FENCHEL DUALITY, FITZPATRICK FUNCTIONS AND THE EXTENSION OF FIRMLY NONEXPANSIVE MAPPINGS

HEINZ H. BAUSCHKE

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Abstract. Recently, S. Reich and S. Simons provided a novel proof of the Kirszbraun-Valentine extension theorem using Fenchel duality and Fitzpatrick functions. In the same spirit, we provide a new proof of an extension result for firmly nonexpansive mappings with an optimally localized range.

Throughout this paper, we assume that $X$ is a real Hilbert space, with inner product $p = \langle \cdot | \cdot \rangle$ and induced norm $\| \cdot \|$, and we denote the identity mapping on $X$ by $\text{Id}$. A mapping $T$ from a subset $D$ of $X$ to $X$ is called firmly nonexpansive if

\begin{equation}
(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2;
\end{equation}

equivalently [13,14], if $2T - \text{Id}$ is nonexpansive (Lipschitz continuous with constant 1), i.e.,

\begin{equation}
(\forall x \in D)(\forall y \in D) \quad \|(2T - \text{Id})x - (2T - \text{Id})y\| \leq \|x - y\|
\end{equation}
or if

\begin{equation}
(\forall x \in D)(\forall y \in D) \quad 0 \leq \langle Tx - Ty | (\text{Id} - T)x - (\text{Id} - T)y \rangle.
\end{equation}

Firmly nonexpansive mappings play an important role in various contexts; see, e.g., [1,2,3,7,8,9,10,15,17,21,22,25]. The Kirszbraun-Valentine theorem (see, e.g., [5,13,16,20,26]) states that any nonexpansive mapping can be extended to a nonexpansive mapping defined on the whole space. A beautiful proof of this result, based on Fenchel duality and Fitzpatrick functions, was recently provided by Reich and Simons [23]. (For further applications of Fitzpatrick functions, see, e.g., [4,24].)

In this note, we refine their technique to obtain a new proof of an extension theorem for firmly nonexpansive mappings where the range of the extension is optimally localized. This extension theorem easily implies the Kirszbraun-Valentine result. Notation not explicitly defined in the following is standard in convex analysis; see, e.g., [27].
Definition 1. Let $D$ be a nonempty subset of $X$ and let $T: D \to X$ be firmly nonexpansive. Then the associated Fitzpatrick function \(12\) $\phi = \phi_T$ is

\[
\phi = (\iota_G + p)^*.
\]

(4) $X \times X \to [-\infty, +\infty] : (x, y) \mapsto \sup_{d \in D} \langle x | d - Td \rangle + \langle y | Td \rangle - \langle d - Td \rangle,$

and we also set $G = G_T = \{ (d - Td, Td) | d \in D \}.$

Proposition 2. Let $D$ be a nonempty subset of $X$, let $T: D \to X$ be firmly nonexpansive, and let $x$ and $y$ be in $X$. Then:

(i) $\phi = (\iota_G + p)^*.$

(ii) The extension $\overline{T}: D \cup \{ y \} \to X$ of $T$ which maps $y$ to $x$ is still firmly nonexpansive if and only if $\phi(y - x, x) \leq p(y - x, x).$

(iii) $\phi$ is convex, lower semicontinuous and proper.

(iv) $\text{conv} G \subset \text{dom} \phi^* \subset \text{conv} G \subset \text{conv} (\text{Id} - T)(D) \times \text{conv} T(D).$

(v) $p \leq \phi^*.$

Proof. Fix $x$ and $y$ in $X.$

(i) By (i), \(\phi \leq p\) on $G$. Hence $\phi$ is proper. The function $\phi$ is convex and lower semicontinuous, as it is a Fenchel conjugate by (i) and (iv). This is clear, since $G = \text{dom}(\iota_G + p)$ and $\phi = (\iota_G + p)^*.$

Fact 3 (Fenchel duality). Let $Y$ be a real Hilbert space and let $L: Y \to X$ be linear and continuous. Let $f: Y \to [-\infty, +\infty]$ and $g: X \to [-\infty, +\infty]$ be convex, lower semicontinuous, and proper such that $g$ is continuous and finite at some point in $\text{dom} f.$

Then

\[
\inf_{y \in Y} \{ f(y) + g(Ly) \} = -\min_{x \in X} \{ f^* (-L^* x) + g^* (x) \}.
\]

Theorem 4. Let $D$ be a nonempty subset of $X$, let $T: D \to X$ be firmly nonexpansive, and let $y \in X$. Then $T$ has a firmly nonexpansive extension $\overline{T}: D \cup \{ y \} \to \text{conv} T(D).$

Proof. Set $\phi = \phi_T$ and $C = \text{conv} T(D)$, and assume first that $y = 0$. In view of Proposition 2(ii) we must show that

\[
\min_{x \in X} \phi(x, -x) + ||x||^2 + \iota_C(x) \leq 0.
\]

Set $f: X \times X \to [-\infty, +\infty] : (x^*, y^*) \mapsto \frac{1}{2} \phi^*(2x^*, 2y^*)$ so that $f^* = \frac{1}{2}\phi$, and let $j = \frac{1}{2} \iota_C$. Now set $g = (j + \iota_C)^*$ and observe (using 14) that $g = j \iota_C = j - (j \square \iota_C) = j - \frac{1}{2} d_C^2$, where $\square$ denotes the infimal convolution and $d_C$ the distance function. Further set $L: X \times X \to X : (x^*, y^*) \mapsto y^* - x^*$. We claim that

\[
\inf_{(x^*, y^*) \in X \times X} f(x^*, y^*) + g(L(x^*, y^*)) \geq 0.
\]

Indeed, pick $(x^*, y^*) \in \text{dom} f$. By
Proposition \cite{iv} \((2x^*, 2y^*) \in \text{dom} \phi^* \subset X \times C\) and hence \(2y^* \in C\). Using Proposition \cite{iv} we deduce that
\begin{align*}
0 &= 4 \langle x^* \mid y^* \rangle + \|y^* - x^*\|^2 - \|x^* + y^*\|^2 \\
&= p(2x^*, 2y^*) + \|y^* - x^*\|^2 - \|y^* - x^*\|^2 - 2y^*\|^2 \\
&\leq \phi^*(2x^*, 2y^*) + \|y^* - x^*\|^2 - d^2_C(y^* - x^*) \\
&= 2(f(x^*, y^*) + g(y^* - x^*)) \\
&= 2(f(x^*, y^*) + g(L(x^*, y^*))).
\end{align*}
Hence \(\inf (f + gL)(X \times X) \geq 0\) and, since \(\text{dom} \ g = X\), Fact \cite{3} now implies that
\begin{equation}
\min_{x \in X} f^*(-L^*x) + g^*(x) \leq 0. 
\end{equation}
Since \(f^* = \frac{1}{2}\phi^*, g^* = j + \iota_C\), and \(L^*: X \to X \times X \colon x \mapsto (-x, x)\), we see that \((13)\) clearly yields \((7)\).

Now assume that \(y \neq 0\). Let \(E = D - y\) and define \(U: E \to X \colon z \mapsto T(z + y)\). Then \(U\) is firmly nonexpansive and \((U(E) = T(D))\). By what we just proved, there exists an extension \(\hat{U}: E \cup \{0\} \to \text{conv} \ U(E) = \text{conv} \ T(D)\). Therefore, \(\hat{T}: D \cup \{y\} \to \text{conv} \ T(D): z \mapsto \hat{U}(z - y)\) is as required. \qed

**Corollary 5.** Let \(D\) be a nonempty subset of \(X\) and let \(T: D \to X\) be firmly nonexpansive. Then \(T\) has a firmly nonexpansive extension \(\hat{T}: X \to \text{conv} \ T(D)\).

**Proof.** Let \(\mathcal{M}\) be the set of all pairs \((U, E)\), where \(D \subseteq E \subseteq X\) and \(U: E \to \text{conv} \ T(D)\) is a firmly nonexpansive extension of \(T\). Partially order \(\mathcal{M}\) via \((U_1, E_1) \preceq (U_2, E_2)\) if \(E_1 \subseteq E_2\) and \(U_2\) extends \(U_1\). Zorn’s lemma guarantees the existence of a maximal element \((\hat{T}, \hat{D})\). Now Theorem \cite{4} shows that \(\hat{D} = X\). \qed

Remark 6 (range localization is optimal). The conclusion that the range of the extension \(\hat{T}\) lies in the closed convex hull of \(T(D)\) cannot be improved upon in general. Indeed, let \(D\) be a nonempty subset of \(X\), let \(T\) be \(\text{Id} \mid_D\), and let \(\hat{T}: X \to X\) be any firmly nonexpansive extension of \(T\). Then \(D = \text{Fix} T \subseteq \text{Fix} \hat{T}\), and the last set is closed and convex \cite{13} \cite{14}. Hence \(C = \text{conv} \ T(D) = \text{conv} \ D \cap \text{Fix} \hat{T} \subset \hat{T}(X)\). In particular, let \(\hat{T}: X \to C\) be any firmly nonexpansive extension of \(T\) as in Corollary \cite{5} Then \(\hat{T}(X) = C\) and \(\hat{T}|C = \text{Id} \mid_C\); therefore, \(\hat{T}\) is the projector onto \(C\).

**Corollary 7** (Kirszbraun-Valentine). Let \(D\) be a nonempty subset of \(X\) and let \(N: D \to X\) be nonexpansive. Then \(N\) has a nonexpansive extension \(\hat{N}: X \to \text{conv} \ N(D)\).

**Proof.** (See also \cite{13} \cite{16} \cite{20} \cite{20} for different proofs and related results.) Let \(T = \frac{1}{2} \text{Id} \mid_D + \frac{1}{2} N\), which is firmly nonexpansive. Corollary \cite{5} guarantees a firmly nonexpansive extension \(\hat{T}: X \to \text{conv} \ T(D)\). Let \(P\) be the (firmly) nonexpansive projector onto \(\text{conv} \ N(D)\). Then \(\hat{N} = P \circ (2\hat{T} - \text{Id})\) is as required. \qed

Remark 8. We do not know whether it is possible to deduce Corollary \cite{5} from Corollary \cite{7}. The following technique of going back and forth between firmly nonexpansive and nonexpansive mappings, utilized in the proof of Corollary \cite{7}, does not work in reverse. Let \(D\) be a nonempty subset of \(X\) and \(T: D \to X\) be firmly nonexpansive. Then \(N = 2T - \text{Id} \mid_D: D \to X\) is nonexpansive, and hence (by
Corollary [7] it has an extension $\tilde{N}: X \to \text{conv } N(D)$. It is tempting to conjecture that $\tilde{T} = \frac{1}{2} \text{Id} + \frac{1}{2} \tilde{N}$ would be an extension of $T$ as in Corollary [3]. However, let us concretely consider $D = \{0\} \subset X$ and $T: D \to X: 0 \mapsto 0$. Then $N = 2T - \text{Id}_{D} = T$ and so $\tilde{N} \equiv 0$. Hence $\tilde{T} = \frac{1}{2} \text{Id}$, which does not satisfy $\tilde{T}(X) \subset \text{conv } T(D) = \{0\}$.

**Remark 9.** The correspondence revealed by Minty [15] between (maximal) monotone operators and firmly nonexpansive mappings (with full domain) provides a reformulation of Theorem [4] in terms of monotone operators (see, e.g., [6, Theorem 2.1]), which in turn relates to the work of Debrunner and Flor [11]. The new proof presented here provides a convex-analytical handle on these results (see also [4]). Furthermore, in the present Hilbert space setting, Reich ([21, Lemma 2.1]) showed that Corollary [5] is equivalent to the following result. Let $A$ be a monotone operator on $X$ with a nonempty graph. Then $A$ has a maximal monotone extension $\tilde{A}$ such that $\text{conv } \text{dom } A = \text{conv } \text{dom } \tilde{A}$. (In fact, his result is about accretive operators in general Banach spaces.) Using [21, Proposition 2.2], it follows that Corollary [5] actually characterizes Hilbert spaces among all Banach spaces of dimension not less than three.

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Department of Mathematics, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia, Canada V1V 1V7

E-mail address: heinz.bauschke@ubc.ca