EXACT MULTIPLICITY RESULT FOR THE PERTURBED SCALAR CURVATURE PROBLEM IN $\mathbb{R}^N$ ($N \geq 3$)

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Abstract. Let $D^{1,2}(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ in the norm $\|u\|_{D^{1,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla u|^2$. Let $N \geq 3$ and define the constants $\alpha_N = N(N-2)$ and $C_N = (N(N-2))^{N-2/4}$. Let $K \in C^2(\mathbb{R}^N)$. We consider the following problem for $\varepsilon \geq 0$:

$$
\begin{aligned}
\text{Find } u \in D^{1,2}(\mathbb{R}^N) \text{ solving:} \\
-\Delta u = \alpha_N (1 + \varepsilon K(x)) u^{\frac{N+2}{N-2}}, & \quad \text{in } \mathbb{R}^N.
\end{aligned}
$$

We show an exact multiplicity result for $(P_\varepsilon)$ for all small $\varepsilon > 0$.

1. Introduction

Let $D^{1,2}(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$
\|u\|_{D^{1,2}(\mathbb{R}^N)}^2 = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2}.
$$

Let $N \geq 3$ and define the constants $\alpha_N = N(N-2)$ and $C_N = (N(N-2))^{N-2/4}$. Let $K \in C^2(\mathbb{R}^N)$. We consider the following problem for $\varepsilon \geq 0$:

$$
\begin{aligned}
\text{Find } u \in D^{1,2}(\mathbb{R}^N) \text{ solving:} \\
-\Delta u = \alpha_N (1 + \varepsilon K(x)) u^{\frac{N+2}{N-2}}, & \quad \text{in } \mathbb{R}^N.
\end{aligned}
$$

We are interested in showing an exact multiplicity result for $(P_\varepsilon)$ for all small $\varepsilon > 0$ (see Theorem 1.4 below).

The above problem is a “perturbed” version of the well-known scalar curvature problem which arises in differential geometry. More precisely, the problem is to find out if a given smooth function $R$ on the $N$-dimensional unit sphere $S^N$ is the scalar curvature function of a metric $g$ on $S^N$ which is conformal to the standard...
metric. This gives rise to the following problem:

\[
\begin{align*}
(P_g) \quad \text{Find } u \in C^2(S^N) \text{ solving:} \\
\frac{4(N-1)}{N-2} \Delta_g u + Ru^{\frac{N+2}{2}} = N(N-1)u, \\
\end{align*}
\]

in \( S^N \).

The above problem has been extensively studied using the background of differential geometry; see the book of T. Aubin [2] for a survey of the available results.

We now assume that \( R \) is a perturbation of the constant, viz, \( R = 1 + \varepsilon K \) for a smooth function \( K \) on \( S^N \) and \( \varepsilon > 0 \) small. Then, using the standard stereographic projection from \( S^N \) to \( \mathbb{R}^N \), it can be checked that \( (P_g) \) is transformed to \( (P) \).

Existence of solutions to \((P)\) was done in [1] using variational methods and finite-dimensional reduction techniques. To describe their result, we make the following assumptions on \( K \):

(K1) \( K \in C^2(\mathbb{R}^N), \|K\|_{L^\infty(\mathbb{R}^N)} + \|\nabla K\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} + \|D^2 K\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N \times \mathbb{R}^N)} < \infty. \)

(K2) (a) There exists \( \rho > 0 \) such that \( \langle \nabla K(x), x \rangle < 0 \) \( \forall |x| \geq \rho \),

(b) \( \langle \nabla K(x), x \rangle \in L^1(\mathbb{R}^N), \int \langle \nabla K(x), x \rangle dx < 0. \)

(K3) The set of all critical points of \( K \), denoted by crit \( (K) \), is finite.

(K4) \( \forall \xi \in \text{crit} \( (K) \), \) there exists \( \beta = \beta_\xi \in [1, N] \) and \( Q_\xi : \mathbb{R}^N \to \mathbb{R} \) depending continuously on \( y \) locally near \( \xi \) such that

\[ A_\xi \overset{\text{def}}{=} \frac{N-2}{2N} \int_{\mathbb{R}^N} Q_\xi(y)U_{1,0}^{\frac{2N}{N-2}}(y)dy \neq 0 \]

and the following relations hold:

\[
\begin{cases}
Q_\xi(\lambda x) = \lambda^\beta Q_\xi(x), & \forall \lambda \geq 0, \\
K(x) = K(y) + Q_\xi(x-y) + o(|x-y|^\beta) & \text{as } x \to y, \\
\sum_{\xi \in \text{crit} (K)} \text{deg}_{\text{loc}}(\nabla K, \xi) \neq (-1)^N. 
\end{cases}
\]

We also recall the following well-known classification result for solutions of \((P)\):

\[ \text{Theorem 1.1. Solutions of } (P_0) \text{ form an } (N+1)\text{-dimensional manifold given by} \]

\[ \mathcal{M} = \{ U_{\delta,y}(x) \overset{\text{def}}{=} C_N \delta^{\frac{N-2}{2}}(\delta^2 + |x-y|^2)^{\frac{N-2}{2}} : (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N \}. \]

We can now state the following existence result.

\[ \text{Theorem 1.2 (I). Let } K \text{ satisfy the assumptions (K1)-(K5). Then there exists } \varepsilon_0 > 0 \text{ such that the following hold:} \]

(i) \((P_\varepsilon) \) has a solution \( \forall \varepsilon \in (0, \varepsilon_0) \),

(ii) \( \forall \varepsilon \in (0, \varepsilon_0), \) there exists \( (\delta_\varepsilon, \nu_\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^N \) such that for any compact set \( A \subset \mathbb{R}^+ \times \mathbb{R}^N, \) we may find a constant \( c(A) > 0 \) such that \( \|u_\varepsilon - U_{\delta_\varepsilon, \nu_\varepsilon}\|_{D^1,2(\mathbb{R}^N)} \leq c(A)\varepsilon. \)

We now define what we mean by a stable zero of a vector field.

\[ \text{Definition 1.3. Let } G : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^{N+1} \text{ be a } C^1 \text{ vector field. We say that} \]

a point \( (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N \) is a stable zero for \( G \) if \( G(\delta, y) = 0 \) and its derivative \( DG(\delta, y) \) is an invertible matrix.
We define the following functional:

\[
\Gamma(\delta, y) = \frac{N - 2}{2N} \int_{\mathbb{R}^N} K(\delta x + y) u_{1,0}^{\frac{2N}{N-2}}(x) \, dx.
\]

For a set \( A \subset \mathbb{R}^+ \times \mathbb{R}^N \), we let \( \mathcal{M}_A = \{ U_{\delta, y} : (\delta, y) \in A \} \). For a function \( u \in D^{1,2} (\mathbb{R}^N) \), let \( d(u, \mathcal{M}_A) = \inf_{(\delta, y) \in A} \| u - U_{\delta, y} \|_{D^{1,2}(\mathbb{R}^N)} \). We can now state the following exact multiplicity result which we will prove later in §4.

**Theorem 1.4.** Let \( K \) satisfy the assumptions (K1)–(K5). We further suppose that \( \nabla \Gamma \) has finitely many zeroes in \( \mathbb{R}^+ \times \mathbb{R}^N \), all of which are stable. Let \( A \subset \mathbb{R}^+ \times \mathbb{R}^N \) be any compact set containing the zeroes of \( \nabla \Gamma \). Then there exist \( \rho_0 = \rho_0(A) > 0 \) and \( \varepsilon_0 = \varepsilon_0(\rho) > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), the problem \( (P_\varepsilon) \) has exactly the same number of solutions \( u \) with \( d(u, \mathcal{M}_A) < \rho_0 \) as the number of zeroes of \( \nabla \Gamma \).

2. Preliminary results

In this section, we recall some of the well-known results concerning the problem \( (P_\varepsilon) \) and its linearized version. Given a solution \( u_\varepsilon \) of \( (P_\varepsilon) \) we consider the following linearization of \( (P_\varepsilon) \) about \( u_\varepsilon \):

\[
(LP_\varepsilon) \quad \begin{cases} -\Delta w = N(N + 2) u_\varepsilon^{4/N - 2} w & \text{in } \mathbb{R}^N, \\ w \in D^{1,2}(\mathbb{R}^N). \end{cases}
\]

Let \( (LP_0)_{\delta, y} \) denote the above linearized problem when \( \varepsilon = 0 \) and \( u_\varepsilon = U_{\delta, y} \) for some \( (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N \). Then, we have the following characterization of solutions of \( (LP_0)_{\delta, y} \).

**Theorem 2.1.** Every solution of \( (LP_0)_{\delta, y} \) is of the form

\[
w = c_0 \frac{\delta U_{\delta, y}}{\partial \delta} + \sum_{k=1}^{N} c_k \frac{\partial U_{\delta, y}}{\partial y_i}
\]

for some \( c_i \in \mathbb{R}, \ i = 0, 1, \ldots, N \).

**Proof.** See (3). \( \square \)

We now recall the following natural decay estimates for solutions of \( (P_\varepsilon) \) and \( (LP_\varepsilon) \).

**Theorem 2.2.** Let \( \{ u_\varepsilon \}_{\varepsilon > 0} \) be a sequence of solutions of \( (P_\varepsilon) \) with \( \sup_{\varepsilon > 0} \| u_\varepsilon \|_{D^{1,2}(\mathbb{R}^N)} < \infty \). Let \( \{ w_\varepsilon \}_{\varepsilon > 0} \) be a sequence of solutions of \( (LP_\varepsilon) \) with \( \sup_{\varepsilon > 0} \| w_\varepsilon \|_{L^\infty(\mathbb{R}^N)} < \infty \). Then we have the following decay estimates:

\[
\sup_{\varepsilon > 0} \sup_{x \in \mathbb{R}^N} \{ |x|^{N-2} u_\varepsilon(x) + |x|^{N-1} |\nabla u_\varepsilon(x)| \} < \infty,
\]

\[
\sup_{\varepsilon > 0} \sup_{x \in \mathbb{R}^N} \{ |x|^{N-2} w_\varepsilon(x) + |x|^{N-1} |\nabla w_\varepsilon(x)| \} < \infty.
\]

**Proof.** Follows in a standard way by using the Kelvin Transform in \( \mathbb{R}^N \) and elliptic regularity. \( \square \)
Finally, we state the very important Pohozaev identity for \((P_z)\):

**Theorem 2.3.** Let \(\{u_{\varepsilon}\}_{\varepsilon > 0}\) be a sequence of solutions of \((P_z)\) with \(\sup_{\varepsilon > 0} \|u_{\varepsilon}\|_{D^{1,2}(\mathbb{R}^N)} < \infty\). Then the following identities hold:

\[
\int_{\mathbb{R}^N} K(x)u_{\varepsilon}^{N+2} \frac{\partial u_{\varepsilon}}{\partial x_i} = 0, \quad i = 1, \ldots, N, \tag{2.2}
\]

\[
\int_{\mathbb{R}^N} K(x)u_{\varepsilon}^{N+2} [(x - y) \cdot \nabla u_{\varepsilon} + \left(\frac{N-2}{2}\right)u_{\varepsilon}] = 0. \tag{2.3}
\]

**Proof.** See [4]. \qed

### 3. Local uniqueness of solutions

We note that if \(\{u_{\varepsilon}\}_{\varepsilon > 0}\) is a sequence of solutions of \((P_z)\) converging (as \(\varepsilon \to 0\)) to \(U_{\delta,y}\) in \(D^{1,2}(\mathbb{R}^N)\), then thanks to Theorems 2.2 and 2.3 we obtain that necessarily \(\nabla \Gamma(\delta, y) = 0\). We show in this section that for all small enough \(\varepsilon > 0\) there is at most one sequence of solutions \(\{u_{\varepsilon}\}\) “bifurcating” from \(U_{\delta,y}\) when \((\delta, y)\) is a stable zero of \(\nabla \Gamma\).

**Theorem 3.1.** Let \((\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N\) be a stable zero of \(\nabla \Gamma\). Let \(\{u_{1,\varepsilon}\}_{\varepsilon > 0}\) and \(\{u_{2,\varepsilon}\}_{\varepsilon > 0}\) be two sequences of solutions to \((P_z)\) with \(\|u_{i,\varepsilon} - U_{\delta,y}\|_{D^{1,2}(\mathbb{R}^N)} \to 0\) as \(\varepsilon \to 0\), \(i = 1, 2\). Then, there exists \(\varepsilon_0 > 0\) such that \(u_{1,\varepsilon} \equiv u_{2,\varepsilon} \forall \varepsilon \in (0, \varepsilon_0)\).

**Proof.** We suppose that \((P_z)\) admits two distinct sequences of solutions \(\{u_{1,\varepsilon}\}_{\varepsilon > 0}\) and \(\{u_{2,\varepsilon}\}_{\varepsilon > 0}\) for some sequence \(\varepsilon_0 \to 0\) for \(i = 1, 2\) with \(\|u_{i,\varepsilon} - U_{\delta,y}\|_{D^{1,2}(\mathbb{R}^N)} \to 0\) as \(\varepsilon \to 0\) and arrive at a contradiction. For notational ease, we let \(u_{i,n} = u_{i,\varepsilon_n}\). Define \(\tilde{w}_n = u_{1,n} - u_{2,n}\). Note that \(\|\tilde{w}\|_{D^{1,2}(\mathbb{R}^N)} \to 0\) as \(n \to \infty\). By Theorem 2.2, we note that \(\{\tilde{w}_n\}\) is a bounded sequence in \(L^\infty(\mathbb{R}^N)\). We let \(w_n = \frac{\tilde{w}_n}{\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^N)}}\).

Let \(x_n \in \mathbb{R}^N\) be such that \(|w_n(x_n)| \geq \frac{1}{2}\). We note that \(w_n\) satisfies

\[-\Delta w_n = N(N + 2)(1 + \varepsilon_n K)c_n w_n,\]

where

\[c_n(x) = \int_0^1 \left(tu_{1,n}(x) + (1-t)u_{2,n}(x)\right)^{4/N-2} dt.\]

Using standard regularity theory, we get that \(w_n \to w\) in \(C^2_{\text{loc}}(\mathbb{R}^N)\) and that \(w\) satisfies

\[-\Delta w = N(N + 2)U_{\delta,y}^{4/N-2} w\text{ in } \mathbb{R}^N.\]

Thanks to Theorem 2.1 we obtain that

\[w = c_0 \frac{\partial U_{\delta,y}}{\partial \delta} + \sum_{k=1}^N c_k \frac{\partial U_{\delta,y}}{\partial y_k}\]

for some \(c_k \in \mathbb{R}, k = 0, 1, \ldots, N\). We now

**Claim.** \(c_k = 0, k = 0, 1, \ldots, N\).
Proof of Claim. From the Pohozaev identity (2.2) we obtain that
\[ \int_{\mathbb{R}^N} K(x) \frac{\partial}{\partial x_j} \left( \frac{2N}{u_{2,n}^{2N}} - \frac{2N}{u_{1,n}^{2N}} \right) = 0, \quad j = 1, \ldots, N. \]

Using integration by parts and the decay estimates in Theorem 2.2, we obtain from the above equation that
\[ \int_{\mathbb{R}^N} \frac{\partial K}{\partial x_j} \left( \frac{2N}{u_{2,n}^{2N}} - \frac{2N}{u_{1,n}^{2N}} \right) = 0, \quad j = 1, \ldots, N. \]

Using the Taylor series, we can write the above equation as
\[ \int_{\mathbb{R}^N} \frac{\partial K}{\partial x_j} \left( \frac{1}{tu_{2,n}(x)} + \frac{1}{u_{1,n}(x)} \right) \frac{N+2}{N-2} \frac{N}{2} w_n(x) dx = 0. \]

Passing to the limit, as \( n \to \infty \), in the above equation we obtain
\[ \int_{\mathbb{R}^N} \frac{\partial K}{\partial x_j} U_{\delta,y}^{N+2} w = 0. \]

Once again, integrating by parts in the above equation, we obtain
\[ \left( \frac{N+2}{N-2} \right) \int_{\mathbb{R}^N} K(x) U_{\delta,y}^{4/N-2} \frac{\partial U_{\delta,y}}{\partial x_j} w dx + \int_{\mathbb{R}^N} K(x) U_{\delta,y}^{\frac{N+2}{N-2}} \frac{\partial w}{\partial x_j} dx = 0. \]

Noting that \( \frac{\partial U_{\delta,y}}{\partial x_j} = -\frac{\partial U_{\delta,y}}{\partial y_j} \), we obtain from the above equation, for \( 1 \leq j \leq N \),
\[ \left( \frac{N+2}{N-2} \right) \int_{\mathbb{R}^N} K(x) U_{\delta,y}^{4/N-2} \frac{\partial U_{\delta,y}}{\partial y_j} w dx + \int_{\mathbb{R}^N} K(x) U_{\delta,y}^{\frac{N+2}{N-2}} \frac{\partial w}{\partial y_j} dx = 0. \]

Integrating by parts on the ball \( B_R(y) \), we have for \( i = 1, 2 \),
\[ \int_{B_R(y)} (x - y) \cdot \nabla (K(x) u_{i,n}^{\frac{2N}{N-2}}) = NR \int_{\partial B_R(y)} K(\sigma) u_{i,n}^{\frac{2N}{N-2}} (\sigma) d\sigma - N \int_{B_R(y)} K(x) u_{i,n}^{\frac{2N}{N-2}} dx. \]

Letting \( R \to \infty \) in the above equation and using the decay estimates in Theorem 2.2 we obtain
\[ \int_{\mathbb{R}^N} (x - y) \cdot \nabla (K(x) u_{i,n}^{\frac{2N}{N-2}}) dx = -N \int_{\mathbb{R}^N} K(x) u_{i,n}^{\frac{2N}{N-2}} dx, \quad i = 1, 2. \]

The above equation for \( i = 1, 2 \) can be rewritten as
\[ \int_{\mathbb{R}^N} (x - y) \cdot \nabla K(x) u_{i,n}^{\frac{2N}{N-2}} dx + \left( \frac{2N}{N-2} \right) \int_{\mathbb{R}^N} K(x) u_{i,n}^{\frac{N+2}{N-2}} (x) (x - y) \cdot \nabla u_{i,n} dx + N \int_{\mathbb{R}^N} K(x) u_{i,n}^{\frac{2N}{N-2}} dx = 0. \]
Using the Pohozaev identity (2.3) in the above equation, we obtain
\[ \int_{\mathbb{R}^N} (x - y) \cdot \nabla K(x) u_{i,n}^{2N} = 0, \quad i = 1, 2. \]

Subtracting the above identity for \( u_{1,n} \) from that for \( u_{2,n} \), dividing by the appropriate norm of \( u_{2,n} - u_{1,n} \) and finally using Taylor's expansion, we obtain
\[ \int_{\mathbb{R}^N} (x - y) \cdot \nabla K(x) \left( \int_0^1 (tu_{1,n}(x) + (1 - t)u_{2,n}(x))^\frac{N+2}{N-2} (1 - t)u_n(x) dx \right) = 0. \]

Letting \( n \to \infty \) in the above equation, we obtain
\[ \int_{\mathbb{R}^N} (x - y) \cdot \nabla K(x) U_n^{\frac{N+2}{N-2}}(x) w(x) dx = 0. \]

We again integrate by parts in the above equation to obtain
\[ \int_{\mathbb{R}^N} K(x) U_n^{\frac{N+2}{N-2}} [Nw + (x - y) \cdot \nabla w] \]
\[ + \left( \frac{N+2}{N-2} \right) \int_{\mathbb{R}^N} K(x) U_n^{\frac{4}{N-2}} (x - y) \cdot \nabla U_n w = 0. \]

A simple calculation (see the Appendix) using the Pohozaev identities in (2.2) and (2.3) shows that, if we let \( c = (c_0, \ldots, c_N)^T \), then (2.2) and (2.3) imply that
\[ D^2 \Gamma(\delta, y) c = 0. \]

Since \( D^2 \Gamma(\delta, y) \) was assumed to be an invertible matrix, this means that \( c = 0 \), thereby proving the Claim.

From the above Claim, it follows that \( w \equiv 0 \) in \( \mathbb{R}^N \). Hence, \( w_n \to 0 \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \) which implies that \( |x_n| \to \infty \) as \( n \to \infty \). Define the following Kelvin Transforms:
\[
\begin{align*}
\hat{u}_{i,n}(x) &= |x|^{2-N} u_{i,n} \left( \frac{x}{|x|^2} \right), i = 1, 2, \\
\hat{w}_n(x) &= |x|^{2-N} w_n \left( \frac{x}{|x|^2} \right), \\
\hat{c}_n(x) &= \frac{1}{N-2} \int_0^1 (t \hat{u}_{1,n}(x) + (1-t)\hat{u}_{2,n}(x)) \left( \frac{x}{|x|^2} \right) dt.
\end{align*}
\]

Then, it can be checked that \( \hat{w}_n \) satisfies the following equation:
\[-\Delta \hat{w}_n(x) = N(N + 2)(1 + \varepsilon_n K(\frac{x}{|x|^2})) \hat{c}_n(x) \hat{w}_n(x) \text{ in } \mathbb{R}^N \setminus \{0\}.\]

Using the decay estimates in Theorem 2.2 we obtain that
\[ \sup_n \sup_{x \in B_1(0) \setminus \{0\}} |\hat{w}_n(x)| < \infty. \]

Since \( \hat{w}_n \to 0 \) in \( C^2_{\text{loc}}(B_1(0) \setminus \{0\}) \), by the dominated convergence theorem, we obtain that \( \hat{w}_n \to 0 \) in \( L^p(B_1(0)) \) for \( p \geq 1 \). It is also easy to see that \( \{\hat{c}_n\} \) is a bounded sequence in \( L^2(B_1(0)) \). Since the capacity of one point set is zero, we can apply the regularity theory to \( \hat{w}_n \) (see Theorem 8.17 of Gilbarg-Trudinger,”Elliptic PDEs of Second Order”) and get that
\[ \|\hat{w}_n\|_{L^\infty(B_{1/2}(0))} \leq c \|\hat{w}_n\|_{L^p(B_1(0))} \forall p > 1. \]
But this implies $\|\tilde{w}_n\|_{L^\infty(B_{1/2}(0))} \to 0$ as $n \to \infty$, contradicting the fact that $|\tilde{w}_n(x_n^*)| \geq \frac{1}{2}$ for all large $n$. This proves the theorem. \hfill $\Box$

4. THE EXACT MULTIPLICITY RESULT

We are now ready to prove the exact multiplicity result stated in Theorem 1.4.

Proof of Theorem 1.4. Let $M$ be the number of zeroes of $\nabla \Gamma$. Appealing to Theorem 1.2 we obtain that there exists $\varepsilon_1 = \varepsilon_1(A) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, the problem $(P_\varepsilon)$ has at least $M$ solutions $\{u_i^\varepsilon\}_{i=1}^M$ and $M$ points $\{(\varepsilon_i, \gamma_i)\}_{i=1}^M \subset A$ such that $u_i^\varepsilon - U_{\varepsilon_i, \gamma_i} \to 0$ in $D^{1,2}(\mathbb{R}^N)$, $i = 1, \ldots, M$, as $\varepsilon \to 0$. For $\mu > 0$ define

\[ S_\mu = \{ u : u \text{ solves } (P_\varepsilon) \text{ for } \varepsilon \in (0, \mu) \} \setminus \{ u_i^\varepsilon \}_{0 < \varepsilon < \mu, 1 \leq i \leq M}. \]

Now define the quantity

\[ \theta_\mu = \inf_{u \in S_\mu} d(u, \mathcal{M}_A). \]

We now have the following claim.

Claim. $\theta_0 = \lim_{\mu \to 0} \theta_\mu > 0$.

We suppose that $\theta_0 = 0$ and derive a contradiction. We may then find sequences $\mu_n \to 0$, $\{u_n\} \subset S_{\mu_n}$ and $\{(\varepsilon_n, \gamma_n)\} \subset A$ such that $\|u_n - U_{\varepsilon_n, \gamma_n}\|_{D^{1,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Let $(\varepsilon_n, \gamma_n) \to (\varepsilon, \gamma) \in A$. Clearly we have $\nabla \Gamma(\varepsilon, \gamma) = 0$ which means that $\{u_n\}$ is a sequence of solutions “bifurcating” from $(\varepsilon, \gamma)$. But using the uniqueness result in Theorem 3.1 we obtain a contradiction since $\{u_n\} \subset S_{\mu_n}$ for all $n$. This proves the Claim.

Therefore, we may choose $\mu_0 > 0$ small enough (but fixed) so that $\theta_\mu \geq \frac{\theta_0}{2}$ for all $0 < \mu < \mu_0$. Also from Theorem 1.2 we obtain that for some constant $c > 1$ and $\varepsilon_2 > 0$ we have $d(u_i^\varepsilon, \mathcal{M}_A) \leq c\varepsilon$, $i = 1, \ldots, M$, $\varepsilon \in (0, \varepsilon_2)$. The theorem now follows by taking $\rho_0 = \frac{\theta_0}{2}$ and $\varepsilon_0 = \min\{\frac{\theta_0}{2c}, \mu_0, \varepsilon_2\}$. \hfill $\Box$

5. APPENDIX

Let $(\delta_0, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ be a stable critical point of $\Gamma$. This in particular means that (see Theorem 1.2) there exists a sequence of solutions $\{u_i^\varepsilon\}$ of $(P_\varepsilon)$ converging to $U_{\delta_0, \gamma_0}$ in $D^{1,2}(\mathbb{R}^N)$ as $\varepsilon \to 0$. As a consequence we have that $U_{\delta_0, \gamma_0}$ satisfies the Pohozaev identities (2.2)–(2.3).

Let

\[ \Lambda(\delta, y) = \int_{\mathbb{R}^N} K(\delta x + y)U_{\delta_0, \gamma_0}^{\frac{N}{2}} \, dx, \quad (\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N. \]

Note that $\Lambda = (\frac{N-2}{2N}) \Gamma$. We can now state

Proposition A.1. Let $K$ satisfy assumptions (K1)–(K5). Let $(\delta_0, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R}^N$ be a stable critical point of $\Gamma$. Let

\[ w = c_0 \frac{\partial U_{\delta_0, \gamma_0}}{\partial \delta} + \sum_{k=1}^N c_k \frac{\partial U_{\delta_0, \gamma_0}}{\partial y_k}. \]

Then (2.2)–(2.3) imply that $D^2 \Gamma(\delta_0, \gamma_0) \cdot (c_0, c_1, \ldots, c_N)^T = 0$. 

Proof. We note that \( U_{\delta, y} \) satisfies the relations:

\[
U_{\delta, y}(x) = \delta^{\frac{N-2}{2}} U_{1,0}\left(\frac{x-y}{\delta}\right),
\]

\[
\nabla U_{\delta, y}(x) = \delta^{-\frac{N}{2}} \nabla U_{1,0}\left(\frac{x-y}{\delta}\right),
\]

\[
(x - y) \cdot \nabla U_{\delta, y} = -\left[ \delta \frac{\partial U_{\delta, y}}{\partial \delta} + \left( \frac{N-2}{2} \right) U_{\delta, y} \right].
\]

We further note that since \((\delta_0, y_0)\) is a critical point of \( \Gamma \), we have that \( U_{\delta_0, y_0} \) satisfies the Pohozaev identities in (2.2)–(2.3). We now compute \( D^2 \Lambda(\delta_0, y_0) \). We have,

\[
\frac{\partial \Lambda}{\partial \delta}(\delta, y) = \int_{\mathbb{R}^N} x \cdot \nabla K(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}(x) dx,
\]

\[
\frac{\partial \Lambda}{\partial y_i}(\delta, y) = \frac{1}{\delta} \int_{\mathbb{R}^N} \frac{\partial K}{\partial x_i}(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}(x) dx, \quad 1 \leq i \leq N.
\]

In (5.6)–(5.7) we make the change of variable \( z = \delta x + y \), use (5.3), integrate by parts and again change back to the \( x \) variable to get

\[
-\delta \frac{\partial \Lambda}{\partial \delta}(\delta, y) = N \int_{\mathbb{R}^N} K(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}(x) dx
\]

\[
+ \left( \frac{2N}{N-2} \right) \int_{\mathbb{R}^N} K(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}(x) (x \cdot \nabla U_{1,0}(x)) dx,
\]

\[
-\frac{\partial \Lambda}{\partial y_i}(\delta, y) = \left( \frac{2N}{N-2} \right) \int_{\mathbb{R}^N} K(\delta x + y) U_{1,0}^{\frac{2N}{N-2}}(x) \frac{\partial U_{1,0}}{\partial x_i}(x) dx, \quad 1 \leq i \leq N.
\]

Differentiating (5.8) on both sides with respect to \( \delta, y_i \) \( (1 \leq i \leq N) \), noting that \( \frac{\partial \Lambda}{\partial \delta}(\delta_0, y_0) = \frac{\partial \Lambda}{\partial y_i}(\delta_0, y_0) = 0 \), changing variables as \( z = \delta x + y \), using (5.3)–(5.5) and integrating by parts, we get

\[
-\frac{\partial^2 \Lambda}{\partial \delta^2}(\delta_0, y_0) = \left( \frac{2N}{N-2} \right) \int_{\mathbb{R}^N} K(z) U_{\delta_0, y_0}^{\frac{N+2}{N-2}}(z) \frac{\partial U_{\delta_0, y_0}}{\partial \delta}(z) dz
\]

\[
+ \int_{\mathbb{R}^N} K(z) U_{\delta_0, y_0}^{\frac{N+2}{N-2}}(z) (z - y_0) \cdot \nabla \left( \frac{\partial U_{\delta_0, y_0}}{\partial \delta} \right) dz
\]

\[
+ \left( \frac{N+2}{N-2} \right) \int_{\mathbb{R}^N} K(z) U_{\delta_0, y_0}^{\frac{N+2}{N-2}}(z) (z - y_0) \cdot \nabla U_{\delta_0, y_0}(z) \frac{\partial U_{\delta_0, y_0}}{\partial \delta}(z) dz,
\]

\[
-\frac{\partial^2 \Lambda}{\partial \delta \partial y_i}(\delta_0, y_0) = \left( \frac{2N}{N-2} \right) \int_{\mathbb{R}^N} K(z) U_{\delta_0, y_0}^{\frac{N+2}{N-2}}(z) \frac{\partial}{\partial z_i} \left( \frac{\partial U_{\delta_0, y_0}}{\partial \delta} \right)(z) dz
\]

\[
+ \left( \frac{N+2}{N-2} \right) \int_{\mathbb{R}^N} K(z) U_{\delta_0, y_0}^{\frac{N+2}{N-2}}(z) \frac{\partial}{\partial z_i} \left( \frac{\partial U_{\delta_0, y_0}}{\partial \delta} \right)(z) dz.
\]
In deriving (5.11) we need to additionally use that the Pohozaev identity (2.2) holds for $U_{\delta_0,y_0}$ in the last step.

Again, for $1 \leq i, j \leq N$, differentiating (5.9) with respect to $y_j$, using (5.3)–(5.5) and integrating by parts, we get

$$- \frac{\partial^2 \Lambda}{\partial y_i \partial y_j}(\delta_0,y_0) = \left( \frac{2N}{N-2} \right) \int_{\mathbb{R}^N} K(z) U^{\frac{N+2}{2}}_{\delta_0,y_0}(z) \frac{\partial}{\partial z_j} \left( \frac{\partial U_{\delta_0,y_0}}{\partial z_i}(z) \right) dz$$

(5.12)

$$+ \left( \frac{N+2}{N-2} \right) \int_{\mathbb{R}^N} K(z) U_{\delta_0,y_0}(z) \frac{\partial U_{\delta_0,y_0}}{\partial z_j}(z) \frac{\partial U_{\delta_0,y_0}}{\partial z_i}(z) dz.$$

Let $w$ be as in (5.2). Using (3.2)–(3.3) we can easily check that (5.10)–(5.12) imply that $D^2 \Gamma(\delta_0,y_0) \cdot (c_0,c_1,\ldots,c_N)^T = 0$.

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**References**


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