ON FRAMES FOR COUNTABLY GENERATED HILBERT $C^*$-MODULES

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Abstract. Let $V$ be a countably generated Hilbert $C^*$-module over a $C^*$-algebra $A$. We prove that a sequence $\{f_i : i \in I\} \subseteq V$ is a standard frame for $V$ if and only if the sum $\sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle$ converges in norm for every $x \in V$ and if there are constants $C, D > 0$ such that $C\|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle |^2 \leq D\|x\|^2$ for every $x \in V$. We also prove that surjective adjointable operators preserve standard frames. A class of frames for countably generated Hilbert $C^*$-modules over the $C^*$-algebra of all compact operators on some Hilbert space is discussed.

1. Introduction and preliminaries

A (right) Hilbert $C^*$-module $V$ over a $C^*$-algebra $A$ (or a Hilbert $A$-module) is a linear space which is a right $A$-module, together with an $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $V \times V$ which is linear in the second and conjugate linear in the first variable such that $V$ is a Banach space with the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We use the symbol $\langle V, V \rangle$ for the closed, two-sided ideal of $A$ spanned by all inner products $\langle x, y \rangle$, $x, y \in V$. $V$ is said to be a full Hilbert $A$-module if $\langle V, V \rangle = A$.

We denote the $C^*$-algebra of all adjointable operators on a Hilbert $C^*$-module $V$ by $B(V)$. We also use $B(V, W)$ to denote the space of all adjointable operators acting between different Hilbert $A$-modules. A good reference for Hilbert $C^*$-modules are the lecture notes of E. C. Lance [12].

The $C^*$-algebra of all bounded operators and the ideal of all compact operators on a Hilbert space $H$ are denoted by $B(H)$ and $K(H)$, respectively.

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [6] as part of their research in non-harmonic Fourier series. A frame for a separable Hilbert space $H$ is defined to be a finite or countable sequence $\{f_i : i \in I\}$ for which there exists constants $C, D > 0$ such that

$$C\|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle |^2 \leq D\|x\|^2, \quad x \in H.$$ 

M. Frank and D. Larson [7][8] generalized this definition to the situation of countably generated Hilbert $C^*$-modules. A frame for a countably generated Hilbert...
A sequence \( \{ f_i : i \in I \} \) (\( I \subseteq \mathbb{N} \) finite or countable) for which there are constants \( C, D > 0 \) such that

\[
C \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D \langle x, x \rangle, \quad x \in V.
\]

We consider standard frames for which the sum in the middle of (1.1) converges in norm for every \( x \in V \). (For non-standard frames the sum in (1.1) converges only weakly for at least one element of \( V \).) The numbers \( C \) and \( D \) are called frame bounds. A frame \( \{ f_i : i \in I \} \) is called a Bessel sequence if the right-hand inequality in (1.1) is called a Bessel sequence with a Bessel bound \( D \).

The frame transform for a standard frame \( \{ f_i : i \in I \} \) is the map \( \theta : V \to \ell_2(A) \) defined by \( \theta(x) = ((f_i(x)))_i \), where \( \ell_2(A) \) denotes a Hilbert \( A \)-module \( \{ (a_i)_i : a_i \in A, \sum_{i \in I} a_i^* a_i \text{ converges in norm} \} \) with pointwise operations and the inner product \( \langle (a_i)_i, (b_i)_i \rangle = \sum_{i \in I} a_i^* b_i \). The frame transform possesses an adjoint operator and realizes an embedding of \( V \) onto an orthogonal summand of \( \ell_2(A) \) (Theorem 4.4). The operator \( S = (\theta^* \theta)^{-1} \in \mathcal{B}(V) \) is said to be the frame operator for a standard frame \( \{ f_i : i \in I \} \). The frame operator is positive, invertible, and is the unique operator in \( \mathcal{B}(V) \) such that the reconstruction formula

\[
x = \sum_{i \in I} f_i \langle S f_i, x \rangle
\]

holds for all \( x \in V \). Let us remark that although M. Frank and D. Larson \cite{2, 8} stated all their results for the unital case, many proofs can be applied to the non-unital situation.

In a countably generated Hilbert \( C^* \)-module over a unital \( C^* \)-algebra, standard frames always exist \cite{2}. Also, a Hilbert \( C^* \)-module over a \( C^* \)-algebra of all compact operators \( K(H) \) on some Hilbert space \( H \) possesses frames; this follows from \cite{2}, where the concept of an orthonormal basis for a Hilbert \( C^* \)-module was discussed.

An element \( v \) of a Hilbert \( A \)-module \( V \) is said to be a basic vector if \( e = \langle v, v \rangle \) is a projection in \( A \) such that \( e A e = C e \). The system of basic vectors \( \{ v_i : i \in I \} \) in \( V \) is said to be an orthonormal basis for \( V \) if \( \langle v_i, v_j \rangle = 0 \) for all \( i \neq j \), and if it generates a dense submodule of \( V \). Every orthonormal basis \( \{ v_i : i \in I \} \) for a Hilbert \( C^* \)-module satisfies \( \langle x, x \rangle = \sum_{i \in I} \langle x, v_i \rangle \langle v_i, x \rangle \) for all \( x \in V \), with the norm convergence (Theorem 1).

A minimal projection in \( K(H) \) is exactly the minimal projection in a countably generated Hilbert \( K(H) \)-module, the condition of the minimality of supporting projections \( e_i = \langle v_i, v_i \rangle, i \in I \), ensures that all orthonormal bases have the same cardinality (Theorem 2). For a countably generated Hilbert \( K(H) \)-module, a set of indices for (all) orthonormal bases is countable. (By choosing an orthonormal basis
\{v_i : i \in I\} such that \langle v_i, v_i \rangle = e, i \in I, for some orthogonal projection e \in \mathbf{K}(H) of rank 1, the last statement proves in the same way as in the Hilbert space case.) So, every orthonormal basis for a countably generated Hilbert \(\mathbf{K}(H)\)-module \(V\) is a standard Parseval frame for \(V\).

The paper is organized as follows.

In Section 2 we study standard frames for arbitrary countably generated Hilbert \(C^*\)-modules. We first show that an adjointable operator between Hilbert \(C^*\)-modules is bounded below with respect to the norm if and only if it is bounded below with respect to the inner product; furthermore, this is equivalent to the surjectivity of its adjoint operator. The first equivalence implies that, in the definition of standard frames, we can replace (1.1) with \(C\|x\|^2 \leq \|\sum_{i \in I} \langle x, f_i \rangle f_i, x\| \leq D\|x\|^2\) for all \(x \in V\) (Theorem 2.6). From the second equivalence we conclude that surjective adjointable operators preserve standard frames (Theorem 2.8).

In Section 3 we discuss standard frames \(\{f_i : i \in I\}\) for which there exists a family of projections \(\{e_i : i \in I\}\) such that \(e_i A e_i = C e_i\) and \(f_i = f_i e_i\) for every \(i \in I\). Surjective images of orthonormal bases are frames of this form. We prove that only a Hilbert \(C^*\)-module \(V\) for which \(\langle V, V\rangle\) is a CCR-algebra admits such frames. Discussion is mainly restricted to countably generated Hilbert \(\mathbf{K}(H)\)-modules, where such frames always exist; moreover, for every orthogonal projection \(e \in \mathbf{K}(H)\) of rank 1, there is a frame \(\{f_i : i \in I\}\) such that \(f_i = f_i e\) for all \(i \in I\). We show that frames \(\{f_i : i \in I\}\) for a countably generated Hilbert \(\mathbf{K}(H)\)-module \(V\) such that \(\langle f_i, f_i \rangle = e, i \in I\), correspond to frames for a Hilbert space \(V_e = \{ve : v \in V\}\) (Theorem 3.3).

2. Some properties of standard modular frames

The results we obtain in this section are the consequences of the statement which generalizes the well known fact: a bounded linear operator between Hilbert spaces is surjective if and only if its adjoint is bounded below.

**Proposition 2.1.** Let \(A\) be a \(C^*\)-algebra, \(V\) and \(W\) Hilbert \(A\)-modules, and \(T \in \mathbf{B}(V, W)\). The following statements are mutually equivalent:

1. \(T\) is surjective.
2. \(T^*\) is bounded below with respect to the norm, i.e., there is \(m > 0\) such that \(\|T^* x\| \geq m\|x\|\) for all \(x \in V\).
3. \(T^*\) is bounded below with respect to the inner product, i.e., there is \(m' > 0\) such that \(\langle T^* x, T^* x \rangle \geq m'\langle x, x \rangle\) for all \(x \in V\).

**Proof.** (1) \(\Rightarrow\) (3): Suppose \(T\) is surjective. Then \(\text{Im } T = W\) is closed. It follows from [12] Theorem 3.2] that \(\text{Im } T^*\) is also closed, \(\text{Ker } T \oplus \text{Im } T^* = V\) and \(\text{Ker } T^* \oplus \text{Im } T = W\). We shall prove that \(TT^*\) is bijective.

If \(TT^* x = 0\) for some \(x \in V\), then \(T^* x \in \text{Ker } T \cap \text{Im } T^* = \{0\}\), hence \(T^* x = 0\). Now \(x \in \text{Ker } T^* = (\text{Im } T)^\perp = W^\perp = \{0\}\) implies \(x = 0\). This proves that \(TT^*\) is injective.

Let \(z\) be an arbitrarily chosen element of \(W\). \(T\) is surjective, so \(z = Ty\) for some \(y \in V\). There are \(y_1 \in \text{Ker } T\) and \(x \in W\) such that \(y = y_1 \oplus T^* x\). Then \(z = Ty = T(y_1 \oplus T^* x) = T T^* x\); therefore \(TT^*\) is surjective.

Since \(TT^*\) is a positive invertible element of the \(C^*\)-algebra \(\mathbf{B}(V)\), we have \(0 \leq (TT^*)^{-1} \leq \|(TT^*)^{-1}\| \text{id}_V \Rightarrow TT^* \geq \|(TT^*)^{-1}\|^{-1} \text{id}_V\),
where id$_V$ stands for the identity operator on $V$. Denoting $m' = ||(TT^*)^{-1}||^{-1}$ we get $TT^* - m' \text{id}_V \geq 0$. By [12] Lemma 4.1, this is equivalent to

$$\langle (TT^* - m' \text{id}_V)x, x \rangle \geq 0$$

for all $x \in V$, i.e., $\langle T^*x, T^*x \rangle \geq m'\langle x, x \rangle$ for all $x \in V$.

The implication (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): Suppose that $T^*$ is bounded below with respect to the norm. Then $T^*$ is clearly injective, and it is easy to see that Im $T^*$ is closed. Then $T$ has the closed range, again by [12] Theorem 3.2, and $W = \text{Ker} T^* \oplus \text{Im} T = \{0\} \oplus \text{Im} T = \text{Im} T$. Hence $T$ is surjective.

**Corollary 2.2.** Let $A$ be a C$^*$-algebra, $V$ a Hilbert $A$-module, and $T \in B(V)$ such that $T = T^*$. The following statements are mutually equivalent:

1. $T$ is surjective.
2. There are $m, M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$ for all $x \in V$.
3. There are $m', M' > 0$ such that $m'\langle x, x \rangle \leq \langle Tx, Tx \rangle \leq M'\langle x, x \rangle$ for all $x \in V$.

**Remark 2.3.** An operator $T \in B(V)$ is said to be coercive if there is a positive constant $m$ such that $\langle T^*x, T^*x \rangle \geq m\langle x, x \rangle$ holds for all $x \in V$. It follows from Proposition [2.1] that coercive operators in $B(V)$ are exactly surjections in $B(V)$.

**Theorem 2.4.** Let $A$ be a C$^*$-algebra, $V$ a countably generated Hilbert $A$-module, $\{f_i : i \in I\}$ a sequence in $V$, and $\theta(x) = (\langle f_i, x \rangle)_{i \in I}$ for $x \in V$. The following statements are mutually equivalent:

1. $\{f_i : i \in I\}$ is a standard frame for $V$.
2. $\theta \in B(V, \ell_2(A))$ and $\theta$ is bounded below.
3. $\theta \in B(V, \ell_2(A))$ and $\theta^*$ is surjective.

**Proof.** It follows from [3] Theorem 4.1] and Proposition 2.1 since

$$\langle \theta x, \theta x \rangle = \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle, \quad x \in V.$$  

Another direct consequence of Proposition [2.1] is that surjective adjointable operators preserve standard frames.

**Theorem 2.5.** Let $A$ be a C$^*$-algebra, $V$ and $W$ countably generated Hilbert $A$-modules, and $T \in B(V, W)$ surjective. If $\{f_i : i \in I\}$ is a standard frame for $V$ with frame bounds $C$ and $D$, then $\{Tf_i : i \in I\}$ is a standard frame for $W$ with frame bounds $\frac{C}{\|TT^*\|_{\ell_2}}$ and $D\|T\|^2$.

**Proof.** Since $\{f_i : i \in I\}$ is a standard frame for $V$, and since $T^*y \in V$ for all $y \in W$, we have

$$C\langle T^*y, T^*y \rangle \leq \sum_{i \in I} \langle T^*y, f_i \rangle \langle f_i, T^*y \rangle \leq D\langle T^*y, T^*y \rangle, \quad y \in W.$$  

From the proof of Proposition [2.1] we have $\langle T^*y, T^*y \rangle \geq ||(TT^*)^{-1}||^{-1}\langle y, y \rangle$ for all $y \in W$, since $T$ is surjective. It follows that

$$\frac{C}{\|TT^*\|^{-1}} \langle y, y \rangle \leq \sum_{i \in I} \langle y, Tf_i \rangle \langle Tf_i, y \rangle \leq D\|T\|^2 \langle y, y \rangle, \quad y \in W.$$  

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We conclude this section with the result which states that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

**Theorem 2.6.** Let \( A \) be a \( C^* \)-algebra, \( V \) a countably generated Hilbert \( A \)-module, and \( \{ f_i : i \in I \} \) a sequence in \( V \) such that \( \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \) converges in norm for every \( x \in V \). Then \( \{ f_i : i \in I \} \) is a standard frame for \( V \) if and only if there are constants \( C, D > 0 \) such that

\[
C \|x\|^2 \leq \left\| \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \right\| \leq D \|x\|^2, \quad x \in V.
\]

**Proof.** Evidently, every standard frame for \( V \) satisfies (2.1).

For the converse we suppose that a sequence \( \{ f_i : i \in I \} \) fulfills (2.1). For an arbitrary \( x \in V \) and a finite \( J \subseteq I \) we define \( x_J = \sum_{i \in J} f_i \langle f_i, x \rangle \). Then

\[
\|x_J\|^2 = \|\langle x_J, x_J \rangle\|^2 = \|\langle x, \sum_{i \in J} f_i \langle f_i, x \rangle \rangle\|^2 = \left\| \sum_{i \in J} \langle x, f_i \rangle \langle f_i, x \rangle \right\|^2
\]

\[
\leq \left\| \sum_{i \in J} \langle x, f_i \rangle \langle f_i, x \rangle \right\| \cdot \left\| \sum_{i \in J} \langle x, f_i \rangle \langle f_i, x \rangle \right\| \leq D \|x_J\|^2 \left\| \sum_{i \in J} \langle x, f_i \rangle \langle f_i, x \rangle \right\|,
\]

therefore

\[
\left\| \sum_{i \in J} f_i \langle f_i, x \rangle \right\|^2 = \|x_J\|^2 \leq D \left\| \sum_{i \in J} \langle x, f_i \rangle \langle f_i, x \rangle \right\|.
\]

Since \( J \) is arbitrary, the series \( \sum_{i \in I} f_i \langle f_i, x \rangle \) converges and

\[
\left\| \sum_{i \in I} f_i \langle f_i, x \rangle \right\|^2 \leq D \left\| \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \right\| \leq D^2 \|x\|^2 \Rightarrow \left\| \sum_{i \in I} f_i \langle f_i, x \rangle \right\| \leq D \|x\|.
\]

Since \( x \in V \) is arbitrarily chosen, the operator

\[
T : V \rightarrow V, \quad x \mapsto \sum_{i \in I} f_i \langle f_i, x \rangle
\]

is well defined, bounded and \( A \)-linear. It is easy to check that \( \langle Tx, y \rangle = \langle x, Ty \rangle \) for all \( x, y \in V \), so \( T \in B(V) \) and \( T = T^* \). From \( \langle Tx, x \rangle = \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \geq 0 \) for all \( x \in V \), it follows that \( T \geq 0 \). Now (2.1) and \( \langle T^{1/2}x, T^{1/2}x \rangle = \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \) imply \( \sqrt{C} \|x\| \leq \|T^{1/2}x\| \leq \sqrt{D} \|x\| \) for all \( x \in V \). By Corollary 2.2 there are constants \( C', D' > 0 \) such that

\[
C'(x, x) \leq \langle T^{1/2}x, T^{1/2}x \rangle = \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D' \langle x, x \rangle, \quad x \in V.
\]

This proves that \( \{ f_i : i \in I \} \) is a standard frame for \( V \). \( \square \)

### 3. On a Class of Frames for Hilbert \( K(H) \)-modules

The existence of standard frames in countably generated Hilbert \( K(H) \)-modules \( V \) follows from the existence of orthonormal bases. If \( T \in B(V) \) is a surjection and \( \{ v_i : i \in I \} \) an orthonormal basis for \( V \), then \( \{ Tv_i : i \in I \} \) is a standard frame for \( V \) which satisfies \( Tv_i = T(v_i e_i) = (Tv_i)e_i \), where \( e_i := \langle v_i, v_i \rangle \) is an orthogonal projection of rank 1 for every \( i \in I \). However, not every standard frame in a Hilbert \( K(H) \)-module is of this type, as we show in the next example.
Example 3.1. Let \( \{v_i : i \in I\} \) be an orthonormal basis for a countably and not finitely generated Hilbert \( K(H) \)-module \( V \) with property \( e_i e_j = 0, i \neq j \), where \( e_i = \langle v_i, v_i \rangle \). (Such a basis can always be constructed by following the procedure described in \( [2, \text{Remark 4(d)}] \).) Let \( I = \bigcup_{j=1}^{\infty} I_j \) be a partition of \( I \) such that \( |I_j| = j \). Let \( f_j = \sum_{i \in I_j} v_i \in V \). Since \( \langle x, v_j \rangle \langle v_i, x \rangle = \langle x, v_j e_j \rangle \langle v_i e_i, x \rangle = \langle x, v_j e_j e_i (v_i, x) = \delta_{ij} \langle x, v_j \rangle \langle v_i, x \rangle \) for all \( x \in V \), we have

\[
\langle x, f_j \rangle \langle f_j, x \rangle = \langle x, \sum_{i \in I_j} v_i \rangle \langle \sum_{i \in I_j} v_i, x \rangle = \sum_{i,j \in I_j} \langle x, v_j \rangle \langle v_i, x \rangle = \sum_{i \in I_j} \langle x, v_i \rangle \langle v_i, x \rangle
\]

and then

\[
\langle x, x \rangle = \sum_{i \in I} \langle x, v_i \rangle \langle v_i, x \rangle = \sum_{j \in J} \langle x, f_j \rangle \langle f_j, x \rangle.
\]

This means that \( \{f_j : j \in J\} \) is a standard Parseval frame for \( V \) such that \( \langle f_j, f_j \rangle = \sum_{i \in I_j} e_i \) is a projection with \( \dim \text{Im} \langle f_i, f_i \rangle = |I_j| = j \) for all \( j \in J \).

Proposition 3.2. Let \( V \) be a countably and not finitely generated Hilbert \( K(H) \)-module. Let \( \{f_i : i \in I\} \) be a standard frame for \( V \) such that \( f_i = f_i e_i \) for some orthogonal projections \( e_i, i \in I \), of rank 1. Then there is an orthonormal basis \( \{v_i : i \in I\} \) and a surjection \( T \in B(V) \) such that \( Tv_i = f_i, i \in I \).

Proof. Let \( C \) and \( D \) be frame bounds. Let \( \{v_i : i \in I\} \) be an orthonormal basis such that \( v_i = v_i e_i, i \in I \). (We may assume that the sets of indices for a standard frame and a basis are the same, since they are both infinite subsets of \( \mathbb{N} \).)

We first show that for every \( x \in V \) the series \( \sum_{i \in I} f_i(v_i, x) \) converges. Let \( J \) be a finite subset of \( I \) and \( x_J = \sum_{i \in J} f_i(v_i, x) \). Then

\[
\|x_J\|^2 = \|\langle x_J, x_J \rangle\|^2 = \|\sum_{i \in J} f_i(v_i, x), x_J\|^2 = \|\sum_{i \in J} \langle x, v_i \rangle \langle f_i, x_J \rangle\|^2
\]

\[
\leq \|\sum_{i \in J} \langle x, v_i \rangle \langle v_i, x \rangle\| \|\sum_{i \in J} \langle x, f_i \rangle \langle f_i, x_J \rangle\| \leq \sum_{i \in J} \|\langle x, v_i \rangle \langle v_i, x \rangle\| \cdot D \|x_J\|^2,
\]

from where we get \( \|x_J\|^2 \leq D \|\sum_{i \in J} (x, v_i) \langle v_i, x \rangle\| \), that is,

\[
\|\sum_{i \in J} f_i(v_i, x)\|^2 \leq D \|\sum_{i \in J} (x, v_i) \langle v_i, x \rangle\|, \text{ for every finite } J \subseteq I.
\]

Since \( \sum_{i \in I} \langle x, v_i \rangle \langle v_i, x \rangle \) converges in norm \( [2, \text{Theorem 1}] \), it follows that the series \( \sum_{i \in I} f_i(v_i, x) \) converges.

Similarly, we check that the series \( \sum_{i \in I} \langle v_i, x \rangle \langle f_i, x \rangle \) converges for every \( x \in V \).

Now we can define the operators \( T, R : V \to V \) by \( Tx = \sum_{i \in I} f_i(v_i, x) \) and \( Rx = \sum_{i \in I} v_i \langle f_i, x \rangle \). It is straightforward to see that \( \langle Tx, y \rangle = \langle x, Ry \rangle \) for all \( x, y \in V \). Therefore \( T \in B(V) \) and \( R = T^* \). From Proposition \( [2, 1] \) and

\[
C \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle = \langle T^* x, T^* x \rangle, \quad x \in V,
\]

it follows that \( T \) is surjective.

It only remains to note that \( Tv_i = f_i(v_i, v_i) = f_i e_i = f_i \) for all \( i \in I \).

A Hilbert \( K(H) \)-module contains a Hilbert space \( V_e \) with respect to the inner product \( \langle x, y \rangle = \text{tr} \langle y, x \rangle \), where ‘tr’ means the trace. More precisely, for a fixed orthogonal projection \( e \in K(H) \) of rank 1, \( V_e \) is given as the set of all \( xe, x \in V \).
Also, for all \( x, y \in V_e \) we obtain that \( \langle x, y \rangle = \langle y, x \rangle e \). \( V_e \) is an invariant subspace for each \( T \) in \( B(V) \) and the map

\[
(3.1) \quad T \mapsto T|V_e, \quad B(V) \to B(V_e)
\]
establishes an isomorphism of \( C^* \)-algebras, where \( B(V_e) \) denotes the \( C^* \)-algebra of all bounded operators on \( V_e \). It is known that a family \( \{ v_i : i \in I \} \subseteq \mathcal{E}_e \) is an orthonormal basis for \( V \) if and only if it is an orthonormal basis for \( V_e \). (The proofs can be found in [2, Remark 4, Theorem 5]). We extend the last statement to a standard frame \( \{ f_i : i \in I \} \) contained in \( \mathcal{E}_e \). First we need a lemma which describes some properties of the isomorphism \( \mathcal{K}_I \).

**Lemma 3.3.** Let \( V \) be a Hilbert \( \mathbb{K}(H) \)-module, \( e \in \mathbb{K}(H) \) an orthogonal projection of rank 1, and \( T \in B(V) \). The following statements hold:

1. \( T \) is bounded below if and only if \( T|V_e \in B(V_e) \) is bounded below.
2. \( T \) is surjective if and only if \( T|V_e \in B(V_e) \) is surjective.

**Proof.** (1) First observe that if \( T|V_e \in B(V_e) \) is a positive operator on the Hilbert space \( V_e \), then \( T \in B(V) \) is a positive element of the \( C^* \)-algebra \( B(V) \). This is a consequence of the fact that the map \( T \mapsto T|V_e \) is an isomorphism of \( C^* \)-algebras.

Suppose \( T_e := T|V_e \) is bounded below. Let \( m > 0 \) be such that \( \|T_e(x_e)\| \geq m\|x_e\| \) for all \( x \in V \). In other words, \( T_e^*T_e - m^2\text{id}_{V_e} \) is a positive operator on the Hilbert space \( V_e \). By the observation from the beginning of the proof, we get \( T^*T - m^2\text{id}_V \geq 0 \), i.e., \( \langle (T^*T - m^2\text{id}_V)x, x \rangle \geq 0 \) for all \( x \in V \). Now we have \( \langle Tx, Tx \rangle \geq m^2\langle x, x \rangle \), and then \( \|Tx\| \geq m\|x\| \) for all \( x \in V \).

The opposite statement is obvious.

(2) It follows from (1) and Proposition 2.1. \( \square \)

**Theorem 3.4.** Let \( V \) be a countably generated Hilbert \( \mathbb{K}(H) \)-module, \( e \in \mathbb{K}(H) \) an orthogonal projection of rank 1, and \( \{ f_i : i \in I \} \) a sequence of vectors in \( \mathcal{E}_e \). Then \( \{ f_i : i \in I \} \) is a standard frame for the Hilbert \( \mathbb{K}(H) \)-module \( V \) with frame bounds \( C \) and \( D \) if and only if \( \{ f_i : i \in I \} \) is a frame for the Hilbert space \( V_e \) with frame bounds \( C \) and \( D \).

**Proof.** Suppose that \( \{ f_i : i \in I \} \) is a standard frame for a Hilbert \( \mathbb{K}(H) \)-module \( V \) with frame bounds \( C \) and \( D \). It means that

\[
C\langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D\langle x, x \rangle, \quad x \in V.
\]

Since \( \langle x, ye \rangle = \langle ye, x \rangle e \) for all \( xe, ye \in \mathcal{E}_e \), by choosing \( xe \) instead of \( x \) in the above inequalities, we get

\[
C(xe, xe)e \leq \sum_{i \in I} \langle xe, f_i \rangle \langle f_i, xe \rangle e \leq D(xe, xe)e, \quad x \in V;
\]

which implies \( C(x, x) \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq D(x, x) \) for all \( x \in \mathcal{E}_e \). It proves that \( \{ f_i : i \in I \} \) is a frame for the Hilbert space \( \mathcal{E}_e \) with frame bounds \( C \) and \( D \).

Now suppose that \( \{ f_i : i \in I \} \subseteq \mathcal{E}_e \) is a frame for the Hilbert space \( \mathcal{E}_e \) with frame bounds \( C \) and \( D \).

First we assume that \( V \) is finitely generated. Let \( \{ v_1, \ldots, v_n \} \subseteq \mathcal{E}_e \) be an orthonormal basis for \( V \) and \( S_e \in B(V_e) \) the frame operator associated to the (Hilbert
space) frame \( \{ f_i : i \in I \} \). Then
\[
x e = \sum_{i \in I} S^2 f_i(xe, S^2 f_i) = \sum_{i \in I} S^2 f_i(S^2 f_i, xe), \quad xe \in V_e.
\]
Since \( x = \sum_{j=1}^n v_i(v_i, x) \) for all \( x \in V \), and since \( v_1, \ldots, v_n \in V_e \), we immediately get that for all \( x \in V \), \( x = \sum_{i \in I} S^2 f_i(S^2 f_i, x) \) holds. This proves that \( \{ S^2 f_i : i \in I \} \) is a Parseval standard frame for \( V \). Let \( S \in B(V) \) be the unique extension of \( S_e \in B(V_e) \). Since \( S_e \) is invertible and positive, \( S \in B(V) \) is also invertible and positive. Therefore \( S^{-\frac{1}{2}} \) preserves standard frames, so the sequence \( \{ S^{-\frac{1}{2}}(S^2 f_i) : i \in I \} = \{ S^ {-\frac{1}{2}} S^2 f_i : i \in I \} = \{ f_i : i \in I \} \) is a standard frame for \( V \).

Now we assume that \( V \) is not finitely generated. Let \( \{ v_i : i \in I \} \) be an orthonormal basis for \( V \) such that \( v_i = v_i e \) for all \( i \in I \). Then \( \{ v_i : i \in I \} \) is an orthonormal basis for the Hilbert space \( V_e \). Let \( T_e : V_e \to V_e \) be the operator defined as \( T_e(xe) = \sum_{i \in I} f_i(v_i, xe) \). As in the proof of Proposition \( \frac{2.2}{2.2} \), we show that \( T_e \) is well defined, \( T_e \in B(V_e) \) and \( T_e \) is surjective. Let \( T \in B(V) \) be the unique extension of \( T_e \) in \( B(V_e) \). By the previous lemma, \( T \) is surjective. Now Theorem \( \frac{2.5}{2.5} \) implies that \( \{ f_i : i \in I \} \) is a standard frame for \( V \), since \( T(v_i) = T_e(v_i) = f_i \) for all \( i \in I \).

The concept of an orthonormal basis has been introduced in Hilbert \( C^* \)-modules over an arbitrary \( C^* \)-algebra. Obviously, there are Hilbert \( C^* \)-modules which do not possess an orthonormal basis. Actually, if a Hilbert \( C^* \)-module \( V \) possesses an orthonormal basis, then \( \langle V, V \rangle \) has to be a \( CCR \)-algebra. We prove this in the next theorem.

**Theorem 3.5.** Let \( A \) be a \( C^* \)-algebra and \( V \) a full countably generated Hilbert \( A \)-module. Let \( \{ e_i : i \in I \} \) be a family of projections in \( A \) such that \( e_i A e_i = C e_i, i \in I \), and \( \{ f_i : i \in I \} \) a standard frame for \( V \) such that \( f_i = f_i e_i, i \in I \). Then \( A \) is a \( CCR \)-algebra. In particular, if \( V \) possesses an orthonormal basis, then \( A \) is a \( CCR \)-algebra.

**Proof.** By the definition of a \( CCR \)-algebra we need to show that for every irreducible representation \( \varphi : A \to B(H) \), \( \varphi(A) \subseteq K(H) \) holds.

Let \( 0 \neq \varphi : A \to B(H) \) be an irreducible representation of \( A \). Then \( e_i A e_i = C e_i \) implies \( \varphi(e_i) \varphi(A) \varphi(e_i) = C \varphi(e_i) \) for every \( i \in I \).

Let \( i \in I \) be such that \( \varphi(e_i) \neq 0 \). Now \( \varphi(e_i) \) is a non-zero projection, so there is a non-zero vector \( \xi_0 \in H \) which belongs to \( \text{Im} \varphi(e_i) \). Then \( \varphi(e_i) \xi_0 = \xi_0 \) and
\[
\varphi(e_i) \varphi(A) \varphi(e_i) \xi_0 = C \varphi(e_i) \xi_0 \Rightarrow \varphi(e_i) \varphi(A) \xi_0 = C \xi_0.
\]
\( \varphi \) is irreducible, therefore it is a cyclic representation of \( A \), and every non-zero vector is cyclic for \( \varphi \). In particular, \( \xi_0 \) is a cyclic vector for \( \varphi \). Therefore
\[
\{ 0 \} \neq \varphi(e_i) H = \varphi(e_i) \varphi(A) \xi_0 \subseteq \varphi(e_i) \varphi(A) \xi_0 = C \xi_0 \Rightarrow \text{Im} \varphi(e_i) = C \xi_0.
\]
This proves that \( \varphi(e_i) \in K(H) \) for every \( i \in I \).

Let \( S \in B(V) \) be the frame operator associated to \( \{ f_i : i \in I \} \). From the reconstruction formula \( \frac{1.2}{1.2} \), we have \( \langle x, y \rangle = \sum_{i \in I} \langle x, S f_i \rangle \langle f_i, y \rangle \) and then
\[
(3.2) \quad \varphi(\langle x, y \rangle) = \sum_{i \in I} \varphi(\langle x, S f_i \rangle) \varphi(\langle f_i, y \rangle), \quad x, y \in V.
\]
From $f_i = f e_i$ and compactness of $\varphi(e_i)$ it follows that
$$\varphi(\langle x, S f_i \rangle \varphi(\langle f_i, y \rangle) = \varphi(\langle x, S f_i \rangle) \varphi(e_i) \varphi(\langle f_i, y \rangle) \in K(H), \quad x, y \in V, i \in I.$$ Finally, we get $\varphi(\langle x, y \rangle) \in K(H)$ for all $x, y \in V$, as the convergence in (3.2) is in norm. Since $V$ is full, we conclude that $\varphi(A) \subseteq K(H)$.

This finishes our proof.

The converse of the previous theorem does not hold. For example, we can take an arbitrary Hilbert $C^*$-module over the $C^*$-algebra $A = C([0, 1])$ of all continuous complex functions on the unit segment $[0, 1]$, $A$ is a CCR-algebra, since it is commutative, and the only projection $e \in A$ which satisfies $e^2 = Ce$ is the constant function 0.

**Remark 3.6.** Frames of subspaces for a separable Hilbert space have been recently introduced and studied in [4]. We can generalize their definition for Hilbert $K(H)$-modules in the following way.

Let $V$ be a countably generated Hilbert $K(H)$-module, $\{W_i : i \in I\}$ ($I \subseteq \mathbb{N}$) a family of closed submodules of $V$, and $\{\lambda_i : i \in I\}$ a family of weights, i.e., a family of positive numbers. We say that $\{W_i : i \in I\}$ is a standard frame of submodules for $V$ with respect to a family of weights $\{\lambda_i : i \in I\}$, if there are constants $C, D > 0$ such that

$$C \langle x, x \rangle \leq \sum_{i \in I} \lambda_i^2 \langle \pi_i(x), \pi_i(x) \rangle \leq D \langle x, x \rangle, \quad x \in V,$$

where $\pi_i \in B(V)$ denotes the orthogonal projection on $W_i$ for every $i \in I$, and convergence of the sum in the middle of (3.3) is in norm.

Let us fix an orthogonal projection $e \in K(H)$ of rank 1. It can be proved that a family of closed submodules $\{W_i : i \in I\}$ is a standard frame of submodules for $V$ with respect to the family of weights $\{\lambda_i : i \in I\}$ if and only if $\{(W_i)_e : i \in I\}$ is a frame of subspaces for $V_e$ with respect to the family of weights $\{\lambda_i : i \in I\}$. Therefore many statements from [4] can be extended to countably generated Hilbert $K(H)$-modules. This will be done in our subsequent paper.

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