ON FRAMES FOR COUNTABLY GENERATED
HILBERT C*-MODULES

LJILJANA ARAMBAŠIĆ

(Communicated by Joseph A. Ball)

Abstract. Let $V$ be a countably generated Hilbert C*-module over a C*-algebra $A$. We prove that a sequence $\{f_i : i \in I\} \subseteq V$ is a standard frame for $V$ if and only if the sum $\sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle$ converges in norm for every $x \in V$ and if there are constants $C, D > 0$ such that $C \|x\|^2 \leq \|\sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle\| \leq D \|x\|^2$ for every $x \in V$. We also prove that surjective adjointable operators preserve standard frames. A class of frames for countably generated Hilbert C*-modules over the C*-algebra of all compact operators on some Hilbert space is discussed.

1. INTRODUCTION AND PRELIMINARIES

A (right) Hilbert C*-module $V$ over a C*-algebra $A$ (or a Hilbert $A$-module) is a linear space which is a right $A$-module, together with an $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $V \times V$ which is linear in the second and conjugate linear in the first variable such that $V$ is a Banach space with the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We use the symbol $\langle V, V \rangle$ for the closed, two-sided ideal of $A$ spanned by all inner products $\langle x, y \rangle$, $x, y \in V$. $V$ is said to be a full Hilbert $A$-module if $\langle V, V \rangle = A$.

We denote the C*-algebra of all adjointable operators on a Hilbert C*-module $V$ by $\mathbf{B}(V)$. We also use $\mathbf{B}(V, W)$ to denote the space of all adjointable operators acting between different Hilbert $A$-modules. A good reference for Hilbert C*-modules are the lecture notes of E. C. Lance [12].

The C*-algebra of all bounded operators and the ideal of all compact operators on a Hilbert space $H$ are denoted by $\mathbf{B}(H)$ and $\mathbf{K}(H)$, respectively.

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [6] as part of their research in non-harmonic Fourier series. A frame for a separable Hilbert space $H$ is defined to be a finite or countable sequence $\{f_i : i \in I\}$ for which there exists constants $C, D > 0$ such that

$$C \|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq D \|x\|^2, \quad x \in H.$$ 

M. Frank and D. Larson [7, 8] generalized this definition to the situation of countably generated Hilbert C*-modules. A frame for a countably generated Hilbert
A $C^*$-module $V$ is a sequence $\{f_i : i \in I\}$ ($I \subseteq \mathbb{N}$ finite or countable) for which there are constants $C, D > 0$ such that
\[
C \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D \langle x, x \rangle, \quad x \in V.
\]

We consider standard frames for which the sum in the middle of (1.1) converges in norm for every $x \in V$. (For non-standard frames the sum in (1.1) converges only weakly for at least one element of $V$.) The numbers $C$ and $D$ are called frame bounds. A frame $\{f_i : i \in I\}$ is called a tight frame if we can choose $C = D$ and a Parseval frame (or a normalized tight frame) if $C = D = 1$. A sequence which satisfies only the right-hand inequality in (1.1) is called a Bessel sequence with a Bessel bound $D$.

The frame transform for a standard frame $\{f_i : i \in I\}$ is the map $\theta : V \to \ell_2(I)$ defined by $\theta(x) = ((\langle f_i, x \rangle))_i$, where $\ell_2(A)$ denotes a Hilbert $A$-module $\{a_i : a_i \in A, \sum_{i \in I} a_i^* a_i \text{ converges in norm}\}$ with pointwise operations and the inner product $\langle (a_i), (b_i) \rangle = \sum_{i \in I} a_i^* b_i$. The frame transform possesses an adjoint operator and realizes an embedding of $V$ onto an orthogonal summand of $\ell_2(I)$ (Theorem 4.4). The operator $S = (\theta^* \theta)^{-1} \in \mathcal{B}(V)$ is said to be the frame operator for a standard frame $\{f_i : i \in I\}$. The frame operator is positive, invertible, and is the unique operator in $\mathcal{B}(V)$ such that the reconstruction formula
\[
x = \sum_{i \in I} f_i (S f_i, x)
\]
holds for all $x \in V$. Let us remark that although M. Frank and D. Larson [7, 8] stated all their results for the unital case, many proofs can be applied to the non-unital situation.

In a countably generated Hilbert $C^*$-module over a unital $C^*$-algebra, standard frames always exist [7]. Also, a Hilbert $C^*$-module over a $C^*$-algebra of all compact operators $\mathcal{K}(H)$ on some Hilbert space $H$ possesses frames; this follows from [2], where the concept of an orthonormal basis for a Hilbert $C^*$-module was discussed.

An element $v$ of a Hilbert $A$-module $V$ is said to be a basic vector if $e = \langle v, v \rangle$ is a projection in $A$ such that $e A e = C e$. The system of basic vectors $\{v_i : i \in I\}$ in $V$ is said to be an orthonormal basis for $V$ if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$, and if it generates a dense submodule of $V$. Every orthonormal basis $\{v_i : i \in I\}$ for a Hilbert $C^*$-module satisfies $\langle x, x \rangle = \sum_{i \in I} \langle x, v_i \rangle \langle v_i, x \rangle$ for all $x \in V$, with the norm convergence (2 Theorem 1).

Recall that, if $A$ is a $C^*$-subalgebra of $\mathcal{K}(H)$ and $e \in A$ a non-zero projection, the condition $e A e = C e$ is equivalent to the minimality of $e$ (i.e., the only subprojections of $e$ in $A$ are 0 and $e$) [11 Lemma 1.4.1]. Minimal projections in $\mathcal{K}(H)$ are exactly orthogonal projections of rank 1.

Clearly, an arbitrary Hilbert $C^*$-module does not possess an orthonormal basis, since there are $C^*$-algebras without projections. It is known that every Hilbert $C^*$-module $V$ over $\mathcal{K}(H)$ possesses an orthonormal basis; furthermore, for a fixed orthogonal projection $e \in \mathcal{K}(H)$ of rank 1, there is an orthonormal basis $\{v : i : i \in I\}$ for $V$ such that $\langle v_i, v_i \rangle = e$ for all $i \in I$.

In a Hilbert $\mathcal{K}(H)$-module, the condition of the minimality of supporting projections $e_i = \langle v_i, v_i \rangle, i \in I$, ensures that all orthonormal bases have the same cardinality (2 Theorem 2). For a countably generated Hilbert $\mathcal{K}(H)$-module, a set of indices for (all) orthonormal bases is countable. (By choosing an orthonormal basis...
\{v_i : i \in I\} such that \langle v_i, v_i \rangle = e, i \in I, for some orthogonal projection e \in \mathbf{K}(H) of rank 1, the last statement proves in the same way as in the Hilbert space case.) So, every orthonormal basis for a countably generated Hilbert \(\mathbf{K}(H)\)-module \(V\) is a standard Parseval frame for \(V\).

The paper is organized as follows.

In Section 2 we study standard frames for arbitrary countably generated Hilbert \(C^*\)-modules. We first show that an adjointable operator between Hilbert \(C^*\)-modules is bounded below with respect to the norm if and only if it is bounded below with respect to the inner product; furthermore, this is equivalent to the surjectivity of its adjoint operator. The first equivalence implies that, in the definition of standard frames, we can replace (1.1) with

\[ C \|x\|^2 \leq \| \sum_{i \in I} \langle x, f_i \rangle (f_i (x)) \| \leq D \|x\|^2 \]

for all \(x \in V\) (Theorem 2.6). From the second equivalence we conclude that surjective adjointable operators preserve standard frames (Theorem 2.6).

In Section 3 we discuss standard frames \{f_i : i \in I\} for which there exists a family of projections \{e_i : i \in I\} such that \(e_i A e_i = C e_i\) and \(f_i = f_i e_i\) for every \(i \in I\). Surjective images of orthonormal bases are frames of this form. We prove that only a Hilbert \(C^*\)-module \(V\) for which \((V, V)\) is a CCR-algebra admits such frames. Discussion is mainly restricted to countably generated Hilbert \(\mathbf{K}(H)\)-modules, where such frames always exist; moreover, for every orthogonal projection \(e \in \mathbf{K}(H)\) of rank 1, there is a frame \(\{f_i : i \in I\}\) such that \(f_i = f_i e\) for all \(i \in I\). We show that frames \{f_i : i \in I\} for a countably generated Hilbert \(\mathbf{K}(H)\)-module \(V\) such that \(\langle f_i, f_i \rangle = e, i \in I,\) correspond to frames for a Hilbert space \(V_e = \{v e : v \in V\}\) (Theorem 3.3).

2. Some properties of standard modular frames

The results we obtain in this section are the consequences of the statement which generalizes the well known fact: a bounded linear operator between Hilbert spaces is surjective if and only if its adjoint is bounded below.

**Proposition 2.1.** Let \(A\) be a \(C^*\)-algebra, \(V\) and \(W\) Hilbert \(A\)-modules, and \(T \in \mathcal{B}(V, W)\). The following statements are mutually equivalent:

1. \(T\) is surjective.
2. \(T^*\) is bounded below with respect to the norm, i.e., there is \(m > 0\) such that \(\|T^* x\| \geq m \|x\|\) for all \(x \in V\).
3. \(T^*\) is bounded below with respect to the inner product, i.e., there is \(m' > 0\) such that \(\langle T^* x, T^* x \rangle \geq m' \langle x, x \rangle\) for all \(x \in V\).

**Proof.** (1) \(\Rightarrow\) (3): Suppose \(T\) is surjective. Then \(\text{Im } T = W\) is closed. It follows from [12, Theorem 3.2] that \(\text{Im } T^*\) is also closed, \(\text{Ker } T \oplus \text{Im } T^* = V\) and \(\text{Ker } T^* \oplus \text{Im } T = W\). We shall prove that \(T T^*\) is bijective.

If \(T T^* x = 0\) for some \(x \in V\), then \(T^* x \in \text{Ker } T \cap \text{Im } T^* = \{0\}\), hence \(T^* x = 0\). Now \(x \in \text{Ker } T^* = (\text{Im } T)^\perp = W^\perp = \{0\}\) implies \(x = 0\). This proves that \(T T^*\) is injective.

Let \(z\) be an arbitrarily chosen element of \(W\). \(T\) is surjective, so \(z = Ty\) for some \(y \in V\). There are \(y_1 \in \text{Ker } T\) and \(x \in W\) such that \(y = y_1 \oplus T^* x\). Then \(z = Ty = T(y_1 \oplus T^* x) = T T^* x\); therefore \(T T^*\) is surjective.

Since \(T T^*\) is a positive invertible element of the \(C^*\)-algebra \(\mathcal{B}(V, W)\), we have

\[0 \leq (T T^*)^{-1} \leq \|(T T^*)^{-1}\| \text{id}_V \Rightarrow T T^* \geq (\|(T T^*)^{-1}\|)^{-1} \text{id}_V,\]
Let $\text{Corollary 2.2.}$

Given that $C$ frame bounds is clearly injective, and it is easy to see that $\text{Im} T = \{0\} \oplus \text{Im} T = \text{Im} T$. Hence $T$ is surjective.

**Corollary 2.2.** Let $A$ be a $C^*$-algebra, $V$ a Hilbert $A$-module, and $T \in B(V)$ such that $T = T^*$. The following statements are mutually equivalent:

1. $T$ is surjective.
2. There are $m, M > 0$ such that $m \|x\| \leq \|Tx\| \leq M \|x\|$ for all $x \in V$.
3. There are $m', M' > 0$ such that $m' \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq M' \langle x, x \rangle$ for all $x \in V$.

**Remark 2.3.** An operator $T \in B(V)$ is said to be coercive if there is a positive constant $m$ such that $\langle T^* x, T^* x \rangle \geq m \langle x, x \rangle$ holds for all $x \in V$. It follows from Proposition 2.1 that coercive operators in $B(V)$ are exactly surjections in $B(V)$.

**Theorem 2.4.** Let $A$ be a $C^*$-algebra, $V$ a countably generated Hilbert $A$-module, $\{f_i : i \in I\}$ a sequence in $V$, and $\theta(x) = (\langle f_i, x \rangle)_{i \in I}$ for $x \in V$. The following statements are mutually equivalent:

1. $\{f_i : i \in I\}$ is a standard frame for $V$.
2. $\theta \in B(V, \ell_2(A))$ and $\theta$ is bounded below.
3. $\theta \in B(V, \ell_2(A))$ and $\theta^*$ is surjective.

**Proof.** It follows from [5, Theorem 4.1] and Proposition 2.1 since

$$\langle \theta x, \theta x \rangle = \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle, \quad x \in V.$$ 

Another direct consequence of Proposition 2.1 is that surjective adjointable operators preserve standard frames.

**Theorem 2.5.** Let $A$ be a $C^*$-algebra, $V$ and $W$ countably generated Hilbert $A$-modules, and $T \in B(V, W)$ surjective. If $\{f_i : i \in I\}$ is a standard frame for $V$ with frame bounds $C$ and $D$, then $\{Tf_i : i \in I\}$ is a standard frame for $W$ with frame bounds $\frac{C}{\|TT^*\|}$ and $D\|T\|^2$.

**Proof.** Since $\{f_i : i \in I\}$ is a standard frame for $V$, and since $T^* y \in V$ for all $y \in W$, we have

$$C \langle T^* y, T^* y \rangle \leq \sum_{i \in I} \langle T^* y, f_i \rangle \langle f_i, T^* y \rangle \leq D \langle T^* y, T^* y \rangle, \quad y \in W.$$ 

From the proof of Proposition 2.1, we have $\langle T^* y, T^* y \rangle \geq \|TT^*\|^{-1} \langle y, y \rangle$ for all $y \in W$, since $T$ is surjective. It follows that

$$\frac{C}{\|TT^*\|^{-1}} \langle y, y \rangle \leq \sum_{i \in I} \langle y, Tf_i \rangle \langle Tf_i, y \rangle \leq D\|T\|^2 \langle y, y \rangle, \quad y \in W.$$ 


We conclude this section with the result which states that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

**Theorem 2.6.** Let $A$ be a $C^*$-algebra, $V$ a countably generated Hilbert $A$-module, and $\{f_i : i \in I\}$ a sequence in $V$ such that $\sum_{i \in J} (x, f_i) \langle f_i, x \rangle$ converges in norm for every $x \in V$. Then $\{f_i : i \in I\}$ is a standard frame for $V$ if and only if there are constants $C, D > 0$ such that

$$C\|x\|^2 \leq \|\sum_{i \in I} (x, f_i) \langle f_i, x \rangle\| \leq D\|x\|^2, \quad x \in V. \quad (2.1)$$

**Proof.** Evidently, every standard frame for $V$ satisfies (2.1).

For the converse we suppose that a sequence $\{f_i : i \in I\}$ fulfills (2.1). For an arbitrary $x \in V$ and a finite $J \subseteq I$ we define $x_J = \sum_{i \in J} f_i(x, f_i)$. Then

$$\|x_J\|^2 = \|(x_J, x_J)\|^2 = \|(x_J, \sum_{i \in J} f_i(x, f_i))\|^2 = \|\sum_{i \in J} (x_J, f_i) \langle f_i, x_J \rangle\|^2 \leq \|\sum_{i \in J} (x_J, f_i) \langle f_i, x_J \rangle\| \cdot \|\sum_{i \in J} (x_J, f_i) \langle f_i, x_J \rangle\| \leq D\|x_J\|^2 \|\sum_{i \in J} (x_J, f_i) \langle f_i, x_J \rangle\|,$$

therefore

$$\|\sum_{i \in J} f_i(x, f_i)\|^2 = \|x_J\|^2 \leq D\|\sum_{i \in J} (x_J, f_i) \langle f_i, x_J \rangle\|.$$

Since $J$ is arbitrary, the series $\sum_{i \in I} f_i(x, f_i)$ converges and

$$\|\sum_{i \in I} f_i(x, f_i)\|^2 \leq D\|\sum_{i \in I} (x, f_i) \langle f_i, x \rangle\| \leq D^2\|x\|^2 \Rightarrow \|\sum_{i \in I} f_i(x, f_i)\| \leq D\|x\|.$$

Since $x \in V$ is arbitrarily chosen, the operator

$$T : V \rightarrow V, \quad x \mapsto \sum_{i \in I} f_i(x, f_i)$$

is well defined, bounded and $A$-linear. It is easy to check that $(Tx, y) = \langle x, Ty \rangle$ for all $x, y \in V$, so $T \in B(V)$ and $T = T^*$. From $(Tx, x) = \sum_{i \in I} (x, f_i) \langle f_i, x \rangle \geq 0$ for all $x \in V$, it follows that $T \geq 0$. Now (2.1) and $\|T^{1/2}x, T^{1/2}x\| = \sum_{i \in I} (x, f_i) \langle f_i, x \rangle$ imply $\sqrt{C}\|x\| \leq \|T^{1/2}x\| \leq \sqrt{D}\|x\|$ for all $x \in V$. By Corollary 2.2 there are constants $C', D' > 0$ such that

$$C'(x, x) \leq \langle T^{1/2}x, T^{1/2}x \rangle = \sum_{i \in I} (x, f_i) \langle f_i, x \rangle \leq D'(x, x), \quad x \in V.$$

This proves that $\{f_i : i \in I\}$ is a standard frame for $V$. \hfill \Box

### 3. ON A CLASS OF FRAMES FOR HILBERT $K(H)$-MODULES

The existence of standard frames in countably generated Hilbert $K(H)$-modules $V$ follows from the existence of orthonormal bases. If $T \in B(V)$ is a surjection and $\{v_i : i \in I\}$ an orthonormal basis for $V$, then $\{Tv_i : i \in I\}$ is a standard frame for $V$ which satisfies $Tv_i = T(v_i e_i) = (Tv_i)e_i$, where $e_i := \langle v_i, v_i \rangle$ is an orthogonal projection of rank 1 for every $i \in I$. However, not every standard frame in a Hilbert $K(H)$-module is of this type, as we show in the next example.
Example 3.1. Let \( \{v_i : i \in I\} \) be an orthonormal basis for a countably and not finitely generated Hilbert \( K(H) \)-module \( V \) with property \( e_i e_j = 0, i \neq j \), where \( e_i = \langle v_i, v_i \rangle \). (Such a basis can always be constructed by following the procedure described in [2, Remark 4(d)].) Let \( I = \bigcup_{j=1}^{\infty} I_j \) be a partition of \( I \) such that \( |I_j| = j \). Let \( f_j = \sum_{i \in I_j} v_i \in V \). Since \( \langle x, v_j \rangle \langle v_i, x \rangle = \delta_{ij} \langle x, v_j \rangle \langle v_j, x \rangle = \delta_{ij} \langle x, v_j \rangle \langle v_i, x \rangle = \delta_{ij} \langle x, v_j \rangle \langle v_i, x \rangle \) for all \( x \in V \), we have
\[
\langle x, f_j \rangle \langle f_j, x \rangle = \langle x, \sum_{i \in I_j} v_i \rangle \sum_{i \in I_j} v_i = \sum_{i,j \in I_j} \langle x, v_j \rangle \langle v_i, x \rangle = \sum_{i \in I_j} \langle x, v_i \rangle \langle v_i, x \rangle
\]
and then
\[
\langle x, x \rangle = \sum_{i \in I} \langle x, v_i \rangle \langle v_i, x \rangle = \sum_{j \in J} \langle x, f_j \rangle \langle f_j, x \rangle.
\]
This means that \( \{f_j : j \in J\} \) is a standard Parseval frame for \( V \) such that \( \langle f_j, f_j \rangle = \sum_{i \in I_j} e_i \) is a projection with \( \dim \text{Im} \langle f_i, f_i \rangle = |I_j| = j \) for all \( j \in J \).

Proposition 3.2. Let \( V \) be a countably and not finitely generated Hilbert \( K(H) \)-module. Let \( \{f_i : i \in I\} \) be a standard frame for \( V \) such that \( f_i = e_i f_i \) for some orthogonal projections \( e_i, i \in I, \) of rank 1. Then there is an orthonormal basis \( \{v_i : i \in I\} \) and a surjection \( T \in B(V) \) such that \( TV_i = f_i, i \in I \).

Proof. Let \( C \) and \( D \) be frame bounds. Let \( \{v_i : i \in I\} \) be an orthonormal basis such that \( v_i = e_i v_i, i \in I \). (We may assume that the sets of indices for a standard frame and a basis are the same, since they are both infinite subsets of \( N \).

We first show that for every \( x \in V \) the series \( \sum_{i \in J} f_i \langle v_i, x \rangle \) converges. Let \( J \) be a finite subset of \( I \) and \( x_J = \sum_{i \in J} f_i \langle v_i, x \rangle \). Then
\[
\|x_J\| = \|\sum_{i \in J} f_i \langle v_i, x \rangle\| = \|\sum_{i \in J} \langle x, v_i \rangle f_i (v_i, x)\| = \|\sum_{i \in J} \langle x, v_i \rangle f_i (v_i, x)\| \leq \|\sum_{i \in J} \langle x, v_i \rangle f_i (v_i, x)\| \cdot D \|x_J\|,
\]
from where we get \( \|x_J\|^2 \leq D \|\sum_{i \in J} \langle x, v_i \rangle f_i (v_i, x)\| \), that is,
\[
\|\sum_{i \in J} f_i \langle v_i, x \rangle\| \leq D \|\sum_{i \in J} \langle x, v_i \rangle f_i (v_i, x)\|, \text{ for every finite } J \subseteq I.
\]
Since \( \sum_{i \in J} \langle x, v_i \rangle f_i \langle v_i, x \rangle \) converges in norm ([2, Theorem 1]), it follows that the series \( \sum_{i \in J} f_i \langle v_i, x \rangle \) converges.

Similarly, we check that the series \( \sum_{i \in J} v_i \langle f_i, x \rangle \) converges for every \( x \in V \).

Now we can define the operators \( T, R : V \rightarrow V \) by \( Tx = \sum_{i \in J} f_i \langle v_i, x \rangle \) and \( R x = \sum_{i \in J} v_i \langle f_i, x \rangle \). It is straightforward to see that \( \langle Tx, y \rangle = \langle x, Ry \rangle \) for all \( x, y \in V \). Therefore \( T \in B(V) \) and \( R = T^* \). From Proposition [2,1] and
\[
C \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle = \langle T^* x, T^* x \rangle, \quad x \in V,
\]

it follows that \( T \) is surjective.

It only remains to note that \( Tv_i = f_i \langle v_i, v_i \rangle = f_i e_i = f_i \) for all \( i \in I \).
Also, for all \(x, y \in V_e\) we obtain that \(\langle x, y \rangle = (y, x)e\). \(V_e\) is an invariant subspace for each \(T\) in \(B(V)\) and the map

\[
T \mapsto T|V_e, \quad B(V) \to B(V_e)
\]

establishes an isomorphism of \(C^*\)-algebras, where \(B(V_e)\) denotes the \(C^*\)-algebra of all bounded operators on \(V_e\). It is known that a family \(\{v_i : i \in I\} \subseteq V_e\) is an orthonormal basis for \(V\) if and only if it is an orthonormal basis for \(V_e\). (The proofs can be found in [2, Remark 4, Theorem 5].) We extend the last statement to a standard frame \(\{f_i : i \in I\}\) contained in \(V_e\). First we need a lemma which describes some properties of the isomorphism [31].

**Lemma 3.3.** Let \(V\) be a Hilbert \(K(H)\)-module, \(e \in K(H)\) an orthogonal projection of rank 1, and \(T \in B(V)\). The following statements hold:

1. \(T\) is bounded below if and only if \(T|V_e \in B(V_e)\) is bounded below.
2. \(T\) is surjective if and only if \(T|V_e \in B(V_e)\) is surjective.

**Proof.** (1) First observe that if \(T|V_e \in B(V_e)\) is a positive operator on the Hilbert space \(V_e\), then \(T \in B(V)\) is a positive element of the \(C^*\)-algebra \(B(V)\). This is a consequence of the fact that the map \(T \mapsto T|V_e\) is an isomorphism of \(C^*\)-algebras.

Suppose \(T_e := T|V_e\) is bounded below. Let \(m > 0\) be such that \(\|T_e(x)e\| \geq m\|xe\|\) for all \(x \in V\). In other words, \(T_e^*T_e - m^2\text{id}_{V_e}\) is a positive operator on the Hilbert space \(V_e\). By the observation from the beginning of the proof, we get \(T^*T - m^2\text{id}_V \geq 0\), i.e., \(\langle (T^*T - m^2\text{id}_V)x, x \rangle \geq 0\) for all \(x \in V\). Now we have \(\|Tx\| \geq m\|x\|\) for all \(x \in V\).

The opposite statement is obvious.

(2) It follows from (1) and Proposition 2.1. \(\square\)

**Theorem 3.4.** Let \(V\) be a countably generated Hilbert \(K(H)\)-module, \(e \in K(H)\) an orthogonal projection of rank 1, and \(\{f_i : i \in I\}\) a sequence of vectors in \(V_e\). Then \(\{f_i : i \in I\}\) is a standard frame for the Hilbert \(K(H)\)-module \(V\) with frame bounds \(C\) and \(D\) if and only if \(\{f_i : i \in I\}\) is a frame for the Hilbert space \(V_e\) with frame bounds \(C\) and \(D\).

**Proof.** Suppose that \(\{f_i : i \in I\}\) is a standard frame for a Hilbert \(K(H)\)-module \(V\) with frame bounds \(C\) and \(D\). It means that

\[
C\langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D\langle x, x \rangle, \quad x \in V.
\]

Since \(\langle xe, ye \rangle = (ye, xe)e\) for all \(xe, ye \in V_e\), by choosing \(xe\) instead of \(x\) in the above inequalities, we get

\[
C(xe, xe)e \leq \sum_{i \in I} \langle xe, f_i \rangle \langle f_i, xe \rangle e \leq D(xe, xe)e, \quad x \in V,
\]

which implies \(C(x, x) \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D(x, x)\) for all \(x \in V_e\). It proves that \(\{f_i : i \in I\}\) is a frame for the Hilbert space \(V_e\) with frame bounds \(C\) and \(D\).

Now suppose that \(\{f_i : i \in I\} \subseteq V_e\) is a frame for the Hilbert space \(V_e\) with frame bounds \(C\) and \(D\).

First we assume that \(V\) is finitely generated. Let \(\{v_1, \ldots, v_n\} \subseteq V_e\) be an orthonormal basis for \(V\) and \(S_e \in B(V_e)\) the frame operator associated to the (Hilbert
The concept of an orthonormal basis has been introduced in Hilbert $C^*$-modules over an arbitrary $C^*$-algebra. Obviously, there are Hilbert $C^*$-modules which do not possess an orthonormal basis. Actually, if a Hilbert $C^*$-module $V$ possesses an orthonormal basis, then $\langle V, V \rangle$ has to be a $CCR$-algebra. We prove this in the next theorem.

**Theorem 3.5.** Let $A$ be a $C^*$-algebra and $V$ a full countably generated Hilbert $A$-module. Let $\{e_i : i \in I\}$ be a family of projections in $A$ such that $e_i A e_i = C e_i, i \in I,$ and $\{f_i : i \in I\}$ a standard frame for $V$ such that $f_i = f_i e_i, i \in I.$ Then $A$ is a $CCR$-algebra. In particular, if $V$ possesses an orthonormal basis, then $A$ is a $CCR$-algebra.

**Proof.** By the definition of a $CCR$-algebra we need to show that for every irreducible representation $\varphi : A \to B(H), \varphi(A) \subseteq K(H)$ holds.

Let $0 \neq \varphi : A \to B(H)$ be an irreducible representation of $A.$ Then $e_i A e_i = C e_i$ implies $\varphi(e_i) \varphi(A) \varphi(e_i) = C \varphi(e_i)$ for every $i \in I.$

Let $i \in I$ be such that $\varphi(e_i) \neq 0.$ Now $\varphi(e_i)$ is a non-zero projection, so there is a non-zero vector $\xi_0 \in H$ which belongs to $\text{Im} \varphi(e_i).$ Then $\varphi(e_i) \xi_0 = \xi_0$ and

$$\varphi(e_i) \varphi(A) \varphi(e_i) \xi_0 = C \varphi(e_i) \xi_0 \Rightarrow \varphi(e_i) \varphi(A) \xi_0 = C \xi_0.$$ 

$\varphi$ is irreducible, therefore it is a cyclic representation of $A,$ and every non-zero vector is cyclic for $\varphi.$ In particular, $\xi_0$ is a cyclic vector for $\varphi.$ Therefore

$$\{0\} \neq \varphi(e_i) H = \varphi(e_i) \varphi(A) \xi_0 \subseteq \varphi(e_i) \varphi(A) \xi_0 = C \xi_0 \Rightarrow \text{Im} \varphi(e_i) = C \xi_0.$$ 

This proves that $\varphi(e_i) \in K(H)$ for every $i \in I.$

Let $S \in B(V)$ be the frame operator associated to $\{f_i : i \in I\}.$ From the reconstruction formula (1.2) we have $\langle x, y \rangle = \sum_{i \in I} \langle x, S f_i \rangle \langle f_i, y \rangle$ and then

$$(3.2) \varphi(\langle x, y \rangle) = \sum_{i \in I} \varphi(\langle x, S f_i \rangle) \varphi(\langle f_i, y \rangle), \quad x, y \in V.$$
From \( f_i = f e_i \) and compactness of \( \varphi(e_i) \) it follows that
\[
\varphi(\langle x, S f_i \rangle) \varphi(\langle f_i, y \rangle) = \varphi(\langle x, S f_i \rangle) \varphi(e_i) \varphi(\langle f_i, y \rangle) \in \mathbf{K}(H), \quad x, y \in V, i \in I.
\]
Finally, we get \( \varphi(\langle x, y \rangle) \in \mathbf{K}(H) \) for all \( x, y \in V \), as the convergence in (3.2) is in norm. Since \( V \) is full, we conclude that \( \varphi(A) \subseteq \mathbf{K}(H) \).

This finishes our proof. \( \square \)

The converse of the previous theorem does not hold. For example, we can take an arbitrary Hilbert \( C^* \)-module over the \( C^* \)-algebra \( A = C([0,1]) \) of all continuous complex functions on the unit segment \([0,1]\), \( A \) is a \( CCR \)-algebra, since it is commutative, and the only projection \( e \in A \) which satisfies \( eAe = Ce \) is the constant function \( 0 \).

Remark 3.6. Frames of subspaces for a separable Hilbert space have been recently introduced and studied in [4]. We can generalize their definition for Hilbert \( K(H) \)-modules in the following way.

Let \( V \) be a countably generated Hilbert \( K(H) \)-module, \( \{W_i : i \in I\} \ (I \subseteq \mathbb{N}) \) a family of closed submodules of \( V \), and \( \{\lambda_i : i \in I\} \) a family of weights, i.e., a family of positive numbers. We say that \( \{W_i : i \in I\} \) is a standard frame of submodules for \( V \) with respect to a family of weights \( \{\lambda_i : i \in I\} \), if there are constants \( C, D > 0 \) such that
\[
C \langle x, x \rangle \leq \sum_{i \in I} \lambda_i^2 \langle \pi_i(x), \pi_i(x) \rangle \leq D \langle x, x \rangle, \quad x \in V,
\]
where \( \pi_i \in \mathcal{B}(V) \) denotes the orthogonal projection on \( W_i \) for every \( i \in I \), and convergence of the sum in the middle of (3.3) is in norm.

Let us fix an orthogonal projection \( e \in \mathbf{K}(H) \) of rank \( 1 \). It can be proved that a family of closed submodules \( \{W_i : i \in I\} \) is a standard frame of submodules for \( V \) with respect to the family of weights \( \{\lambda_i : i \in I\} \) if and only if \( \{(W_i)_e : i \in I\} \) is a frame of subspaces for \( V_e \) with respect to the family of weights \( \{\lambda_i : i \in I\} \). Therefore many statements from [4] can be extended to countably generated Hilbert \( K(H) \)-modules. This will be done in our subsequent paper.

Acknowledgement

The author would like to thank Professors Damir Bakić and Boris Guljaš for helpful discussions. Thanks are also due to the referee for useful comments.

References


Department of Mathematics, University of Zagreb, Blienička c. 30, 10000 Zagreb, Croatia

E-mail address: ljsekul@math.hr