A HILBERT C*-MODULE NOT ANTI-ISOMORPHIC TO ITSELF

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Abstract. We study the complexification of real Hilbert C*-modules over real C*-algebras. We give an example of a Hilbert Ac-module that is not the complexification of any Hilbert A-module, where A is a real C*-algebra.

1. Introduction

The Hilbert C*-module theory has, so far, mainly been focused on Hilbert C*-modules over complex C*-algebras. However, one can verify that, with some appropriate modification, most Hilbert C*-module results still hold for real Hilbert C*-modules over real C*-algebras. Also, it would be interesting if we could build up a connection between real Hilbert C*-modules and complex Hilbert C*-modules. A standard way to do this is to consider the complexification of real Hilbert C*-modules. We show that for every real Hilbert A-module E over a real C*-algebra A, there exists a unique Ac-valued inner product on Ec = E + iE that extends the original A-valued inner product on E, and Ec equipped with this Ac-valued inner product will be a Hilbert Ac-module.

It is well known that Hilbert C*-modules can be represented as concrete ones, the same as C*-algebras [2], [7], [11]. The existence of C*-algebras not isomorphic to the complexification of any real C*-algebra has been known for some time. Early examples are due to Raeburn and P. Green, and several examples are given in [9]. Also, A. Connes showed that there are von Neumann factors, which are not isomorphic, as von Neumann algebras, to their opposites [4].

On the other hand, every complex Hilbert space is the complexification of some real Hilbert space and one would like to think that Hilbert C*-modules behave like Hilbert spaces. Despite the fact that we can show that every Hilbert K(H)-module is the complexification of some real Hilbert C*-module, we give an example of a Hilbert Ac-module that is not the complexification of any Hilbert A-module.

2. Real Hilbert C*-modules

As in the complex case, we can define real Hilbert C*-modules over real C*-algebras just as the usual Hilbert C*-modules by replacing the field C with the field R.
Note that the definition conditions require the inner product to be linear in both its first and second variables.

If \( \pi \) denotes a faithful \(*\)-representation of a real \( C^* \)-algebra \( A \) on a real Hilbert space \( H \), we can, in a similar way to the complex case [11], identify the Hilbert \( C^* \)-module \( E \) as a concrete Hilbert \( \pi(A) \)-module of \( B(H,K) \) for some real Hilbert space \( K \) by the isometry \( \Phi : E \to B(H,K), x \mapsto \Phi_x \). Here \( \Phi_x(h) = x \otimes h \), for all \( x \in E, h \in H \), and \( \Phi \) satisfies \( (\Phi_x)^*\Phi_y = \pi(\langle x, y \rangle) \), \( \Phi_{xa} = \Phi_x \pi(a) \). Here \( K = E \otimes H \) is the completion of the pre-Hilbert space \( E \otimes H \) by the following inner product:

\[
\langle x \otimes h, y \otimes g \rangle = \langle h, \pi((x,y))g \rangle \quad \text{for all } x, y \in E, h, g \in H.
\]

3. The complexification of real Hilbert \( C^* \)-modules

If \( A \) is a real algebra and \( E \) is a real right \( A \)-module, then there is a natural right \( A_c \)-module structure on the complexification \( E_c \) of \( E \), given by

\[
(x + iy)(a + ib) = (xa - yb) + i(xb + ya),
\]

for all \( x, y \in E, a, b \in A \).

**Proposition 3.1.** Let \( A \) be a real \( C^* \)-algebra and let \( (E, \langle \cdot, \cdot \rangle) \) be a real Hilbert \( A \)-module. Then there is a (unique) \( A_c \)-valued inner product on \( E_c \), which keeps the original \( A \)-valued inner product on \( E \), and \( E_c \) equipped with this \( A_c \)-valued inner product will be a Hilbert \( A_c \)-module.

**Proof.** There is a natural (unique) \( A_c \)-valued inner product on \( E_c \), given by

\[
\langle x + iy, x' + iy' \rangle_c = \langle x, x' \rangle + \langle y, y' \rangle + i(x, y') - iy(x', y).
\]

All of the circumstances based on which \( (E_c, \langle \cdot, \cdot \rangle_c) \) will be a Hilbert \( C^* \)-module over \( A_c \) can be deduced directly, except the condition

\[
(x + iy, x + iy)_c \geq 0,
\]

for all \( x, y \in E \).

Without loss of generality, we can assume that \( A \) is a concrete real \( C^* \)-algebra in \( B(H) \) for some real Hilbert space \( H \). Therefore, we can choose a real Hilbert space \( K \) such that \( E \) can be regarded as a concrete real Hilbert \( A \)-module of \( B(H,K) \); therefore,

\[
(x + iy, x + iy)_c = \langle x, x \rangle + \langle y, y \rangle + i\langle x, y \rangle - \langle y, x \rangle = x^*x + y^*y + i(x^*y - y^*x) = (x + iy)^*(x + iy) \in A_c^+.
\]

**Remark 3.2.** Let \( A \) be a real \( C^* \)-algebra. It is natural to ask whether or not every Hilbert \( A_c \)-module \( E \) can be obtained as the complexification of some real Hilbert \( A \)-module. We can characterize complex Hilbert \( C^* \)-modules that can be expressed as the complexification of some real Hilbert \( C^* \)-modules by the following:

The Hilbert \( A_c \)-module \( E \) is the complexification of some real Hilbert \( A \)-module if and only if there exists a conjugate linear \( "-\) on \( E \) such that \( -^2 = id, \overline{\langle \xi, \eta \rangle} = \langle \xi, \eta \rangle \) and \( \overline{\xi a} = \overline{\xi} \overline{a} \) for all \( \xi, \eta \in F, a \in A_c \).
However, the most interesting thing is the question of which Hilbert \( A_c \)-modules are complexifications. In particular, which real \( C^* \)-algebras \( A \) have the property that every Hilbert \( A_c \)-module is a complexification of some Hilbert \( A \)-module.

Let \( E \) be a Hilbert \( C^* \)-module over an arbitrary \( C^* \)-algebra \( A \). We recall the definition of an orthonormal basis for Hilbert \( C^* \)-modules from [1]. An element \( v \in E \) is said to be a basic vector if \( e = \langle v, v \rangle \) is a minimal projection in \( A \), in the sense \( c_Ae = Ce \). A system \((v_\lambda), \lambda \in \Lambda \), in \( E \) is said to be an orthonormal basis for \( E \) if it generates a dense submodule of \( E \) and each \( v_\lambda \) is a basic vector and \( \langle v_\lambda, v_\mu \rangle = 0 \) for all \( \lambda \neq \mu \).

**Proposition 3.3.** Let \( A \) be a real \( C^* \)-algebra. Then every Hilbert \( A_c \)-module that has an orthonormal basis is a complexification of some Hilbert \( A \)-module.

**Proof.** Let \( E \) be a Hilbert \( A \)-module and let \((v_\lambda), \lambda \in \Lambda \), be an orthonormal basis for \( E \). Then \( x = \sum_\lambda \langle v_\lambda, x \rangle v_\lambda \) for all \( x \in E \), by [1] Theorem 1.

If \( "−" \) is the conjugation on \( A \), and \( e_\lambda = \langle v_\lambda, v_\lambda \rangle \), then \( e_\lambda\overline{e_\lambda} = \alpha_\lambda e_\lambda \) for some \( \alpha_\lambda \in \mathbb{C} \) by minimality of \( e_\lambda \). Also, \( \alpha_\lambda \) is positive since \( e_\lambda\overline{e_\lambda} \) is a positive element in \( A \). We define \( \varphi = \sum_\lambda \sqrt{\alpha_\lambda} \langle v_\lambda, x \rangle \) for all \( x \in E \). One can easily check that the operation \( "−" \) has properties mentioned in the previous remark. \( \square \)

It is proved in [1] Theorem 4 that each Hilbert \( C^* \)-module over the \( C^* \)-algebra of (not necessarily all) compact operators on some complex Hilbert space possesses an orthonormal basis. Then we have

**Corollary 3.4.** Let \( A \) be an arbitrary real \( C^* \)-algebra of (not necessarily all) compact operators on some real Hilbert space. Then every Hilbert \( A_c \)-module is a complexification of some Hilbert \( A \)-module.

Now, we are going to show that Corollary 3.4 is not necessarily true for all real \( C^* \)-algebras.

**Proposition 3.5.** Let \( A \) be a real \( C^* \)-algebra, in some \( M_n(A_c) \) of which there exists a projection \( P \) that is not Murray-von Neumann equivalent to the conjugate projection \( \overline{P} \), where \( \overline{P} = [\overline{p}_{ij}] \) whenever \( P = [p_{ij}] \). Then \( PA_c^n \) will be a Hilbert \( A_c \)-module that is not the complexification of any real Hilbert \( A \)-module.

**Proof.** Assume, to reach a contradiction, that \( PA_c^n \) is the complexification of a real Hilbert \( A \)-module. Thus there exists the conjugate \( A_c \)-linear map \( \overline{\cdot} : PA_c^n \to PA_c^n \), mentioned in Remark 3.2. Now, since \( \overline{P[0...1j...0]} \in PA_c^n \), there exist the elements \( c_{1,j}, ..., c_{n,j} \in A_c \) such that \( \overline{P[0...1j...0]} = P[c_{1,j}, ..., c_{n,j}] \); therefore,

\[
\overline{P[a_1, ..., a_n]}^t = P[c_{1,j}, ..., c_{n,j}]^t
\]

for all \( a_1, ..., a_n \in A \). We let \( T = P[c_{i,j}] \); then we have

\[
P[a_1, ..., a_n]^t = \overline{T[a_1, ..., a_n]}^t = \overline{T[a_1, ..., a_n]}^t = \overline{T[a_1, ..., a_n]}^t = T^t\overline{a_1, ..., a_n} = T^t\overline{a_1, ..., a_n}.
\]

Then, \( P = TT \).

Also, we have

\[
\langle P[a_1, ..., a_n]^t, P[b_1, ..., b_n]^t \rangle = \langle P[a_1, ..., a_n]^t, P[b_1, ..., b_n]^t \rangle.
\]
Then
\[
\langle [a_1, \ldots, a_n]^t, T^*T [b_1, \ldots, b_n]^t \rangle = \langle [a_1, \ldots, a_n]^t, P [b_1, \ldots, b_n]^t \rangle
\]
for all \(a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}_c\), where in the last step we used the fact that the Hilbert \(\mathcal{A}_c\)-module \(\mathcal{A}_c^n\) is the complexification of the real Hilbert \(\mathcal{A}\)-module \(\mathcal{A}^n\). Then \(T^*T = \mathcal{T}\) and furthermore,
\[
\mathcal{T} = \mathcal{PT} = \mathcal{P} \mathcal{T} = T^*T \mathcal{T} = T^*P = T^*.
\]
Therefore, \(P\) is Murray-von Neumann equivalent to the conjugate projection \(\mathcal{T}\), and this contradicts the choice of \(P\). \(\square\)

Certainly, \(P\mathcal{A}_c^n\) is a finitely generated Hilbert \(\mathcal{A}_c\)-module that is not the complexification of any real Hilbert \(\mathcal{A}\)-module.

J. L. Boersema [3] developed the functor \(KT(\mathcal{A})\) that is based on projections in \(\mathcal{A}_c\) that are equivalent to their conjugate projection. There is a natural transformation from \(KT_0(\mathcal{A})\) to \(K_0(\mathcal{A}_c)\) that is not surjective in general. Thus there are projections in some \(M_n(\mathcal{A}_c)\) that are not Murray-von Neumann equivalent to their conjugate projection. Therefore, there exists a source of examples satisfying the hypothesis of the above proposition.

For a concrete example, we can consider \(\mathcal{A} = C(S^2, \mathbb{R})\) and hence \(\mathcal{A}_c = C(S^2, \mathbb{C})\). Then we look at \(K_0(\mathcal{A}_c) = \mathbb{Z} \oplus \mathbb{Z}\). The first summand is for the unit and the second summand is for the Bott element. The natural conjugation transformation \(c : K_0(\mathcal{A}_c) \to K_0(\mathcal{A}_c)\) fixes the first generator, but is multiplication by \(-1\) on the second generator. Thus the Bott projection is not equivalent to its conjugate.

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