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EXACT MULTIPLICITY RESULT FOR THE PERTURBED SCALAR CURVATURE PROBLEM IN $\mathbb{R}^N$ $(N \geq 3)$

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Abstract. Let $D^{1,2}(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ in the norm $\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2$. Let $N \geq 3$ and define the constants $\alpha_N = N(N-2)$ and $C_N = (N(N-2))^{\frac{N-2}{4}}$. Let $K \in C^2(\mathbb{R}^N)$. We consider the following problem for $\varepsilon \geq 0$:

$$(P_\varepsilon) \qquad \left\{ \begin{array}{l} \text{Find } u \in D^{1,2}(\mathbb{R}^N) \text{ solving :} \\ \begin{array}{lcl} -\Delta u & = & \alpha_N(1+\varepsilon K(x))u^{\frac{N+2}{N-2}}, \\ u & > & 0 \end{array} \end{array} \right\} \text{ in } \mathbb{R}^N.$$

We show an exact multiplicity result for (P_ε) for all small $\varepsilon > 0$.

1. Introduction

Let $D^{1,2}(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}.$$

Let $N \geq 3$ and define the constants $\alpha_N = N(N-2)$ and $C_N = (N(N-2))^{\frac{N-2}{4}}$. Let $K \in C^2(\mathbb{R}^N)$. We consider the following problem for $\varepsilon \geq 0$:

$$(P_\varepsilon) \qquad \left\{ \begin{array}{l} \text{Find } u \in D^{1,2}(\mathbb{R}^N) \text{ solving :} \\ \begin{array}{lcl} -\Delta u & = & \alpha_N(1+\varepsilon K(x))u^{\frac{N+2}{N-2}}, \\ u & > & 0 \end{array} \end{array} \right\} \text{ in } \mathbb{R}^N.$$

We are interested in showing an exact multiplicity result for (P_ε) for all small $\varepsilon > 0$ (see Theorem 1.4 below).

The above problem is a "perturbed" version of the well-known scalar curvature problem which arises in differential geometry. More precisely, the problem is to find out if a given smooth function R on the N-dimensional unit sphere S^N is the scalar curvature function of a metric g on S^N which is conformal to the standard

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metric. This gives rise to the following problem:

$$(P_\varepsilon)_g \qquad \left\{ \begin{array}{l} \text{Find } u \in C^2(S^N) \text{ solving:} \\ \frac{4(N-1)}{N-2}\Delta_g u + Ru^{\frac{N+2}{N-2}} = N(N-1)u, \\ \qquad\qquad\qquad\qquad u > 0 \end{array} \right\} \text{ in } S^N.$$

The above problem has been extensively studied using the background of differential geometry; see the book of T. Aubin [2] for a survey of the available results.

We now assume that R is a perturbation of the constant, viz, $R = 1 + \varepsilon K$ for a smooth function K on S^N and $\varepsilon > 0$ small. Then, using the standard streographic projection from S^N to $\mathbb{R}^N$, it can be checked that $(P_\varepsilon)_g$ is transformed to (P_ε).

Existence of solutions to (P_ε) was done in [1] using variational methods and finite-dimensional reduction techniques. To describe their result, we make the following assumptions on K:

(K1) $K \in C^2(\mathbb{R}^N), \|K\|_{L^\infty(\mathbb{R}^N)} + \|\nabla K\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} + \|D^2K\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N\times\mathbb{R}^N)} < \infty$.

(K2) (a) There exists $\rho > 0$ such that $\langle \nabla K(x), x\rangle < 0 \quad \forall |x| \geq \rho$,
(b) $\langle \nabla K(x), x\rangle \in L^1(\mathbb{R}^N)$, $\int\limits_{\mathbb{R}^N} \langle \nabla K(x), x\rangle dx < 0$.

(K3) The set of all critical points of K, denoted by crit (K), is finite.

(K4) $\forall \xi \in$ crit (K), there exists $\beta = \beta_\xi \in [1, N]$ and $Q_\xi : \mathbb{R}^N \to \mathbb{R}$ depending continuously on y locally near ξ such that

$$A_\xi \stackrel{def}{=} \frac{N-2}{2N} \int\limits_{\mathbb{R}^N} Q_\xi(y) U_{1,0}^{\frac{2N}{N-2}}(y) dy \neq 0$$

and the following relations hold:

$$\left\{ \begin{array}{lll} Q_\xi(\lambda x) & = & \lambda^\beta Q_\xi(x), \quad \forall \lambda \geq 0, \\ K(x) & = & K(y) + Q_\xi(x-y) + o(|x-y|^\beta) \text{ as } x \to y, \end{array} \right.$$

(K5) $\sum\limits_{\xi \in \text{ crit } (K)} \deg_{loc}(\nabla K, \xi) \neq (-1)^N$.

We also recall the following well-known classification result for solutions of (P_0) :

Theorem 1.1. *Solutions of (P_0) form an $(N+1)$-dimensional manifold given by*

$$\mathcal{M} = \{U_{\delta,y}(x) \stackrel{def}{=} C_N \delta^{\frac{N-2}{2}} (\delta^2 + |x-y|^2)^{\frac{2-N}{2}} : (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N\}.$$

We can now state the following existence result.

Theorem 1.2 ([1])**.** *Let K satisfy the assumptions* (K1)–(K5)*. Then there exists $\varepsilon_0 > 0$ such that the following hold:*

(i) *(P_ε) has a solution $\forall \varepsilon \in (0, \varepsilon_0)$,*

(ii) *$\forall \varepsilon \in (0, \varepsilon_0)$, there exists $(\delta_\varepsilon, y_\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^N$ such that for any compact set $A \subset \mathbb{R}^+ \times \mathbb{R}^N$, we may find a constant $c(A) > 0$ such that $\|u_\varepsilon - U_{\delta_\varepsilon, y_\varepsilon}\|_{D^{1,2}(\mathbb{R}^N)} \leq c(A)\varepsilon$.*

We now define what we mean by a stable zero of a vector field.

Definition 1.3. Let $\mathbf{G} : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^{N+1}$ be a C^1 vector field. We say that a point $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N$ is a stable zero for $\mathbf{G}$ if $\mathbf{G}(\delta, y) = \mathbf{O}$ and its derivative $D\mathbf{G}(\delta, y)$ is an invertible matrix.

We define the following functional:

$$\Gamma(\delta, y) = \frac{N-2}{2N} \int_{\mathbb{R}^N} K(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}(x) dx. \tag{1.1}$$

For a set $A \subset \mathbb{R}^+ \times \mathbb{R}^N$, we let $\mathcal{M}_A = \{U_{\delta,y} : (\delta, y) \in A\}$. For a function $u \in D^{1,2}(\mathbb{R}^N)$, let $d(u, \mathcal{M}_A) = \inf_{(\delta,y)\in A} \|u - U_{\delta,y}\|_{D^{1,2}(\mathbb{R}^N)}$. We can now state the following exact multiplicity result which we will prove later in §4.

Theorem 1.4. *Let K satisfy the assumptions* (K1)–(K5). *We further suppose that $\nabla\Gamma$ has finitely many zeroes in $\mathbb{R}^+ \times \mathbb{R}^N$, all of which are stable. Let $A \subset \mathbb{R}^+ \times \mathbb{R}^N$ be any compact set containing the zeroes of $\nabla\Gamma$. Then there exist $\rho_0 = \rho_0(A) > 0$ and $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the problem (P_ε) has exactly the same number of solutions u with $d(u, \mathcal{M}_A) < \rho_0$ as the number of zeroes of $\nabla\Gamma$.*

2. Preliminary results

In this section, we recall some of the well-known results concerning the problem (P_ε) and its linearized version. Given a solution u_ε of (P_ε) we consider the following linearization of (P_ε) about u_ε:

$$((LP)_\varepsilon) \qquad \begin{cases} -\Delta w &= N(N+2)u_\varepsilon^{4/N-2} w \text{ in } \mathbb{R}^N, \\ w &\in D^{1,2}(\mathbb{R}^N). \end{cases}$$

Let $(LP_0)_{\delta,y}$ denote the above linearized problem when $\varepsilon = 0$ and $u_\varepsilon = U_{\delta,y}$ for some $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N$. Then, we have the following characterization of solutions of $(LP_0)_{\delta,y}$.

Theorem 2.1. *Every solution of $(LP_0)_{\delta,y}$ is of the form*

$$w = c_0 \frac{\delta U_{\delta,y}}{\partial \delta} + \sum_{k=1}^{N} c_k \frac{\partial U_{\delta,y}}{\partial y_i} \tag{2.1}$$

for some $c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, N$.

Proof. See ([3]). □

We now recall the following natural decay estimates for solutions of (P_ε) and $(LP)_\varepsilon$.

Theorem 2.2. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence of solutions of (P_ε) with $\sup_{\varepsilon>0} \|u_\varepsilon\|_{D^{1,2}(\mathbb{R}^N)} < \infty$. Let $\{w_\varepsilon\}_{\varepsilon>0}$ be a sequence of solutions of $(LP)_\varepsilon$ with either $\sup_{\varepsilon>0} \|w_\varepsilon\|_{L^\infty(\mathbb{R}^N)} < \infty$ or $\sup_{\varepsilon>0} \|w_\varepsilon\|_{D^{1,2}(\mathbb{R}^N)} < \infty$. Then we have the following decay estimates:*

$$\sup_{\varepsilon>0} \sup_{x\in\mathbb{R}^N} \{|x|^{N-2} u_\varepsilon(x) + |x|^{N-1} |\nabla u_\varepsilon(x)|\} < \infty,$$
$$\sup_{\varepsilon>0} \sup_{x\in\mathbb{R}^N} \{|x|^{N-2} |w_\varepsilon(x)| + |x|^{N-1} |\nabla w_\varepsilon(x)|\} < \infty.$$

Proof. Follows in a standard way by using the Kelvin Transform in $\mathbb{R}^N$ and elliptic regularity. □

Finally, we state the very important Pohozaev identity for (P_ε):

Theorem 2.3. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence of solutions of (P_ε) with $\sup_{\varepsilon>0} \|u_\varepsilon\|_{D^{1,2}(\mathbb{R}^N)} < \infty$. Then the following identities hold:*

$$\int_{\mathbb{R}^N} K(x) u_\varepsilon^{\frac{N+2}{N-2}} \frac{\partial u_\varepsilon}{\partial x_i} = 0, \quad i = 1, \ldots, N, \tag{2.2}$$

$$\int_{\mathbb{R}^N} K(x) u_\varepsilon^{\frac{N+2}{N-2}} [(x-y).\nabla u_\varepsilon + (\frac{N-2}{2}) u_\varepsilon] = 0. \tag{2.3}$$

Proof. See [4]. □

3. Local uniqueness of solutions

We note that if $\{u_\varepsilon\}_{\varepsilon>0}$ is a sequence of solutions of (P_ε) converging (as $\varepsilon \to 0$) to $U_{\delta,y}$ in $D^{1,2}(\mathbb{R}^N)$, then thanks to Theorems 2.2 and 2.3 we obtain that necessarily $\nabla\Gamma(\delta, y) = \mathbf{0}$. We show in this section that for all small enough $\varepsilon > 0$ there is at most one sequence of solutions $\{u_\varepsilon\}$ "bifurcating" from $U_{\delta,y}$ when (δ, y) is a stable zero of $\nabla\Gamma$.

Theorem 3.1. *Let $(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N$ be a stable zero of $\nabla\Gamma$. Let $\{u_{1,\varepsilon}\}_{\varepsilon>0}$ and $\{u_{2,\varepsilon}\}_{\varepsilon>0}$ be two sequences of solutions to (P_ε) with $\|u_{i,\varepsilon} - U_{\delta,y}\|_{D^{1,2}(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0, i = 1, 2$. Then, there exists $\varepsilon_0 > 0$ such that $u_{1,\varepsilon} \equiv u_{2,\varepsilon}$ $\forall \varepsilon \in (0, \varepsilon_0)$.*

Proof. We suppose that (P_ε) admits two distinct sequences of solutions $\{u_{1,\varepsilon_n}\}$ and $\{u_{2,\varepsilon_n}\}$ for some sequence $\varepsilon_n \to 0$ for $i = 1, 2$ with $\|u_{i,\varepsilon_n} - U_{\delta,y}\|_{D^{1,2}(\mathbb{R}^N)} \to 0$ as $\varepsilon_n \to 0$ and arrive at a contradiction. For notational ease, we let $u_{i,n} = u_{i,\varepsilon_n}$. Define $\tilde{w}_n = u_{1,n} - u_{2,n}$. Note that $\|\tilde{w}\|_{D^{1,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. By Theorem 2.2, we note that $\{\tilde{w}_n\}$ is a bounded sequence in $L^\infty(\mathbb{R}^N)$. We let $w_n = \frac{\tilde{w}_n}{\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^N)}}$. Let $x_n \in \mathbb{R}^N$ be such that $|w_n(x_n)| \geq \frac{1}{2}$. We note that w_n satisfies

$$-\Delta w_n = N(N+2)(1 + \varepsilon_n K) c_n w_n,$$

where

$$c_n(x) = \int_0^1 (t u_{1,n}(x) + (1-t) u_{2,n}(x))^{4/N-2} dt.$$

Using standard regularity theory, we get that $w_n \to w$ in $C^2_{loc}(\mathbb{R}^N)$ and that w satisfies

$$-\Delta w = N(N+2) U_{\delta,y}^{4/N-2} w \text{ in } \mathbb{R}^N.$$

Thanks to Theorem 2.1 we obtain that

$$w = c_0 \frac{\partial U_{\delta,y}}{\partial \delta} + \sum_{k=1}^N c_k \frac{\partial U_{\delta,y}}{\partial y_k} \tag{3.1}$$

for some $c_k \in \mathbb{R}, k = 0, 1, \ldots, N$. We now

Claim. $c_k = 0, k = 0, 1, \ldots, N$.

Proof of Claim. From the Pohozaev identity (2.2) we obtain that

$$\int_{\mathbb{R}^N} K(x)\frac{\partial}{\partial x_j}\left(u_{2,n}^{\frac{2N}{N-2}} - u_{1,n}^{\frac{2N}{N-2}}\right) = 0, j = 1, \ldots, N.$$

Using integration by parts and the decay estimates in Theorem 2.2, we obtain from the above equation that

$$\int_{\mathbb{R}^N} \frac{\partial K}{\partial x_j}\left(u_{2,N}^{\frac{2N}{N-2}} - u_{1,n}^{\frac{2N}{N-2}}\right) = 0, j = 1, \ldots, N.$$

Using the Taylor series, we can write the above equation as

$$\int_{\mathbb{R}^N} \frac{\partial K}{\partial x_j}\left(\int_0^1 (tu_{2,n}(x) + (1-t)u_{1,n}(x))^{\frac{N+2}{N-2}} dt\right) w_n(x)dx = 0.$$

Passing to the limit, as $n \to \infty$, in the above equation we obtain

$$\int_{\mathbb{R}^N} \frac{\partial K}{\partial x_j} U_{\delta,y}^{\frac{N+2}{N-2}} w = 0.$$

Once again, integrating by parts in the above equation, we obtain

$$(\frac{N+2}{N-2})\int_{\mathbb{R}^N} K(x)U_{\delta,y}^{4/N-2}\frac{\partial U_{\delta,y}}{\partial x_j} w dx + \int_{\mathbb{R}^N} K(x)U_{\delta,y}^{\frac{N+2}{N-2}}\frac{\partial w}{\partial x_j} dx = 0.$$

Noting that $\frac{\partial U_{\delta,y}}{\partial x_j} = -\frac{\partial U_{\delta,y}}{\partial y_j}$, we obtain from the above equation, for $1 \le j \le N$,

$$\left(\frac{N+2}{N-2}\right)\int_{\mathbb{R}^N} K(x)U_{\delta,y}^{4/N-2}\frac{\partial U_{\delta,y}}{\partial y_j} w + \int_{\mathbb{R}^N} K(x)U_{\delta,y}^{\frac{N+2}{N-2}}\frac{\partial w}{\partial y_j} = 0. \tag{3.2}$$

Integrating by parts on the ball $B_R(y)$, we have for $i = 1, 2$,

$$\begin{aligned}\int_{B_R(y)} (x-y)\cdot\nabla(K(x)u_{i,n}^{\frac{2N}{N-2}}) &= NR\int_{\partial B_R(y)} K(\sigma)u_{i,n}^{\frac{2N}{N-2}}(\sigma)d\sigma \\ &\quad -N\int_{B_R(y)} K(x)u_{i,n}^{\frac{2N}{N-2}}dx.\end{aligned}$$

Letting $R \to \infty$ in the above equation and using the decay estimates in Theorem 2.2 we obtain

$$\int_{\mathbb{R}^N} (x-y)\cdot\nabla(K(x)u_{i,n}^{\frac{2N}{N-2}}(x))dx = -N\int_{\mathbb{R}^N} K(x)u_{i,n}^{\frac{2N}{N-2}}(x)dx, \quad i = 1, 2.$$

The above equation for $i = 1, 2$ can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}^N} (x-y)\cdot\nabla K(x)u_{i,n}^{\frac{2N}{N-2}}(x)dx + \left(\frac{2N}{N-2}\right)\int_{\mathbb{R}^N} K(x)u_{i,n}^{\frac{N+2}{N-2}}(x)(x-y)\cdot\nabla u_{i,n}(x)dx \\ + N\int_{\mathbb{R}^N} K(x)u_{i,n}^{\frac{2N}{N-2}}(x)dx = 0.\end{aligned}$$

Using the Pohozaev identity (2.3) in the above equation, we obtain

$$\int_{\mathbb{R}^N} (x-y)\cdot\nabla K(x) u_{i,n}^{\frac{2N}{N-2}} = 0, \quad i = 1, 2.$$

Subtracting the above identity for $u_{1,n}$ from that for $u_{2,n}$, dividing by the appropriate norm of $u_{2,n} - u_{1,n}$ and finally using Taylor's expansion, we obtain

$$\int_{\mathbb{R}^N} (x-y).\nabla K(x)(\int_0^1 (tu_{1,n}(x) + (1-t)u_{2,n}(x))^{\frac{N+2}{N-2}} dt) w_n(x) dx = 0.$$

Letting $n \to \infty$ in the above equation, we obtain

$$\int_{\mathbb{R}^N} (x-y).\nabla K(x) U_{\delta,y}^{\frac{N+2}{N-2}}(x) w(x) dx = 0.$$

We again integrate by parts in the above equation to obtain

$$\int_{\mathbb{R}^N} K(x) U_{\delta,y}^{\frac{N+2}{N-2}} [Nw + (x-y).\nabla w]$$

$$+ \left(\frac{N+2}{N-2}\right) \int_{\mathbb{R}^N} K(x) U_{\delta,y}^{4/N-2} (x-y)\cdot\nabla U_{\delta,y} w = 0. \tag{3.3}$$

A simple calculation (see the Appendix) using the Pohozaev identities in (2.2)–(2.3) shows that, if we let $\mathbf{c} = (c_0, \ldots, c_N)^T$, then (3.2) and (3.3) imply that

$$D^2\Gamma(\delta, y)\mathbf{c} = \mathbf{0}.$$

Since $D^2\Gamma(\delta, y)$ was assumed to be an invertible matrix, this means that $\mathbf{c} = \mathbf{0}$, thereby proving the Claim.

From the above Claim, it follows that $w \equiv 0$ in $\mathbb{R}^N$. Hence, $w_n \to 0$ in $C^2_{loc}(\mathbb{R}^N)$ which implies that $|x_n| \to \infty$ as $n \to \infty$. Define the following Kelvin Transforms:

$$\left.\begin{array}{rcl} \widehat{u}_{i,n}(x) & = & |x|^{2-N} u_{i,n}\left(\frac{x}{|x|^2}\right), i = 1, 2, \\ \widehat{w}_n(x) & = & |x|^{2-N} w_n\left(\frac{x}{|x|^2}\right), \\ \widehat{c}_n(x) & = & \int_0^1 (t\widehat{u}_{1,n}(x) + (1-t)\widehat{u}_{2,n}(x))^{\frac{4}{N-2}} dt \end{array}\right\} x \in \mathbb{R}^n \setminus \{0\}.$$

Then, it can be checked that $\widehat{w}_n$ satisfies the following equation:

$$-\Delta\widehat{w}_n(x) = N(N+2)(1 + \varepsilon_n K(\frac{x}{|x|^2}))\widehat{c}_n(x)\widehat{w}_n(x) \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Using the decay estimates in Theorem 2.2 we obtain that

$$\sup_n \sup_{x \in B_1(0)\setminus\{0\}} |\widehat{w}_n(x)| < \infty.$$

Since $\widehat{w}_n \to 0$ in $C^2_{loc}(B_1(0) \setminus \{0\})$, by the dominated convergence theorem, we obtain that $\widehat{w}_n \to 0$ in $L^p(B_1(0))$ $\forall p \geq 1$. It is also easy to see that $\{\widehat{c}_n\}$ is a bounded sequence in $L^2(B_1(0))$. Since the capacity of one point set is zero, we can apply the regularity theory to $\widehat{w}_n$ (see Theorem 8.17 of Gilbarg-Trudinger,"Elliptic PDEs of Second Order") and get that

$$\|\widehat{w}_n\|_{L^\infty(B_{1/2}(0))} \leq c\|\widehat{w}_n\|_{L^p(B_1(0))} \ \forall\, p > 1.$$

But this implies $\|\widehat{w}_n\|_{L^\infty(B_{1/2}(0))} \to 0$ as $n \to \infty$, contradicting the fact that $|\widehat{w}_n(\frac{x_n}{|x_n|^2})| \geq \frac{1}{2}$ for all large n. This proves the theorem. □

4. The exact multiplicity result

We are now ready to prove the exact multiplicity result stated in Theorem 1.4.

Proof of Theorem 1.4. Let M be the number of zeroes of $\nabla\Gamma$. Appealing to Theorem 1.2 we obtain that there exists $\varepsilon_1 = \varepsilon_1(A) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, the problem (P_ε) has at least M solutions $\{u^i_\varepsilon\}_{i=1}$ and M points $\{(\delta_i, y_i)\}_{i=1}^M \subset A$ such that $u^i_\varepsilon - U_{\delta_i,y_i} \to 0$ in $D^{1,2}(\mathbb{R}^N), i = 1, \ldots, M$, as $\varepsilon \to 0$. For $\mu > 0$ define

$$\mathcal{S}_\mu = \{u : u \text{ solves } (P_\varepsilon) \text{ for } \varepsilon \in (0, \mu)\} \setminus \{u^i_\varepsilon\}_{0<\varepsilon<\mu,\ 1\leq i\leq M}.$$

Now define the quantity

$$\theta_\mu = \inf_{u\in\mathcal{S}_\mu} d(u, \mathcal{M}_A).$$

We now have the following claim.

Claim. $\theta_0 = \lim_{\mu\to 0} \theta_\mu > 0$.

We suppose that $\theta_0 = 0$ and derive a contradiction. We may then find sequences $\mu_n \to 0, \{u_n\} \subset \mathcal{S}_{\mu_n}$ and $\{(\delta_n, y_n)\} \subset A$ such that $\|u_n - U_{\delta_n,y_n}\|_{D^{1,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Let $(\delta_n, y_n) \to (\delta, y) \in A$. Clearly we have $\nabla\Gamma(\delta, y) = \mathbf{0}$ which means that $\{u_n\}$ is a sequence of solutions "bifurcating" from (δ, y). But using the uniqueness result in Theorem 3.1 we obtain a contradiction since $\{u_n\} \subset \mathcal{S}_{\mu_n}$ for all n. This proves the Claim.

Therefore, we may choose $\mu_0 > 0$ small enough (but fixed) so that $\theta_\mu \geq \frac{\theta_0}{2}$ for all $0 < \mu < \mu_0$. Also from Theorem 1.2 we obtain that for some constant $c > 1$ and $\varepsilon_2 > 0$ we have $d(u^i_\varepsilon, \mathcal{M}_A) \leq c\varepsilon, i = 1, \ldots, M, \varepsilon \in (0, \varepsilon_2)$. The theorem now follows by taking $\rho_0 = \frac{\theta_0}{2}$ and $\varepsilon_0 = \min\{\frac{\theta_0}{2c}, \mu_0, \varepsilon_2\}$. □

5. Appendix

Let $(\delta_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ be a stable critical point of Γ. This in particular means that (see Theorem 1.2) there exists a sequence of solutions $\{u_\varepsilon\}$ of (P_ε) converging to U_{δ_0,y_0} in $D^{1,2}(\mathbb{R}^N)$ as $\varepsilon \to 0$. As a consequence we have that U_{δ_0,y_0} satisfies the Pohozaev identitites (2.2)–(2.3).

Let

$$\Lambda(\delta, y) = \int_{\mathbb{R}^N} K(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}\, dx, \quad (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N. \tag{5.1}$$

Note that $\Lambda = (\frac{N-2}{2N})\Gamma$. We can now state

Proposition A.1. *Let K satisfy assumptions* (K1)–(K5). *Let $(\delta_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ be a stable critical point of Γ. Let*

$$w = c_0 \frac{\partial U_{\delta_0,y_0}}{\partial \delta} + \sum_{k=1}^N c_k \frac{\partial U_{\delta_0,y_0}}{\partial y_k}\ . \tag{5.2}$$

Then (3.2)–(3.3) *imply that $D^2\Gamma(\delta_0, y_0) \cdot (c_0, c_1, \ldots, c_N)^T = \mathbf{0}$.*

Proof. We note that $U_{\delta,y}$ satisfies the relations:

$$U_{\delta,y}(x) = \delta^{\frac{2-N}{2}} U_{1,0}\left(\frac{x-y}{\delta}\right), \tag{5.3}$$

$$\nabla U_{\delta,y}(x) = \delta^{-\frac{N}{2}} \nabla U_{1,0}\left(\frac{x-y}{\delta}\right), \tag{5.4}$$

$$(x-y)\cdot\nabla U_{\delta,y} = -\left[\delta\frac{\partial U_{\delta,y}}{\partial\delta} + \left(\frac{N-2}{2}\right)U_{\delta,y}\right]. \tag{5.5}$$

We further note that since (δ_0, y_0) is a critical point of Γ, we have that U_{δ_0,y_0} satisfies the Pohozaev identities in (2.2)–(2.3). We now compute $D^2\Lambda(\delta_0, y_0)$. We have,

$$\frac{\partial\Lambda}{\partial\delta}(\delta,y) = \int\limits_{\mathbb{R}^N} x\cdot\nabla K(\delta x+y)U_{1,0}^{\frac{2N}{N-2}}(x)dx, \tag{5.6}$$

$$\frac{\partial\Lambda}{\partial y_i}(\delta,y) = \frac{1}{\delta}\int\limits_{\mathbb{R}^N}\frac{\partial K}{\partial x_i}(\delta x+y)U_{1,0}^{\frac{2N}{N-2}}(x)dx, \quad 1\le i\le N. \tag{5.7}$$

In (5.6)–(5.7) we make the change of variable $z = \delta x + y$, use (5.3), integrate by parts and again change back to the x variable to get

$$\begin{aligned} -\delta\frac{\partial\Lambda}{\partial\delta}(\delta,y) &= N\int\limits_{\mathbb{R}^N} K(\delta x+y)U_{1,0}^{\frac{2N}{N-2}}(x)dx \\ &\quad + \left(\frac{2N}{N-2}\right)\int\limits_{\mathbb{R}^N} K(\delta x+y)U_{1,0}^{\frac{N+2}{N-2}}(x)(x\cdot\nabla U_{1,0}(x))dx, \end{aligned} \tag{5.8}$$

$$-\delta\frac{\partial\Lambda}{\partial y_i}(\delta,y) = \left(\frac{2N}{N-2}\right)\int\limits_{\mathbb{R}^N} K(\delta x+y)U_{1,0}^{\frac{N+2}{N-2}}(x)\frac{\partial U_{1,0}}{\partial x_i}(x)dx, \quad 1\le i\le N. \tag{5.9}$$

Differentiating (5.8) on both sides with respect to δ, y_i $(1\le i\le N)$, noting that $\frac{\partial\Lambda}{\partial\delta}(\delta_0,y_0) = \frac{\partial\Lambda}{\partial y_i}(\delta_0,y_0) = 0$, changing variables as $z = \delta x + y$, using (5.3)–(5.5) and integrating by parts, we get

$$\begin{aligned} -\delta_0\frac{\partial^2\Lambda}{\partial\delta^2}(\delta_0,y_0) &= \left(\frac{2N}{N-2}\right)\Bigg[N\int\limits_{\mathbb{R}^N} K(z)U_{\delta_0,y_0}^{\frac{N+2}{N-2}}(z)\frac{\partial U_{\delta_0,y_0}}{\partial\delta}(z)dz \\ &\quad + \int\limits_{\mathbb{R}^N} K(z)U_{\delta_0,y_0}^{\frac{N+2}{N-2}}(z)(z-y_0)\cdot\nabla\left(\frac{\partial U_{\delta_0,y_0}}{\partial\delta}\right)dz \\ &\quad + \left(\frac{N+2}{N-2}\right)\int\limits_{\mathbb{R}^N} K(z)U_{\delta_0,y_0}^{\frac{4}{N-2}}(z)(z-y_0)\cdot\nabla U_{\delta_0,y_0}(z)\frac{\partial U_{\delta_0,y_0}}{\partial\delta}(z)dz\Bigg], \end{aligned} \tag{5.10}$$

$$\begin{aligned} -\frac{\partial^2\Lambda}{\partial\delta\partial y_i}(\delta_0,y_0) &= \left(\frac{2N}{N-2}\right)\Bigg[\int\limits_{\mathbb{R}^N} K(z)U_{\delta_0,y_0}^{\frac{N+2}{N-2}}(z)\frac{\partial}{\partial z_i}\left(\frac{\partial U_{\delta_0},y_0}{\partial\delta}(z)\right)dz \\ &\quad + \left(\frac{N+2}{N-2}\right)\int\limits_{\mathbb{R}^N} K(z)U_{\delta_0,y_0}^{\frac{4}{N-2}}(z)\frac{\partial U_{\delta_0,y_0}}{\partial\delta}(z)\frac{\partial U_{\delta_0,y_0}}{\partial z_i}\,dz\Bigg]. \end{aligned} \tag{5.11}$$

In deriving (5.11) we need to additionally use that the Pohozaev identity (2.2) holds for U_{δ_0,y_0} in the last step.

Again, for $1 \le i, j \le N$, differentiating (5.9) with respect to y_j, using (5.3)–(5.5) and integrating by parts, we get

$$\begin{aligned} -\frac{\partial^2 \Lambda}{\partial y_i \partial y_j}(\delta_0, y_0) &= \left(\frac{2N}{N-2}\right)\Bigg[\int_{\mathbb{R}^N} K(z) U_{\delta_0,y_0}^{\frac{N+2}{N-2}}(z) \frac{\partial}{\partial z_j}\left(\frac{\partial U_{\delta_0,y_0}}{\partial z_i}(z)\right) dz \\ &\quad + \left(\frac{N+2}{N-2}\right)\int_{\mathbb{R}^N} K(z) U_{\delta_0,y_0}^{\frac{4}{N-2}}(z) \frac{\partial U_{\delta_0,y_0}}{\partial z_j^0}(z) \frac{\partial U_{\delta_0,y_0}}{\partial z_i}(z) dz\Bigg]. \end{aligned} \tag{5.12}$$

Let w be as in (5.2). Using (3.2)–(3.3) we can easily check that (5.10)–(5.12) imply that $D^2\Gamma(\delta_0, y_0) \cdot (c_0, c_1, \ldots, c_N)^T = \mathbf{0}$. □

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