aut}}(X_n)$. Indeed, the matrix $u$ is a corepresentation of $A_{\text{aut}}(X_n)$ in the sense of Woronowicz [11], and the element $\chi(u)$ is its character.

Lemma 2.1. Let $v \in M_n(A)$ be a magic biunitary matrix, with $n \geq 4$. Assume that there is a unital linear form $\varphi : A \to \mathbb{C}$ such that

$$\varphi(\chi(v)^k) = \frac{1}{k+1} \binom{2k}{k}$$

for any $k$. Then the representation $\pi : A_{\text{aut}}(X_n) \to A$ defined by $u_{ij} \to v_{ij}$ is inner faithful.

Proof. The numbers in the statement are the Catalan numbers, appearing as multiplicities in the representation theory of $SO(3)$. The result will follow from the following fact from [3]. The finite-dimensional irreducible corepresentations of $A_{\text{aut}}(X_n)$ can be arranged in a sequence $\{r_k\}$, such that their fusion rules are the same as those for representations of $SO(3)$:

$$r_k \otimes r_s = r_{|k-s|} + r_{|k-s|+1} + \ldots + r_{k+s}.$$

Let $h : A_{\text{aut}}(X_n) \to \mathbb{C}$ be the Haar functional constructed by Woronowicz in [11]. Also consider the character of the fundamental corepresentation of $A_{\text{aut}}(X_n)$:

$$\chi(u) = u_{11} + u_{22} + \ldots + u_{nn}.$$

The Poincaré series of $A_{\text{aut}}(X_n)$ is defined by the following formula:

$$f(z) = \sum_{k=0}^{\infty} h(\chi(u)^k) z^k.$$

By the above result, this is equal to the Poincaré series for $SO(3)$:

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^k.$$
The assumption of the lemma says that the equality \( \varphi \pi = h \) holds on all powers of \( \chi(u) \). By linearity, this equality must hold on the algebra \( \langle \chi(u) \rangle \) generated by \( \chi(u) \). Now by positivity of \( h \) it follows that the restriction of \( \pi \) to this algebra is injective.

On the other hand, once again from fusion rules, we see that \( \chi(u) \) generates the algebra of characters \( A_{aut}(X_n)_{central} \) constructed by Woronowicz in [11].

Summing up, we know that \( \pi \) is faithful on \( A_{aut}(X_n)_{central} \).

Now consider the “minimal model” construction in [2]. This is the factorisation of \( \pi \) into a Hopf \( \mathbb{C}^* \)-algebra morphism \( A_{aut}(X_n) \rightarrow H \), and an inner faithful representation \( H \rightarrow A \):

\[ A_{aut}(X_n) \rightarrow H \rightarrow A. \]

Since \( \pi \) is faithful on \( A_{aut}(X_n)_{central} \), so is the map on the left. By Woronowicz’s analogue of the Peter-Weyl theory in [11], it follows that the map on the left is an isomorphism. Thus \( \pi \) coincides with the map on the right, which is by definition inner faithful.

It is possible to reformulate this result by using notions from Voiculescu’s free probability theory [8]. A non-commutative \( \mathbb{C}^* \)-probability space is a pair \((A, \varphi)\) consisting of a unital \( \mathbb{C}^* \)-algebra \( A \) together with a positive unital linear form \( \varphi : A \rightarrow \mathbb{C} \).

Associated to a self-adjoint element \( x \in A \) is its spectral measure \( \mu_x \). This is a probability measure on the spectrum of \( x \), defined by the formula

\[ \varphi(f(x)) = \int_{\mathbb{R}} f(t) \, d\mu_x(t). \]

This equality must hold for any continuous function \( f \) on the spectrum of \( x \). By density we can restrict our attention to polynomials \( f \in \mathbb{C}[X] \), and then by linearity it is enough to have this equality for monomials \( f(t) = t^k \). We say that \( \mu_x \) is uniquely determined by its moments,

\[ \varphi(x^k) = \int_{\mathbb{R}} t^k \, d\mu_x(t). \]

The following notion plays a central role in free probability. See [8], page 26.

**Definition 2.3.** An element \( x \) in a non-commutative \( \mathbb{C}^* \)-probability space is called semicircular if its spectral measure is \( \mu_x(t) = (2\pi)^{-1/2} \sqrt{4 - t^2} \, dt \) on \([-2, 2] \), and 0 elsewhere.

In terms of moments, we must have the following equalities, for any \( k \):

\[ \varphi(x^k) = \frac{1}{2\pi} \int_{-2}^{2} t^k \sqrt{4 - t^2} \, dt. \]

The integral is 0 when \( k \) is odd, and equal to a Catalan number when \( k \) is even,

\[ \varphi(x^{2k}) = \frac{1}{k+1} \binom{2k}{k}. \]

We get in this way a reformulation of the above lemma.

**Theorem 2.1.** A magic biunitary matrix whose character has the same spectral measure as the square of a semicircular element produces an inner faithful representation of Wang’s algebra.
The assumption $n \geq 4$ was removed, because it is superfluous. Indeed, for $n = 1, 2, 3$ finite dimensionality of $A_{\text{aut}}(X_n)$ implies that the spectrum of any $\chi(v)$ is discrete.

3. Geometric constructions

Consider the Pauli matrices,
\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]
These satisfy the relations for quaternions $i^2 = j^2 = k^2 = -1$, $ij = ji = -k$, etc.

To any $x \in SU(2)$ we associate the following matrix:
\[
x(x) = \begin{pmatrix} x & xi & xj & xk \\ ix & ixi & ixj & ixx \\ jx & jxi & jxj & jxk \\ kx & kxi & kxj & kxx \end{pmatrix}.
\]
Each row and each column of this matrix is an orthogonal basis of $M_2(\mathbb{C}) \cong \mathbb{C}^4$ with respect to the inner product
\[
\langle x, y \rangle = \frac{1}{2} \text{Tr}(xy^*),
\]
since $i, j, k$ are skew-adjoints. Thus the matrix of corresponding orthogonal projections is a magic biunitary.

**Theorem 3.1.** There is an inner faithful representation
\[
\pi : A_{\text{aut}}(X_4) \to \mathbb{C}(SU(2), M_4(\mathbb{C}))
\]
mapping the universal $4 \times 4$ magic biunitary matrix to the $4 \times 4$ matrix
\[
v(x) = \begin{pmatrix} P_x & P_{xi} & P_{xj} & P_{xk} \\ P_{ix} & ixi & ixj & ixx \\ P_{jx} & jxi & jxj & jxk \\ P_{kx} & kxi & kxj & kxx \end{pmatrix},
\]
where for $y \in SU(2)$ we denote by $P_y$ the orthogonal projection onto the space $\mathbb{C}y \subset M_2(\mathbb{C})$, and we regard it as a continuous function of $y$, with values in $M_2(M_2(\mathbb{C})) \cong M_4(\mathbb{C})$.

**Proof.** We have to compute the character of $v = v(x)$:
\[
\chi(v) = P_x + P_{ixi} + P_{jxj} + P_{kxk}.
\]
We make the convention that Greek letters designate quaternions in $\{1, i, j, k\}$. We decompose $x$ as a sum with real coefficients $x = \sum x_{\alpha} \alpha$. We have the following formula for $\chi(v)$:
\[
\chi(v) = \sum_{\alpha} P_{\alpha x\alpha}.
\]
With the notations $\alpha \beta = (-1)^{N(\alpha, \beta)} \beta \alpha$ and $\alpha^2 = (-1)^{N(\alpha)}$ we can compute $\alpha x\alpha$:
\[
\alpha x\alpha = \sum_{\beta} (-1)^{N(\alpha, \beta) + N(\alpha)} x_{\beta} \beta.
\]
Now using the above-mentioned canonical scalar product on $M_2(\mathbb{C})$, this gives the following formula for $P_{\alpha \alpha} \beta$, after cancelling the $(-1)^{2N(\alpha)} = 1$ term:
\[
\langle P_{\alpha \alpha \beta} \gamma, \gamma \rangle = (-1)^{N(\alpha, \beta) + N(\alpha, \gamma)} x_{\beta} x_\gamma.
\]

Now summing over $\alpha$ gives the formula of the character $\chi(v)$:
\[
\langle \chi(v) \beta, \gamma \rangle = \sum_{\alpha} (-1)^{N(\alpha, \beta) + N(\alpha, \gamma)} x_{\beta} x_\gamma.
\]

The coefficient of $x_{\beta} x_\gamma$ can be computed by using the multiplication table of quaternions,
\[
\sum_{\alpha} (-1)^{N(\alpha, \beta) + N(\alpha, \gamma)} = 4 \delta_{\beta, \gamma}.
\]

Thus $\chi(v)$ is a diagonal matrix, having the numbers $4x^2_\beta$ on the diagonal:
\[
\chi(v) = \text{diag}(4x^2_\beta).
\]

Consider the linear form $\varphi = \int \otimes tr$, where the integral is with respect to the Haar measure of $SU(2)$, and $tr$ is the normalised trace of $4 \times 4$ matrices, meaning $1/4$ times the usual trace. The moments of $\chi(v)$ with respect to $\varphi$ are computed as follows:
\[
\int tr(\chi(v)^k) dx = 4^{k-1} \sum_{\beta} \int x^2_\beta dx.
\]

By symmetry reasons the four integrals are all equal, say to the first one,
\[
\int tr(\chi(v)^k) dx = 4^k \int x^2_1 dx.
\]

It follows that $\chi(v)$ has the same spectral measure as $4x^2_1$,
\[
\mu_{\chi(v)} = \mu_{4x^2_1}.
\]

But the variable $2x_1$ is semicircular. This can be seen in many ways, for instance by direct computation, after identifying $SU(2)$ with the real sphere $S^3$, or by using the fact that $2x_1 = \text{Tr}(x)$ is the character of the fundamental representation of $SU(2)$, whose moments are computed using Clebsch-Gordon rules. The result now follows by applying Theorem 2.1. 

The construction of $\pi$ has the following generalisation. Consider the Clifford algebra $Cl(\mathbb{R}^s)$. This is a finite-dimensional algebra, having a basis formed by products $e_{i_1} \cdots e_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq s$, with multiplicative structure given by $e^2_i = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$.

It is convenient to use the notation $e_I = e_{i_1} \cdots e_{i_k}$ with $I = (i_1, \ldots, i_k)$.

As an example, the Clifford algebra $Cl(\mathbb{R}^2)$ is spanned by the elements $e_{\emptyset} = 1$, $e_1$, $e_2$ and $e_{12} = e_1 e_2$. The generators $e_1$, $e_2$ are subject to the relations $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$. Now these relations are satisfied by the Pauli matrices $i$, $j$, and the corresponding representation of $Cl(\mathbb{R}^2)$ turns out to be faithful. That is, we have the following identifications:
\[
\begin{align*}
e_{\emptyset} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & e_1 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & e_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & e_{12} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\end{align*}
\]

We can label as well indices of $4 \times 4$ matrices by elements of the set $\{\emptyset, 1, 2, 12\}$. With these notations, the representation in Theorem 3.1 is given by $\pi(u_{IJ}) = P_{e_I x e_J}$. 

The same formula works for an arbitrary number $s$.

**Theorem 3.2.** There is a representation $\pi_n : A_{aut}(X_n) \to C(G_n, M_n(\mathbb{C}))$ mapping the universal $n \times n$ magic biunitary matrix to the $n \times n$ matrix

$$v = (P_{e_i e_j})_{I,J}$$

where $n = 2^s$, the unitary group of the Clifford algebra $Cl(\mathbb{R}^s)$ is denoted $G_n$, and the algebra of endomorphisms of $Cl(\mathbb{R}^s)$ is identified with $M_n(\mathbb{C})$.

The first part of proof of Theorem 3.1 extends to this general situation. We get that $\chi(v)$ is diagonal, with eigenvalues $\{nx_j^2\}$. This does not seem to be related to semicircular elements when $s \geq 3$. The representation $\pi_n$ probably comes from an inner faithful representation of a quotient of $A_{aut}(X_n)$, corresponding to a “subgroup” of the quantum permutation group.

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