FENCHEL DUALITY, FITZPATRICK FUNCTIONS AND THE EXTENSION OF FIRMLY NONEXPANSIVE MAPPINGS

HEINZ H. BAUSCHKE

(Communicated by Jonathan M. Borwein)

Abstract. Recently, S. Reich and S. Simons provided a novel proof of the Kirszbraun-Valentine extension theorem using Fenchel duality and Fitzpatrick functions. In the same spirit, we provide a new proof of an extension result for firmly nonexpansive mappings with an optimally localized range.

Throughout this paper, we assume that $X$ is a real Hilbert space, with inner product $p = \langle \cdot | \cdot \rangle$ and induced norm $\| \cdot \|$, and we denote the identity mapping on $X$ by $\text{Id}$. A mapping $T$ from a subset $D$ of $X$ to $X$ is called firmly nonexpansive if

\begin{equation}
(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|^2 + \| (\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2;
\end{equation}

equivalently \[13,14\], if $2T - \text{Id}$ is nonexpansive (Lipschitz continuous with constant 1), i.e.,

\begin{equation}
(\forall x \in D)(\forall y \in D) \quad \| (2T - \text{Id})x - (2T - \text{Id})y\| \leq \|x - y\|
\end{equation}

or if

\begin{equation}
(\forall x \in D)(\forall y \in D) \quad 0 \leq \langle Tx - Ty | (\text{Id} - T)x - (\text{Id} - T)y\rangle.
\end{equation}

Firmly nonexpansive mappings play an important role in various contexts; see, e.g., \[1,2,3,7,8,9,10,15,17,21,22,25\]. The Kirszbraun-Valentine theorem (see, e.g., \[5,13,16,20,26\]) states that any nonexpansive mapping can be extended to a nonexpansive mapping defined on the whole space. A beautiful proof of this result, based on Fenchel duality and Fitzpatrick functions, was recently provided by Reich and Simons \[23\]. (For further applications of Fitzpatrick functions, see, e.g., \[4,24\].) In this note, we refine their technique to obtain a new proof of an extension theorem for firmly nonexpansive mappings where the range of the extension is optimally localized. This extension theorem easily implies the Kirszbraun-Valentine result. Notation not explicitly defined in the following is standard in convex analysis; see, e.g., \[27\].
**Definition 1.** Let $D$ be a nonempty subset of $X$ and let $T : D \to X$ be firmly nonexpansive. Then the associated Fitzpatrick function $\phi_T$ is

$\phi = \phi_T$ in $X \times X \to ]-\infty, +\infty] : (x, y) \mapsto \sup_{d \in D} \langle x \mid d - Td \rangle + \langle y \mid Td \rangle - \langle d - Td \rangle$,

and we also set $G = G_T = \{(d - Td, Td) \mid d \in D\}$.

**Proposition 2.** Let $D$ be a nonempty subset of $X$, let $T : D \to X$ be firmly nonexpansive, and let $x$ and $y$ be in $X$. Then:

(i) $\phi = (\iota_G + p)^\ast$.

(ii) The extension $\overline{\phi} : D \cup \{y\} \to X$ of $\phi$ which maps $y$ to $x$ is still firmly nonexpansive if and only if $\phi(x - x, x) \leq p(y - x, x)$.

(iii) $\phi$ is convex, lower semicontinuous and proper.

(iv) $\text{conv} G \subset \text{dom} \phi \ast \subset \text{conv} \{\text{Id} - T\}(D) \times \text{conv} T(D)$.

(v) $p \leq \phi^\ast$.

**Proof.** Fix $x$ and $y$ in $X$. (i) For every $d \in D$, we have

$\langle x \mid d - Td \rangle + \langle y \mid Td \rangle - \langle d - Td \rangle = \langle (d - Td, Td) \mid (x, y) \rangle - (p + \iota_G)(d - Td, Td)$,

from which the identity follows by supremizing over $d \in D$. (ii) This is a consequence of (ii). By (ii) $\phi \leq p$ on $G$. Hence $\phi$ is proper. The function $\phi$ is convex and lower semicontinuous, as it is a Fenchel conjugate by (i). (iv): $\phi$ is convex and lower semicontinuous, as it is a Fenchel conjugate by (i). (v) In view of (ii), $p(G - G) \subset [0, +\infty]$. Suppose that $(x, y) \in \text{conv} G$, say it is a finite convex combination $(x, y) = \sum_{i \in I} \lambda_i (x_i, y_i)$ of elements in $G$. Then $\sum_{i \in I} \lambda_i p(x_i, y_i) = p(x, y) + \frac{1}{2} \sum_{i, j \in I} p((x, y) - (x_j, y_j)) \geq p(x, y)$, hence $p \leq \text{conv}(\iota_G + p)$. Since $p$ is continuous, it follows that $p \leq \text{conv}(\iota_G + p) = (\iota_G + p)^\ast = \phi^\ast$. □

**Fact 3** (Fenchel duality). Let $Y$ be a real Hilbert space and let $L : Y \to X$ be linear and continuous. Let $f : Y \to ]-\infty, +\infty]$ and $g : X \to ]-\infty, +\infty]$ be convex, lower semicontinuous, and proper such that $g$ is continuous and finite at some point in $L \text{dom} f$. Then

$\inf_{y \in Y} \{f(y) + g(Ly)\} = -\min_{x \in X} \{f^\ast(-L^\ast x) + g^\ast(x)\}$.

**Proof.** See, e.g., [27, Corollary 2.8.5]. □

**Theorem 4.** Let $D$ be a nonempty subset of $X$, let $T : D \to X$ be firmly nonexpansive, and let $y \in X$. Then $T$ has a firmly nonexpansive extension $\overline{T} : D \cup \{y\} \to \text{conv} T(D)$.

**Proof.** Set $\phi = \phi_T$ and $C = \text{conv} T(D)$, and assume first that $y = 0$. In view of Proposition 2(ii) we must show that

$\min_{x \in X} \phi(x, -x) + \|x\|^2 + \iota_C(x) \leq 0$.

Set $f : X \times X \to ]-\infty, +\infty] : (x, y) \mapsto \frac{1}{2} \phi^\ast(2x, 2y)$ so that $f^\ast = \frac{1}{2} \phi^\ast$, and let $j = \frac{1}{2} \| \cdot \|^2$. Now set $g = (j + \iota_C)^\ast$ and observe (using [14]) that $g = j \iota_{d_C} = j - (j \boxplus \iota_C) = j - \frac{1}{2} d_C^2$, where $\boxplus$ denotes the infimal convolution and $d_C$ the distance function. Further set $L : X \times X \to X : (x^*, y^*) \mapsto y^* - x^*$. We claim that $\inf_{(x^*, y^*) \in X \times X} f(x^*, y^*) + g(L(x^*, y^*)) \geq 0$. Indeed, pick $(x^*, y^*) \in \text{dom} f$. By
Proposition \([\mathbf{iv}]\) \((2x^*, 2y^*) \in \text{dom } \phi^* \subset X \times C\) and hence \(2y^* \in C\). Using Proposition \([\mathbf{iv}]\) we deduce that

\begin{align}
0 &= 4 \langle x^* | y^* \rangle + \|y^* - x^*\|^2 - \|x^* + y^*\|^2 \\
&= p(2x^*, 2y^*) + \|y^* - x^*\|^2 - \|\langle y^* - x^* \rangle - 2y^*\|^2 \\
&\leq \phi^*(2x^*, 2y^*) + \|y^* - x^*\|^2 - d^2_C(y^* - x^*) \\
&= 2 \langle f(x^*, y^*) + g(y^* - x^*) \rangle \\
&= 2 \langle f(x^*, y^*) + g(L(x^*, y^*)) \rangle.
\end{align}

Hence \(\inf(f + gL)(X \times X) \geq 0\) and, since \(\text{dom } g = X\), Fact \([\mathbf{iii}]\) now implies that

\begin{equation}
\min_{x \in X} f^*(-L^*x) + g^*(x) \leq 0.
\end{equation}

Since \(f^* = \frac{1}{\lambda} \phi\), \(g^* = j + \iota_C\), and \(L^*: X \to X \times X: x \mapsto (-x, x)\), we see that \([\mathbf{13}]\) clearly yields \([\mathbf{7}]\).

Now assume that \(y \neq 0\). Let \(E = D - y\) and define \(U: E \to X: z \mapsto T(z + y)\). Then \(U\) is firmly nonexpansive and \(U(E) = T(D)\). By what we just proved, there exists an extension \(\tilde{U}: E \cup \{0\} \to \text{conv } U(E) = \text{conv } T(D)\). Therefore, \(\tilde{T}: D \cup \{y\} \to \text{conv } T(D): z \mapsto \tilde{U}(z - y)\) is as required. \(\square\)

**Corollary 5.** Let \(D\) be a nonempty subset of \(X\) and let \(T: D \to X\) be firmly nonexpansive. Then \(T\) has a firmly nonexpansive extension \(\hat{T}: X \to \text{conv } T(D)\).

**Proof.** Let \(\mathcal{M}\) be the set of all pairs \((U, E)\), where \(D \subset E \subset X\) and \(U: E \to \text{conv } T(D)\) is a firmly nonexpansive extension of \(T\). Partially order \(\mathcal{M}\) via \((U_1, E_1) \preceq (U_2, E_2)\) if \(E_1 \subset E_2\) and \(U_2\) extends \(U_1\). Zorn’s lemma guarantees the existence of a maximal element \((\hat{T}, \hat{D})\). Now Theorem \([\mathbf{4}]\) shows that \(\hat{D} = X\). \(\square\)

**Remark 6** (range localization is optimal). The conclusion that the range of the extension \(\hat{T}\) lies in the closed convex hull of \(T(D)\) cannot be improved upon in general. Indeed, let \(D\) be a nonempty subset of \(X\), let \(T\) be Id \(\mid_D\), and let \(\hat{T}: X \to X\) be any firmly nonexpansive extension of \(T\). Then \(D = \text{Fix } T \subset \text{Fix } \hat{T}\), and the last set is closed and convex \([\mathbf{13}, \mathbf{14}]\). Hence \(C = \text{conv } T(D) = \text{conv } D \subset \text{Fix } \hat{T} \subset \hat{T}(X)\). In particular, let \(\hat{T}: X \to C\) be any firmly nonexpansive extension of \(T\) as in Corollary \([\mathbf{5}]\) Then \(\hat{T}(X) = C\) and \(\hat{T}|C = \text{Id } |_C\); therefore, \(\hat{T}\) is the projector onto \(C\).

**Corollary 7** (Kirszbraun-Valentine). Let \(D\) be a nonempty subset of \(X\) and let \(N: D \to X\) be nonexpansive. Then \(N\) has a nonexpansive extension \(\tilde{N}: X \to \text{conv } N(D)\).

**Proof.** (See also \([\mathbf{13}, \mathbf{16}, \mathbf{20}, \mathbf{26}]\) for different proofs and related results.) Let \(T = \frac{1}{2} \text{Id } \mid_D + \frac{1}{2} N\), which is firmly nonexpansive. Corollary \([\mathbf{5}]\) guarantees a firmly nonexpansive extension \(\hat{T}: X \to \text{conv } T(D)\). Let \(P\) be the (firmly) nonexpansive projector onto \(\text{conv } N(D)\). Then \(\tilde{N} = P \circ (2\hat{T} - \text{Id})\) is as required. \(\square\)

**Remark 8.** We do not know whether it is possible to deduce Corollary \([\mathbf{5}]\) from Corollary \([\mathbf{7}]\). The following technique of going back and forth between firmly nonexpansive and nonexpansive mappings, utilized in the proof of Corollary \([\mathbf{7}]\), does not work in reverse. Let \(D\) be a nonempty subset of \(X\) and \(T: D \to X\) be firmly nonexpansive. Then \(N = 2T - \text{Id } \mid_D: D \to X\) is nonexpansive, and hence (by
Corollary [7] it has an extension \( \tilde{N} : X \to \text{conv} N(D) \). It is tempting to conjecture that \( \tilde{T} = \frac{1}{2} \text{Id} + \frac{1}{2} \tilde{N} \) would be an extension of \( T \) as in Corollary [5]. However, let us concretely consider \( D = \{0\} \subset X \) and \( T : D \to X : 0 \mapsto 0 \). Then \( N = 2T - \text{Id}_D = T \) and so \( \tilde{N} \equiv 0 \). Hence \( \tilde{T} = \frac{1}{2} \text{Id} \), which does not satisfy \( \tilde{T}(X) \subset \text{conv} T(D) = \{0\} \).

**Remark 9.** The correspondence revealed by Minty [18] between (maximal) monotone operators and firmly nonexpansive mappings (with full domain) provides a reformulation of Theorem [1] in terms of monotone operators (see, e.g., [6, Theorem 2.1]), which in turn relates to the work of Debrunner and Flor [11]. The new proof presented here provides a convex-analytical handle on these results (see also [4]). Furthermore, in the present Hilbert space setting, Reich ([21, Lemma 2.1]) showed that Corollary [5] is equivalent to the following result. Let \( A \) be a monotone operator on \( X \) with a nonempty graph. Then \( A \) has a maximal monotone extension \( \hat{A} \) such that \( \text{conv} \text{dom} A = \text{conv} \text{dom} \hat{A} \). (In fact, his result is about accretive operators in general Banach spaces.) Using [21, Proposition 2.2], it follows that Corollary [5] actually characterizes Hilbert spaces among all Banach spaces of dimension not less than three.

**ACKNOWLEDGMENT**

The author wishes to thank the referee for his insightful comments. H. H. Bauschke’s work was partially supported by the Natural Sciences and Engineering Research Council of Canada.

**REFERENCES**


Department of Mathematics, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia, Canada V1V 1V7

E-mail address: heinz.bauschke@ubc.ca

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use