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SPECTRUM OF A COMPACT WEIGHTED COMPOSITION OPERATOR

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ABSTRACT. For ψ analytic in the open unit disk and φ an analytic map from the unit disk into itself, the weighted composition operator $C_{\psi,\varphi}$ is the operator on the weighted Hardy space $H^2(\beta)$ given by $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$. This paper discusses the spectrum of $C_{\psi,\varphi}$ when it is compact on a certain class of weighted Hardy spaces and when the composition map φ has a fixed point inside the open unit disk.

INTRODUCTION

A *weighted composition operator* $C_{\psi,\varphi}$ maps an analytic function f on the unit disk D into $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$. These operators come up naturally. For example, Forelli showed that an isometry on H^p for $1 < p < \infty$ and $p \neq 2$ is a weighted composition operator [4]. Though somewhat unrelated to the spaces discussed here, backward weighted shifts on sequence spaces are weighted composition operators, too.

To avoid $C_{\psi,\varphi}$ being merely a multiplication operator, the composition map φ is taken to be different from identity. Theorem 1, which is the main result of this paper, asserts that if $C_{\psi,\varphi}$ is a compact operator on the weighted Hardy space $H^2(\beta)$ and if the Denjoy-Wolff point of φ is inside the open unit disk, then the spectrum of $C_{\psi,\varphi}$ is the set $\{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \cdots\}$, where a is the Denjoy-Wolff point of φ. A similar result that applies only to the classical Hardy space has appeared in recent work of Shapiro and Smith [7, pages 76-77] and Clifford and Dabkowski [2, pages 186-187]. Also the doctoral dissertation of Hammond [6] contains a similar result obtained by using a different technique. In addition, the proofs of Lemmas 2 and 3 were inspired by material in [3, pages 270-275], and the proof of Theorem 2 was inspired by the proof of Theorem 7.14 of [3].

The proof of Theorem 1 is divided into two parts. First we show that if λ is an eigenvalue, then it must be of the form $\psi(a)(\varphi'(a))^j$ for some non-negative integer j.

Next we identify the actual eigenvalues among these numbers by finding eigenvalues of the adjoint operator $C^*_{\psi,\varphi}$. First we find a finite-dimensional invariant subspace of the adjoint. This allows us to represent the adjoint as a 2×2 upper

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triangular block matrix. Eigenvalues of $C^*_{\psi,\varphi}$ can be found by looking at this block matrix.

After finding the spectrum when the composition map has a fixed point inside the unit disk, we classify a class of compact weighted composition operators where the composition map φ must have its Denjoy-Wolff point inside the open unit disk. Thus the spectrum for this class of operators can be computed using Theorem 1.

Compact weighted composition operators on the Hardy space $H^2(D)$, where the composition map has fixed points on the unit circle, are discussed in [1]. We find the spectrum of such operators.

Preliminaries

Let f be an analytic map on the open unit disk D given by the Taylor series $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and let $\{\beta(j)\}_{j=0}^{\infty}$ be a sequence of positive numbers where $\liminf_{j\to\infty} \beta(j)^{1/j} \geq 1$. Then f is in the space $H^2(\beta)$ if and only if $\sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2 < \infty$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=0}^{\infty} b_j z^j$ are two functions in $H^2(\beta)$, an inner product can be defined by

$$\langle f, g\rangle = \sum_{j=0}^{\infty} a_j \overline{b_j} \beta(j)^2.$$

The space $H^2(\beta)$ becomes a Hilbert space with this inner product [3, page 14]. If $\beta(j) = 1$ for all j the space $H^2(\beta)$ is known as the Hardy Hilbert space, if $\beta(j)^2 = 1/(j+1)$ it is known as the Bergman Hilbert space, and if $\beta(j)^2 = j+1$ it is the Dirichlet Hilbert space.

Let w be a point on the open unit disk. Define

$$K_w(z) = \sum_{j=0}^{\infty} \frac{z^j \overline{w}^{\,j}}{\beta(j)^2}.$$

Then K_w is in $H^2(\beta)$ and $\|K_w\|^2 = \sum_{j=0}^{\infty} |w|^{2j}/\beta(j)^2$ (see [3, page16]). Thus $\|K_w\|$ is an increasing function of $|w|$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then

$$\langle f, K_w\rangle = \sum_{j=0}^{\infty} \frac{a_j w^j \beta(j)^2}{\beta(j)^2} = f(w).$$

Therefore

$$\langle f, K_w\rangle = f(w)$$

for all f, and K_w is known as the point evaluation kernel at w. It can be easily shown that $C^*_{\psi,\varphi} K_w = \overline{\psi(w)} K_{\varphi(w)}$.

Let φ be an analytic map from the open unit disk into itself and let ζ be a point on the unit circle. Then $\lim_{r\to 1^-} \varphi(r\zeta)$ exists for almost all ζ on the unit circle; by abusing the notation we will denote this limit by $\varphi(\zeta)$. If the limit is ζ we say ζ *is a fixed point of* φ on the unit circle. Similarly if $\lim_{r\to 1^-} \varphi'(r\zeta)$ exists for some ζ on the unit circle, then we will denote this limit by $\varphi'(\zeta)$ [3, page 50].

Suppose φ is a map from the unit disk into itself which is different from the identity. Then it can have at most one fixed point inside the open unit disk. If φ has a fixed point in the open unit disk, the absolute value of the derivative of φ at the fixed point will be less than or equal to 1. If φ does not have a fixed point in the open disk, there is at least one fixed point ζ on the unit circle and there is exactly one fixed point on the unit circle with $|\varphi'(\zeta)| \leqslant 1$. When φ is not an elliptic automorphism this distinguished fixed point, a, where $|a| \leq 1$, $\varphi(a) = a$, and $\varphi'(a) \leq 1$, is known as the **Denjoy-Wolff** point of φ (see [3, pages 50-59]).

In the lemma below we identify all possible eigenvalues of $C_{\psi,\varphi}$.

Lemma 1. *Suppose $C_{\psi,\varphi}$ is a compact operator on the weighted Hardy space $H^2(\beta)$. If the Denjoy-Wolff point a of the composition map φ is inside the open unit disk, then the set*

$$\{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \cdots\}$$

contains the spectrum.

Proof. A compact operator on an infinite-dimensional space is not invertible, hence 0 is in the spectrum. Now we show that if λ is an eigenvalue, then λ must be of the form $\psi(a)(\varphi'(a))^n$. If λ is a non-zero eigenvalue, then

$$\psi(z)f(\varphi(z)) = \lambda f(z) \tag{1}$$

for some non-zero f which is holomorphic on the unit disk. Let f have a zero of order n at a. If $n = 0$ put $z = a$, and we get that $\lambda = \psi(a)$. For $n > 0$ differentiate (1) n times. Then,

$$\sum_{j=0}^{n-1} \alpha_j(z) f^{(j)}(\varphi(z)) + \psi(z) f^{(n)}(\varphi(z))(\varphi'(z))^n = \lambda f^{(n)}(z), \tag{2}$$

where $f^{(k)}$ stands for the k^{th} derivative of f and α_j's are functions which consist of various products of derivatives of ψ and φ. The exact values of these are not important to us. Now let $z = a$. Since f has a zero of order of n at a, all the terms in the sum $\sum_{j=0}^{n-1}$ vanish, and this yields $\lambda = \psi(a)(\varphi'(a))^n$. The spectrum of a compact operator consists only of eigenvalues, and the above computation shows that only possible eigenvalues are of the form $\psi(a)(\varphi'(a))^n$. This completes the proof. □

To find the spectrum we need to find which of the numbers discussed in Lemma 1 are really eigenvalues. To do that, we look at the adjoint of this operator and a special class of continuous linear functionals on $H^2(\beta)$.

Definition 1. Let j be a positive integer. If

$$K_a^{[j]}(z) = \frac{d^j K_a(z)}{d\overline{w}^j},$$

then $K_a^{[j]}$ is in $H^2(\beta)$ and $\langle f, K_a^{[j]} \rangle = f^{(j)}(a)$ where $f^{(j)}(a)$ is the j^{th} derivative of f at a. The function $K_a^{[j]}$ is known as the j^{th} derivative evaluation kernel at a (see [3, Theorem 2.16]).

Lemma 2. *Let m be a positive integer. Then the span of $\{K_a, K_a^{[1]}, \cdots, K_a^{[m]}\}$ is an invariant subspace of $C^*_{\psi,\varphi}$.*

Proof. Let f be any function on $H^2(\beta)$. Then

$$\langle f, C^*_{\psi,\varphi} K_a^{[n]} \rangle = \langle C_{\psi,\varphi} f, K_a^{[n]} \rangle \tag{3}$$

$$= \langle \psi f \circ \varphi, K_a^{[n]} \rangle. \tag{4}$$

Now $\psi f \circ \varphi$ is differentiated n times and evaluated at a. This results in

$$(5) \qquad \langle f, C^*_{\psi,\varphi} K_a^{[n]} \rangle = \sum_{j=0}^{n-1} \alpha_j(a) f^{(j)}(a) + \psi(a) f^{(n)}(a)(\varphi'(a))^n.$$

Now (5) can be written as

$$(6) \qquad \langle f, C^*_{\psi,\varphi} K_a^{[n]} \rangle = \sum_{j=0}^{n-1} \alpha_j(a) \langle f, K_a^{[j]} \rangle + \psi(a)(\varphi'(a))^n \langle f, K_a^{[n]} \rangle.$$

Now using the algebraic properties of the inner product on the right-hand side of (6), we get

$$(7) \qquad \langle f, C^*_{\psi,\varphi} K_a^{[n]} \rangle = \langle f, \quad \sum_{j=0}^{n-1} \overline{\alpha_j(a)} K_a^{[j]} + \overline{\psi(a)(\varphi'(a))^n} K_a^{[n]} \rangle.$$

Since f was arbitrary, this proves that

$$(8) \qquad C^*_{\psi,\varphi} K_a^{[n]} = \sum_{j=0}^{n-1} \overline{\alpha_j(a)} K_a^{[j]} + \overline{\psi(a)(\varphi'(a))^n} K_a^{[n]}.$$

Note that the order of the highest derivative on the right-hand side does not exceed n. This completes the proof. □

Lemma 3. *Suppose $C_{\psi,\varphi}$ is a bounded operator on $H^2(\beta)$, and the Denjoy-Wolff point a of φ is inside the open unit disk. Then $\overline{\psi(a)(\varphi(a))^j}$ is in the spectrum of the adjoint operator $C^*_{\psi,\varphi}$ for any non-negative integer j.*

Proof. First we use (8) to compute the matrix representation of $C^*_{\psi,\varphi}$ restricted to the subspace discussed in Lemma 2. Let $\mathbf{K}_m$ denote the span of $\{K_a, K_a^{[1]}, \cdots, K_a^{[m]}\}$. This spanning set is also linearly independent and hence is a basis. The matrix of the operator restricted to $\mathbf{K}_m$ with respect to this basis is

$$\begin{pmatrix} \overline{\psi(a)} & * & * & * & \cdots & * \\ 0 & \overline{\psi(a)\varphi'(a)} & * & * & \cdots & * \\ 0 & 0 & \overline{\psi(a)(\varphi'(a))^2} & * & \cdots & * \\ 0 & 0 & 0 & \overline{\psi(a)(\varphi'(a))^3} & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \overline{\psi(a)(\varphi'(a))^m} \end{pmatrix}.$$

Let us call this matrix A_m. Then A_m is an $(m+1) \times (m+1)$ upper triangular matrix. The *'s denote $\overline{\alpha_j(a)}$'s.

The subspace $\mathbf{K}_m$ is finite dimensional and therefore is closed. The Hardy space can be decomposed as $H^2(\beta) = \mathbf{K}_m \oplus \mathbf{K}_m^\perp$. The block matrix of $C^*_{\psi,\varphi}$ with respect to this decomposition is

$$\begin{pmatrix} A_m & B \\ 0 & C_m \end{pmatrix}.$$

The fact that $\mathbf{K}_m$ is invariant under $C^*_{\psi,\varphi}$ makes the lower left corner of this decomposition 0. Since there is a 0 at the lower left and the subspace $\mathbf{K}_m$ is finite dimensional, the spectrum of $C^*_{\psi,\varphi}$ is the union of the spectrum of A_m and the spectrum of C_m [3, page 270]. Since A_m is a finite-dimensional upper triangular matrix, its spectrum consists of the diagonal values. Therefore we can conclude that the spectrum of $C^*_{\psi,\varphi}$ contains the set

$$\{\overline{\psi(a)}, \overline{\psi(a)(\varphi'(a))}, \overline{\psi(a)(\varphi'(a))^2}, \cdots, \overline{\psi(a)(\varphi'(a))^m}\}.$$

But m is arbitrary, therefore we can see that any number of the form $\overline{\psi(a)(\varphi(a))^j}$, where j is a non-negative integer, is in the spectrum of $C^*_{\psi,\varphi}$. This completes the proof. $\square$

Theorem 1. *Let $C_{\psi,\varphi}$ be a compact operator on the weighted Hardy space $H^2(\beta)$. If the Denjoy-Wolff point of φ is inside the disk, then the spectrum of $C_{\psi,\varphi}$ is the set*

$$\{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \cdots\},$$

where a is the Denjoy-Wolff point of φ.

Proof. If the Denjoy-Wolff point of φ is a and is inside the open disk, then for any positive integer j the number $\psi(a)(\varphi'(a))^j$ is in the spectrum by Lemma 3. Clearly 0 is in the spectrum. Thus the spectrum contains the set

$$\{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \cdots\}.$$

By Lemma 1 this set contains the spectrum. This completes the proof. $\square$

Interestingly, it is possible for $C_{\psi,\varphi}$ to be compact even when ψ is unbounded. For example if $\psi(z) = ((1-z)/(1+z))^{1/4}$ and $\varphi(z) = (1+z)/2$, then $C_{\psi,\varphi}$ is compact and also if $\psi(z) = (1+z)^{-1/4}$ and $\varphi(z) = z/2$, then $C_{\psi,\varphi}$ is compact. This and other conditions for compactness are included in [5] and is the subject of a forthcoming paper by the author.

As we will see in the next theorem compactness of $C_{\psi,\varphi}$ for some weight functions ψ force φ to have a fixed point inside the disk.

Theorem 2. *Let $C_{\psi,\varphi}$ be a compact operator on the space $H^2(\beta)$ where*

$$\sum 1/\beta(j)^2 = \infty. \tag{9}$$

If $\liminf_{r\to 1^-} |\psi(r\zeta)| > 0$ for any ζ on the unit circle, then the composition map φ will have its Denjoy-Wolff point inside the open unit disk.

Proof. By contradiction. Suppose φ does not have a fixed point inside the disk. Then φ must have its Denjoy-Wolff point on the unit circle [3, pages 50-59]. Let us call this point ζ. Let r belong to the interval $(0, 1)$.

Now we apply the adjoint of $C_{\psi,\varphi}$ to the normalized kernel function $K_{r\zeta}/||K_{r\zeta}||$:

$$||C^*_{\psi,\varphi}\frac{K_{r\zeta}}{||K_{r\zeta}||}|| \quad = \quad |\psi(r\zeta)|\frac{||K_{\varphi(r\zeta)}||}{||K_{r\zeta}||}. \tag{10}$$

Since ζ is the Denjoy-Wolff point $|\varphi(r\zeta)| \geqslant |r\zeta|$ (see [3, page 49]). But $||K_w||$ is an increasing function of $|w|$, therefore $||K_{\varphi(r\zeta)}|| \geqslant ||K_{r\zeta}||$. Now by using (10) we get

$$||C^*_{\psi,\varphi}\frac{K_{r\zeta}}{||K_{r\zeta}||}|| \quad \geqslant \quad |\psi(r\zeta)|. \tag{11}$$

Normalized kernel functions $K_{r\zeta}/||K_{r\zeta}||$ converge weakly to zero as r tends to 1 (see [3, Theorem 2.17]). Since $C^*_{\psi,\varphi}$ is compact, the left-hand side of (11) must tend to zero as r tends 1, but the right-hand side will be larger than some positive δ_ζ by our hypothesis about ψ. This is a contradiction. Therefore φ cannot have its

Denjoy-Wolff point on the unit circle. Hence the Denjoy-Wolff point of φ must be inside the open unit disk. $\square$

Theorem 2 and Theorem 1 give us the following corollary.

Corollary 1. *Suppose $C_{\psi,\varphi}$ is a compact operator on $H^2(\beta)$ where $\sum 1/\beta(j)^2 = \infty$. If $\liminf_{r\to 1^-} |\psi(r\zeta)| > 0$ for any ζ on the unit circle, then the spectrum of $C_{\psi,\varphi}$ is the set*

$$\{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \cdots\},$$

where a is the Denjoy-Wolff point of φ.

Lemma 3.3 of [1] gives a method to find ψ so that $C_{\psi,\varphi}$ is compact on the Hardy space $H^2(D)$ when φ has fixed points on the boundary. We discuss the spectrum for such operators. First we quote the lemma.

Lemma 4. *Suppose ψ is continuous on the closed unit disk with $\psi(1) = 0$ and suppose further that the function $\theta \to \psi(e^{i\theta})$ is differentiable at $\theta = 0$. Then for every non-automorphic linear fractional φ of the unit disk with fixed point at 1, the operator $C_{\psi,\varphi}$ is compact on $H^2(D)$.*

If φ has a fixed point inside the disk in addition to the one on the unit circle, then Theorem 1 gives the spectrum. Therefore we compute the spectrum when φ has no fixed points inside the open unit disk. We will denote the composition of φ with itself n times by φ_n. That is $\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi$ (n times).

Theorem 3. *Suppose ψ and φ satisfy the hypothesis in Lemma 4, and φ has no fixed points inside the open unit disk. Then $\sigma(C_{\psi,\varphi}) = \{0\}$.*

Proof. We will show that the spectral radius of this operator is 0. Since φ is a non-automorphic linear fractional map with a fixed point at 1, it takes the unit circle to a circle tangent to unit circle at 1. The map φ has no fixed points inside the open unit disk, so 1 is the Denjoy-Wolff point.

Let $\epsilon > 0$. There is a $\delta > 0$ such that $|\psi(z)| < \epsilon$ whenever $|z-1| < \delta$ and z is in the closed unit disk. Let $W = \{z : |z-1| < \delta, |z| \le 1\}$. Clearly W is open in $\overline{D}$. Let $U = \varphi(D)$. Then $\overline{U}$ is tangent to the unit circle at 1. Let $V = \overline{U - W}$; then V is a compact subset of the open unit disk. Therefore the sequence $\{\varphi_n\}$ converges uniformly to 1 on V. Consider a point $e^{i\theta}$ on the unit circle. Then $\varphi(e^{i\theta})$ is either in W or V. If $\varphi(e^{i\theta})$ is in V, then there is an N that does not depend on θ such that $\varphi_j(e^{i\theta})$ is in W for all $j > N$. If $\varphi(e^{i\theta})$ is not in V consider the sequence $\{\varphi_j(e^{i\theta})\}_{j=1}^{\infty}$; either $\varphi_j(e^{i\theta})$ is in W for all j or $\varphi_j(e^{i\theta})$ will be in V for some j. If $\varphi_j(e^{i\theta})$ is in V for some j, take j' to be the smallest integer such that $\varphi_j(e^{i\theta})$ is in V. Then $\varphi_j(e^{i\theta})$ is in W for all $j > j' + N$. Therefore for any $e^{i\theta}$ on the unit circle at most N terms of the sequence $\{\varphi_j(e^{i\theta})\}_{j=1}^{\infty}$ will be outside W. Hence at most N terms of the sequence $\{|\psi(\varphi_j(e^{i\theta}))|\}_{j=1}^{\infty}$ will be larger than ϵ from the sequence for any $e^{i\theta}$. Also ψ is bounded on the closed unit disk, therefore $|\psi(\varphi_j(e^{i\theta}))| < M$

for some $M > 0$. Now if f is in $H^2(D)$ and $n > N$, then

$$\begin{aligned} ||C^n_{\psi,\varphi}(f)||^2 &= \int_T |\psi(e^{i\theta})|^2 |\psi(\varphi(e^{i\theta}))|^2 \cdots |\psi(\varphi_{n-1}(e^{i\theta}))|^2 |f(\varphi_n(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ &\leq \epsilon^{2(n-N-1)} M^{2(N+1)} \int_T |f(\varphi_n(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ &= \epsilon^{2(n-N-1)} M^{2(N+1)} ||C_{\varphi_n}(f)||^2 \\ &\leq \epsilon^{2(n-N-1)} M^{2(N+1)} ||C_{\varphi_n}||^2 ||(f)||^2, \end{aligned} \tag{12}$$

thus

$$||C^n_{\psi,\varphi}|| \leq \epsilon^{(n-N-1)} M^{(N+1)} ||C_{\varphi_n}||,$$

but $C_{\varphi_n} = C^n_\varphi$, therefore

$$||C^n_{\psi,\varphi}|| \leq \epsilon^{(n-N-1)} M^{(N+1)} ||C^n_\varphi||.$$

Hence for all n large enough,

$$\begin{aligned} ||C^n_{\psi,\varphi}||^{1/n} &\leq \epsilon 2 ||C^n_\varphi||^{1/n} \\ &\leq \epsilon 2 ||C_\varphi||. \end{aligned} \tag{13}$$

Thus we can see that the spectral radius of the operator is 0. Therefore $\sigma(C_{\psi,\varphi}) = \{0\}$. This completes the proof. □

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