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A HILBERT C^*-MODULE NOT ANTI-ISOMORPHIC TO ITSELF

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Abstract. We study the complexification of real Hilbert C^*-modules over real C^*-algebras. We give an example of a Hilbert $\mathcal{A}_c$-module that is not the complexification of any Hilbert $\mathcal{A}$-module, where $\mathcal{A}$ is a real C^*-algebra.

1. Introduction

The Hilbert C^*-module theory has, so far, mainly been focused on Hilbert C^*-modules over complex C^*-algebras. However, one can verify that, with some appropriate modification, most Hilbert C^*-module results still hold for real Hilbert C^*-modules over real C^*-algebras. Also, it would be interesting if we could build up a connection between real Hilbert C^*-modules and complex Hilbert C^*-modules. A standard way to do this is to consider the complexification of real Hilbert C^*-modules. We show that for every real Hilbert $\mathcal{A}$-module E over a real C^*-algebra $\mathcal{A}$, there exists a unique $\mathcal{A}_c$-valued inner product on $E_c = E\dot{+}iE$ that extends the original $\mathcal{A}$-valued inner product on E, and E_c equipped with this $\mathcal{A}_c$-valued inner product will be a Hilbert $\mathcal{A}_c$-module.

It is well known that Hilbert C^*-modules can be represented as concrete ones, the same as C^*-algebras [2], [7], [11]. The existence of C^*-algebras not isomorphic to the complexification of any real C^*-algebra has been known for some time. Early examples are due to Raeburn and P. Green, and several examples are given in [9]. Also, A. Connes showed that there are von Neumann factors, which are not isomorphic, as von Neumann algebras, to their opposites [4].

On the other hand, every complex Hilbert space is the complexification of some real Hilbert space and one would like to think that Hilbert C^*-modules behave like Hilbert spaces. Despite the fact that we can show that every Hilbert $K(H)$-module is the complexification of some real Hilbert C^*-module, we give an example of a Hilbert $\mathcal{A}_c$-module that is not the complexification of any Hilbert $\mathcal{A}$-module.

2. Real Hilbert C^*-modules

As in the complex case, we can define real Hilbert C^*-modules over real C^*-algebras just as the usual Hilbert C^*-modules by replacing the field $\mathbb{C}$ with the field $\mathbb{R}$.

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Note that the definition conditions require the inner product to be linear in both its first and second variables.

If π denotes a faithful $*$-representation of a real C^*-algebra $\mathcal{A}$ on a real Hilbert space H, we can, in a similar way to the complex case [11], identify the Hilbert C^*-module E as a concrete Hilbert $\pi(\mathcal{A})$-module of $B(H,K)$ for some real Hilbert space K by the isometry $\Phi : E \to B(H,K)$, $x \mapsto \Phi_x$. Here $\Phi_x(h) = x \otimes h$, for all $x \in E, h \in H$, and Φ satisfies $(\Phi_x)^*\Phi_y = \pi(\langle x, y\rangle), \Phi_{xa} = \Phi_x\pi(a)$. Here $K = \overline{E \otimes H}$ is the compeletion of the pre-Hilbert space $E \otimes H$ by the following inner product:

$$\langle x \otimes h, y \otimes g\rangle = \langle h, \pi(\langle x, y\rangle)g\rangle \quad \text{for all } x, y \in E, h, g \in H.$$

3. The complexification of real Hilbert C^*-modules

If $\mathcal{A}$ is a real algebra and E is a real right $\mathcal{A}$-module, then there is a natural right $\mathcal{A}_c$-module structure on the complexification E_c of E, given by

$$(x + iy)(a + ib) = (xa - yb) + i(xb + ya),$$

for all $x, y \in E$, $a, b \in \mathcal{A}$.

Proposition 3.1. *Let $\mathcal{A}$ be a real C^*-algebra and let $(E, \langle .,. \rangle)$ be a real Hilbert $\mathcal{A}$-module. Then there is a (unique) $\mathcal{A}_c$-valued inner product on E_c, which keeps the original $\mathcal{A}$-valued inner product on E, and E_c equipped with this $\mathcal{A}_c$-valued inner product will be a Hilbert $\mathcal{A}_c$-module.*

Proof. There is a natural (unique) $\mathcal{A}_c$-valued inner product on E_c, given by

$$\langle x + iy, x' + iy'\rangle_c = \langle x, x'\rangle + \langle y, y'\rangle + i\langle x, y'\rangle - i\langle x', y\rangle.$$

All of the circumstances based on which $(E_c, \langle .,. \rangle_c)$ will be a Hilbert C^*-module over $\mathcal{A}_c$ can be deduced directly, except the condition

$$\langle x + iy, x + iy\rangle_c \geq 0,$$

for all $x, y \in E$.

Without loss of generality, we can assume that $\mathcal{A}$ is a concrete real C^*-algebra in $B(H)$ for some real Hilbert space H. Therefore, we can choose a real Hilbert space K such that E can be regarded as a concrete real Hilbert A-module of $B(H,K)$; therefore,

$$\begin{aligned}\langle x + iy, x + iy\rangle_c &= \langle x, x\rangle + \langle y, y\rangle + i(\langle x, y\rangle - \langle y, x\rangle)\\ &= x^*x + y^*y + i(x^*y - y^*x) = (x + iy)^*(x + iy) \in \mathcal{A}_c^+.\end{aligned}$$

□

Remark 3.2. Let $\mathcal{A}$ be a real C^*-algebra. It is natural to ask whether or not every Hilbert $\mathcal{A}_c$-module E can be obtained as the complexification of some real Hilbert $\mathcal{A}$-module. We can characterize complex Hilbert C^*-modules that can be expressed as the complexification of some real Hilbert C^*-modules by the following:

The Hilbert $\mathcal{A}_c$-module E is the complexification of some real Hilbert $\mathcal{A}$-module if and only if there exists a conjugate linear "$-$" on E such that $-^2 = id$, $\langle \overline{\xi}, \overline{\eta}\rangle = \overline{\langle \xi, \eta\rangle}$ and $\overline{\xi a} = \overline{\xi}\overline{a}$ for all $\xi, \eta \in F$, $a \in \mathcal{A}_c$.

However, the most interesting thing is the question of which Hilbert $\mathcal{A}_c$-modules are complexifications. In particular, which real C^*-algebras $\mathcal{A}$ have the property that every Hilbert $\mathcal{A}_c$-module is a complexification of some Hilbert $\mathcal{A}$-module.

Let E be a Hilbert C^*-module over an arbitrary C^*-algebra $\mathcal{A}$. We recall the definition of an orthonormal basis for Hilbert C^*-modules from [1]. An element $\upsilon \in E$ is said to be a basic vector if $e = \langle \upsilon, \upsilon \rangle$ is a minimal projection in $\mathcal{A}$, in the sense $e\mathcal{A}e = \mathbb{C}e$. A system $(\upsilon_\lambda), \lambda \in \Lambda$, in E is said to be an orthonormal basis for E if it generates a dense submodule of E and each υ_λ is a basic vector and $\langle \upsilon_\lambda, \upsilon_\mu \rangle = 0$ for all $\lambda \neq \mu$.

Proposition 3.3. *Let $\mathcal{A}$ be a real C^*-algebra. Then every Hilbert $\mathcal{A}_c$-module that has an orthonormal basis is a complexification of some Hilbert $\mathcal{A}$-module.*

Proof. Let E be a Hilbert $\mathcal{A}_c$-module and let $(\upsilon_\lambda), \lambda \in \Lambda$, be an orthonormal basis for E. Then $x = \sum_\lambda \upsilon_\lambda \langle \upsilon_\lambda, x \rangle$ for all $x \in E$, by [1, Theorem 1].

If "$-$" is the conjugation on $\mathcal{A}_c$ and $e_\lambda = \langle \upsilon_\lambda, \upsilon_\lambda \rangle$, then $e_\lambda \overline{e_\lambda} e_\lambda = \alpha_\lambda e_\lambda$ for some $\alpha_\lambda \in \mathbb{C}$ by minimality of e_λ. Also, α_λ is positive since $e_\lambda \overline{e_\lambda} e_\lambda$ is a positive element in $\mathcal{A}_c$. We define $\overline{x} = \sum_\lambda \frac{1}{\sqrt{\alpha_\lambda}} \upsilon_\lambda \overline{\langle \upsilon_\lambda, x \rangle}$ for all $x \in E$. One can easily check that the operation "$-$" has properties mentioned in the previous remark. $\square$

It is proved in [1, Theorem 4] that each Hilbert C^*-module over the C^*-algebra of (not necessarily all) compact operators on some complex Hilbert space possesses an orthonormal basis. Then we have

Corollary 3.4. *Let $\mathcal{A}$ be an arbitrary real C^*-algebra of (not necessarily all) compact operators on some real Hilbert space. Then every Hilbert $\mathcal{A}_c$-module is a complexification of some Hilbert $\mathcal{A}$-module.*

Now, we are going to show that Corollary 3.4 is not necessarily true for all real C^*-algebras.

Proposition 3.5. *Let $\mathcal{A}$ be a real C^*-algebra, in some $M_n(\mathcal{A}_c)$ of which there exists a projection P that is not Murray-von Neumann equivalent to the conjugate projection $\overline{P}$, where $\overline{P} = [\overline{p_{i,j}}]$ whenever $P = [p_{i,j}]$. Then $P\mathcal{A}_c^n$ will be a Hilbert $\mathcal{A}_c$-module that is not the complexification of any real Hilbert $\mathcal{A}$-module.*

Proof. Assume, to reach a contradiction, that $P\mathcal{A}_c^n$ is the complexification of a real Hilbert $\mathcal{A}$-module. Thus there exists the conjugate $\mathcal{A}_c$-linear map $- : P\mathcal{A}_c^n \to P\mathcal{A}_c^n$, mentioned in Remark 3.2. Now, since $\overline{P[0...1_j...0]^t} \in P\mathcal{A}_c^n$, there exist the elements $c_{1,j}, ..., c_{n,j} \in \mathcal{A}_c$ such that $\overline{P[0...1_j...0]^t} = P[c_{1,j}, ..., c_{n,j}]^t$; therefore,

$$\overline{P[a_1, ..., a_n]^t} = P[c_{i,j}][\overline{a_1}, ..., \overline{a_n}]^t$$

for all $a_1, ..., a_n \in \mathcal{A}$. We let $T = P[c_{i,j}]$; then we have

$$\begin{aligned} P[a_1, ..., a_n]^t &= \overline{\overline{P[a_1, ..., a_n]^t}} = \overline{T[\overline{a_1}, ..., \overline{a_n}]^t} = \overline{PT[\overline{a_1}, ..., \overline{a_n}]^t} \\ &= T\overline{T}[a_1, ..., a_n]^t. \end{aligned}$$

Then, $P = T\overline{T}$.

Also, we have

$$\langle \overline{P[a_1, ..., a_n]^t}, \overline{P[b_1, ..., b_n]^t} \rangle = \overline{\langle P[a_1, ..., a_n]^t, P[b_1, ..., b_n]^t \rangle}.$$

Then

$$\begin{aligned}\langle[\overline{a_1},...,\overline{a_n}]^t, T^*T[\overline{b_1},...,\overline{b_n}]^t\rangle &= \overline{\langle[a_1,...,a_n]^t, P[b_1,...,b_n]^t\rangle}\\ &= \langle[\overline{a_1},...,\overline{a_n}]^t, \overline{P}[\overline{b_1},...,\overline{b_n}]^t\rangle,\end{aligned}$$

for all $a_1,...,a_n,b_1,...,b_n \in \mathcal{A}_c$, where in the last step we used the fact that the Hilbert $\mathcal{A}_c$-module $\mathcal{A}_c^n$ is the complexification of the real Hilbert $\mathcal{A}$-module $\mathcal{A}^n$. Then $T^*T = \overline{P}$ and furthermore,

$$\overline{T} = \overline{PT} = \overline{P}\ \overline{T} = T^*T\overline{T} = T^*P = T^*.$$

Therefore, P is Murray-von Neumann equivalent to the conjugate projection $\overline{P}$, and this contradicts the choice of P. □

Certainly, $P\mathcal{A}_c^n$ is a finitely generated Hilbert $\mathcal{A}_c$-module that is not the complexification of any real Hilbert $\mathcal{A}$-module.

J. L. Boersema [3] developed the functor $KT(\mathcal{A})$ that is based on projections in $\mathcal{A}_c$ that are equivalent to their conjugate projection. There is a natural transformation from $KT_0(\mathcal{A})$ to $K_0(\mathcal{A}_c)$ that is not surjective in general. Thus there are projections in some $M_n(\mathcal{A}_c)$ that are not Murray-von Neumann equivalent to their conjugate projection. Therefore, there exists a source of examples satisfying the hypothesis of the above proposition.

For a concrete example, we can consider $\mathcal{A} = C(S^2,\mathbb{R})$ and hence $\mathcal{A}_c = C(S^2,\mathbb{C})$. Then we look at $K_0(\mathcal{A}_c) = \mathbb{Z}\oplus\mathbb{Z}$. The first summand is for the unit and the second summand is for the Bott element. The natural conjugation transformation $c : K_0(\mathcal{A}_c) \to K_0(\mathcal{A}_c)$ fixes the first generator, but is multiplication by -1 on the second generator. Thus the Bott projection is not equivalent to its conjugate.

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