ZERO DISTRIBUTION OF MÜNTZ EXTREMAL POLYNOMIALS IN $L_p[0,1]$

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Abstract. Let $\{\lambda_j\}_{j=0}^{\infty}$ be a sequence of distinct positive numbers. Let $1 \leq p \leq \infty$ and let $T_{n,p} = T_{n,p}\{\lambda_0,\lambda_1,\lambda_2,\ldots,\lambda_n\}(x)$ denote the $L_p$ extremal Müntz polynomial in $[0,1]$ with exponents $\lambda_0,\lambda_1,\lambda_2,\ldots,\lambda_n$. We investigate the zero distribution of $\{T_{n,p}\}_{n=1}^{\infty}$. In particular, we show that if
$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \alpha > 0,$$
then the normalized zero counting measure of $T_{n,p}$ converges weakly as $n \to \infty$ to
$$\frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^\alpha (1-t^\alpha)}} dt,$$
while if $\alpha = 0$ or $\infty$, the limiting measure is a Dirac delta at 0 or 1, respectively.

1. Introduction and Results

Let $\lambda_1, \lambda_2, \ldots$ be a sequence of distinct positive numbers. An expression of the form

$$(1.1) \quad \sum_{j=0}^{n} c_j x^{\lambda_j}$$

is called a Müntz polynomial. The name refers, of course, to the famous theorem of Müntz that if $\inf_j \lambda_j > 0$, these polynomials are dense in $L_p$ spaces iff
$$\sum_{j=0}^{\infty} \frac{1}{\lambda_j} = \infty.$$  

Müntz polynomials share many of the properties of ordinary algebraic polynomials. The most fundamental is that a polynomial of the form (1.1) has at most $n$ distinct zeros in $(0, \infty)$, or is identically zero.

Müntz extremal polynomials are generalizations of classical orthogonal and Chebyshev polynomials. They have been investigated by, amongst others, Borwein and Erdelyi [2] and Milovanovic and his coworkers [3]. Let $1 \leq p \leq \infty$. We denote by
\[ T_{n,p}(x) = T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}(x) \text{ the linear combination of } \{x^{\lambda_j}\}_{j=0}^n \text{ with the coefficient of } x^{\lambda_n} \text{ equal to 1, satisfying} \]

\[ \|T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}\|_{L^p[0,1]} = \min_{c_0,\ldots,c_{n-1}} \|x^{\lambda_n} - \sum_{j=0}^{n-1} c_j x^{\lambda_j}\|_{L^p[0,1]} . \]

It is known that \( T_{n,p} \) exists and is unique, has exactly \( n \) distinct (and simple) zeros in \((0,1)\), and the zeros of \( T_{n,p} \) and \( T_{n+1,p} \) interlace. Moreover, if we swap \( \lambda_n \) with some \( \lambda_j \), the extremal polynomial changes only by a non-zero multiplicative constant. Thus when dealing with a fixed \( n \), and studying zeros of extremal polynomials, we may assume that \( \{\lambda_j\}_{j=0}^n \) are in increasing order. However, we shall not need to assume that \( \{\lambda_j\}_{j=0}^\infty \) is increasing. Concerning the zeros as \( n \to \infty \), an important result of Borwein [2, Thm. 4.1.1, p. 155] asserts that the corresponding Müntz polynomials are dense iff the maximum spacing between successive zeros of \( T_{n,p} \) has limit 0 as \( n \to \infty \). Saff and Varga [6] studied the related zero distribution of lacunary incomplete polynomials.

In this paper, we study the asymptotic zero distribution of \( \{T_{n,p}\}_{n=1}^\infty \). Let \( \nu_n \) denote the normalized zero counting measure of \( T_{n,p} \), so that

\[ \nu_n([a,b]) = \frac{1}{n} \times \text{number of zeros of } T_{n,p} \text{ in } [a,b]. \]

In the case of polynomials, where \( \lambda_j = j \), \( j \geq 0 \), it is a classical result [5, pp. 169–170], [7, Thm. 3.4.1, p. 84 and Thm. 3.6.1, p. 98] that for \( 0 \leq a < b \leq 1 \),

\[ \lim_{n \to \infty} \nu_n([a,b]) = \int_a^b \frac{dx}{\pi \sqrt{x(1-x)}}. \]

Equivalently we write

\[ d\nu_n \overset{s}{\to} \frac{dx}{\pi \sqrt{x(1-x)}}, \quad n \to \infty, \]

and say that \( d\nu_n \) converges weakly to the arcsine distribution on \([0,1]\). This type of result has been studied in detail for the case \( p = 2 \) of orthogonal polynomials, and when there is a weight \( w \) in the norm in (1.2). The monograph of Stahl and Totik [7] gives a comprehensive account, while the monograph of Andrievskii and Blatt [1] considers discrepancy, or rate of convergence, to the limiting distribution.

In a loose sense, our conclusion is that when \( \lim_{n \to \infty} \lambda_n/n \) exists, all the possible zero distributions are those provided by

\[ \lambda_j = \alpha j, \quad j \geq 0, \]

for some \( \alpha \in [0,\infty] \). Extremal polynomials for these exponents are essentially \( L_p \) extremal polynomials with the substitution of variable \( x = t^\alpha \). Accordingly, we define for \( 0 < \alpha < \infty \), a probability measure on \((0,1)\),

\[ d\mu_\alpha(t) = \frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}} dt. \]

For \( \alpha = 0 \), we set

\[ d\mu_0 = d\delta_0, \]
a unit mass at 0, and for $\alpha = \infty$, we set
\begin{equation}
(1.5) \quad d\mu_{\infty} = d\delta_{1},
\end{equation}
a unit mass at 1. We prove:

**Theorem 1.1.** Let $1 \leq p \leq \infty$, $0 \leq \alpha \leq \infty$, and let $\{\lambda_j\}_{j=0}^{\infty}$ denote a sequence of distinct positive numbers with
\begin{equation}
(1.6) \quad \lim_{j \to \infty} \frac{\lambda_j}{j} = \alpha.
\end{equation}
Then if $0 \leq a \leq b \leq 1$,
\begin{equation}
(1.7) \quad \lim_{n \to \infty} \nu_n([a, b]) = \mu_{\alpha}([a, b]),
\end{equation}
that is,
\begin{equation}
(1.8) \quad d\nu_n \star\to d\mu_{\alpha}, \quad n \to \infty.
\end{equation}

Remarks. (a) An interesting feature of the theorem is that asymptotic zero distribution has no relation to the density of Müntz polynomials—in stark contrast to the Borwein-Erdelyi result on spacing. Thus if $\lambda_n = n \log n, n \geq 2$, then the corresponding Müntz polynomials are dense, while the asymptotic zero distribution is a Dirac delta at 1. If $\lambda_n = n^2, n \geq 0$, then the limiting zero distribution is still a Dirac delta at 1, but the corresponding Müntz polynomials are not dense.

(b) We can somewhat weaken the hypothesis (1.6): roughly speaking we can ignore $o(n)$ of the exponents in $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$. To make this more precise, assume $\alpha < \infty$. We write
\begin{equation}
(1.9) \quad \lim_{j \to \infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha
\end{equation}
if for each $\varepsilon \in (0, 1)$, there exists for large enough $n$, a set
\begin{equation}
(1.10) \quad S_{n, \varepsilon} \subset \{0, 1, 2, \ldots, n\}
\end{equation}
with at most $\varepsilon n$ elements such that
\begin{equation}
(1.11) \quad j \in \{0, 1, 2, \ldots, n\} \setminus S_{n, \varepsilon} \Rightarrow \left| \frac{\lambda_j}{j} - \alpha \right| < \varepsilon.
\end{equation}
In the case $\alpha = \infty$, we replace this with the fact that for each $K > 0$, there exists for large enough $n$, a set $S_{n, \varepsilon} \subset \{0, 1, 2, \ldots, n\}$ with at most $\varepsilon n$ elements such that
\begin{equation}
(1.12) \quad j \in \{0, 1, 2, \ldots, n\} \setminus S_{n, \varepsilon} \Rightarrow \frac{\lambda_j}{j} > K.
\end{equation}

**Theorem 1.2.** Let $1 \leq p \leq \infty$, $0 \leq \alpha \leq \infty$, and $\{\lambda_j\}_{j=0}^{\infty}$ denote a sequence of distinct positive numbers with
\begin{equation}
(1.13) \quad \lim_{j \to \infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha.
\end{equation}
Then the conclusion (1.7) of Theorem 1.1 persists.

We shall also show that one cannot ignore more than $o(n)$ exponents in $\{\lambda_j\}_{j=0}^{n}$ without affecting the zero distribution.
Theorem 1.3. Let $1 \leq p \leq \infty$ and $\varepsilon \in (0, 1)$. Let $\{\lambda_j\}_{j=0}^{\infty}$, $\{\gamma_j\}_{j=0}^{\infty}$, $\{\rho_j\}_{j=0}^{\infty}$ denote sequences of distinct positive numbers with

\begin{equation}
\lim_{j \to \infty} \frac{\gamma_j}{j} = 0, \quad \lim_{j \to \infty} \frac{\rho_j}{j} = \infty.
\end{equation}

Assume also that for large enough $n$, there is the disjoint union

\begin{equation}
\{\lambda_j\}_{j=0}^{n} := \{\gamma_j\}_{j=0}^{k(n)} \cup \{\rho_j\}_{j=0}^{\ell(n)},
\end{equation}

where

\[ \lim_{n \to \infty} \frac{k(n)}{n} = \varepsilon. \]

Then

\begin{equation}
d\nu_n \to^* \varepsilon \, d\mu_0 + (1 - \varepsilon) \, d\mu_\infty, \quad n \to \infty.
\end{equation}

We are not sure if this result generalizes to the case where $0$ and $\infty$ are replaced in (1.12) by other limits. What is clear is that for a general choice of $\{\lambda_j\}_{j=0}^{\infty}$, the asymptotic zero distribution can be quite complicated, and there need not be a weak limit. For example, by adjoining sufficiently large blocks of exponents $\{a_j\}_{j=n}^{\infty}$, one may construct $\{\lambda_n\}_{n=0}^{\infty}$ such that each $\mu_n, n \in [0, \infty]$, is a weak limit of some subsequence of $\{\nu_n\}$. We prove the results in the next section.

2. Proofs

We begin with some notation. We abbreviate $T_{n,p} \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ as $T_{n,p} \{\lambda_0 \cdots \lambda_n\}$ and $Z_p (\lambda_0 \cdots \lambda_n) [a, b]$ denote the total number of zeros of $T_{n,p} \{\lambda_0 \cdots \lambda_n\} (x)$ in $[a, b]$. We say that $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a refinement of $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ if

\[ \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}. \]

The main tools of the proof are interlacing properties of successive Chebyshev polynomials, monotonicity properties with respect to the exponents, and zero distribution for the specific choice $\{\alpha_j\}_{j=0}^{\infty}$.

Lemma 2.1. Let $\{\gamma_j\}_{j=0}^{m}$ be distinct positive numbers and let $\{\lambda_j\}_{j=0}^{n}$ be distinct positive numbers.

(a) Suppose that $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a refinement of $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then for $[a, b] \subset [0, 1]$,

\begin{equation}
|Z_p (\lambda_0 \cdots \lambda_n) [a, b] - Z_p (\gamma_0 \cdots \gamma_m) [a, b]| \leq 2(m - n).
\end{equation}

(b) Suppose that $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k\} \subset \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ have $\ell$ exponents in common. Then for $[a, b] \subset [0, 1]$,

\begin{equation}
|Z_p (\lambda_0 \cdots \lambda_n) [a, b] - Z_p (\gamma_0 \cdots \gamma_k) [a, b]| \leq 2(n + k + 2 - 2\ell).
\end{equation}

Proof. (a) We may rewrite $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}$ as $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m\}$. Since any subset of $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_m}\}$ is a Chebyshev system on $[\varepsilon, 1]$ for any $0 < \varepsilon < 1$, the zeros of $T_{n,p} \{\lambda_0 \cdots \lambda_j\} (x)$ and $T_{n,p} \{\lambda_0 \cdots \lambda_j+1\} (x)$ interlace [4] Corollary 1.1, p. 2. It then follows that for every interval $[a, b]$,

\[ |Z_p (\lambda_0 \cdots \lambda_j) [a, b] - Z_p (\lambda_0 \cdots \lambda_j+1) [a, b]| \leq 2. \]

Applying this for $j = n, n + 1, \ldots, m$ gives (2.1).

(b) We may find a refinement of both $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ consisting of $n + k + 2 - \ell$ elements. Applying (a) to the refinement and each of
the sets \( \{ \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k \} \) and \( \{ \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n \} \), and then combining the two inequalities gives the result. \( \square \)

Apart from interlacing, we shall also use the lexicographic property.

**Lemma 2.2.** Let \( \{ \lambda_j \}_{j=0}^n \) be a sequence of distinct positive numbers and let \( \{ \gamma_j \}_{j=0}^n \) be a sequence of distinct positive numbers with

\[
(2.3) \quad \lambda_j \leq \gamma_j, \quad 0 \leq j \leq n.
\]

Then for \( 0 \leq a \leq 1 \),

\[
(2.4) \quad Z_p (\lambda_0 \cdots \lambda_n) [a, 1] \leq Z_p (\gamma_0 \cdots \gamma_n) [a, 1].
\]

**Proof.** We may assume that the two sets have \( n \) exponents in common. For then, one can apply the result for this special case \( n \) times, using monotonicity each time. Let \( 0 < \varepsilon < 1 \). Then in \( [\varepsilon, 1] \), the combined set of powers \( \{ x^{\lambda_j} \}_{j=0}^n \cup \{ x^{\gamma_j} \}_{j=0}^n \) (with duplicates deleted, and exponents placed in increasing order) is a Descartes system.

If \( T_{n,p} \{ \lambda_0 \cdots \lambda_n \} (x) \) and \( T_{n,p} \{ \gamma_0 \cdots \gamma_n \} (x) \) denote the corresponding Müntz extremal polynomials on \( [\varepsilon, 1] \), it is known that the zeros of \( T_{n,p} \{ \lambda_0 \cdots \lambda_n \} (x) \) lie to the left of those of \( T_{n,p} \{ \gamma_0 \cdots \gamma_n \} (x) \), in the sense that the \( j \)-th smallest zero of the former Müntz polynomial is \( \leq \) the \( j \)-th smallest zero of the latter Müntz polynomial. For \( p = \infty \), a proof of this is given in the book of Borwein and Erdélyi \[2\] Thm. 3.3.4, pp. 116–117. For \( 1 < p \leq \infty \), a proof is given in Pinkus and Ziegler \[4\] Thm. 5.1, p. 13, while when \( p = 1 \), we can apply the remarks there (or a continuity argument involving \( p \to 1^+ \)). As \( \varepsilon \to 0^+ \), \( T_{n,p} \{ \gamma_0 \cdots \gamma_n \} (x) \) must converge uniformly to \( T_{n,p} \{ \gamma_0 \cdots \gamma_n \} (x) \) because of uniqueness of \( T_{n,p} \{ \gamma_0 \cdots \gamma_n \} (x) \), and the fact that the extremal error increases as \( [\varepsilon, 1] \) grows to \( [0, 1] \). Hence the zeros of \( T_{n,p} \{ \lambda_0 \cdots \lambda_n \} (x) \) lie to the left of those of \( T_{n,p} \{ \gamma_0 \cdots \gamma_n \} (x) \), and \( (2.4) \) follows. \( \square \)

The next result asserts essentially that if for “most” indices \( j \), we have \( \lambda_j \leq \gamma_j \), then the asymptotic proportion of zeros in \([a, 1]\) of extremal polynomials with exponents \( \{ \lambda_j \} \) does not exceed that for \( \{ \gamma_j \} \).

**Lemma 2.3.** Let \( \{ \lambda_j \}_{j=0}^\infty \) and \( \{ \gamma_j \}_{j=0}^\infty \) be sequences of distinct positive numbers with the following property: for each \( \varepsilon > 0 \), there exists for large enough \( n \), a set

\[
(2.5) \quad S_{n,\varepsilon} \subset \{ 0, 1, 2, \ldots, n \}
\]

with at most \( \varepsilon n \) elements such that

\[
(2.6) \quad j \in \{ 0, 1, 2, \ldots, n \} \setminus S_{n,\varepsilon} \Rightarrow \lambda_j \leq \gamma_j.
\]

Then for \( 0 \leq a \leq 1 \),

\[
(2.7) \quad \limsup_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [a, 1] \leq \limsup_{n \to \infty} \frac{1}{n} Z_p (\gamma_0 \cdots \gamma_n) [a, 1]
\]

and

\[
(2.8) \quad \liminf_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [a, 1] \leq \liminf_{n \to \infty} \frac{1}{n} Z_p (\gamma_0 \cdots \gamma_n) [a, 1].
\]
Proof. Let us fix $\varepsilon > 0$, $n$ large, and let $S_{n,\varepsilon}$ be as in the statement. We define for the given $n$, a modified set of exponents $\{\lambda^*_j\}_{j=0}^n$ by

$$
\lambda_j^* = \begin{cases} 
\lambda_j, & j \in \{0, 1, 2, \ldots, n\} \setminus S_{n,\varepsilon}, \\
\gamma_j, & j \in S_{n,\varepsilon}.
\end{cases}
$$

Then

$$
\lambda_j^* \leq \gamma_j, \quad 0 \leq j \leq n.
$$

By the previous lemma, for $0 \leq a \leq 1$,

$$
Z_p (\lambda^*_0 \cdots \lambda^*_n) [a, 1] \leq Z_p (\gamma_0 \cdots \gamma_n) [a, 1].
$$

Also $\{\lambda^*_j\}_{j=0}^n$ and $\{\lambda_j\}_{j=0}^n$ have at least $1 + n (1 - \varepsilon)$ elements in common, so by Lemma 2.1(b),

$$
|Z_p (\lambda^*_0 \cdots \lambda^*_n) [a, 1] - Z_p (\lambda_0 \cdots \lambda_n) [a, 1]| \leq 4\varepsilon n + 4.
$$

Combining these inequalities gives

$$
Z_p (\lambda_0 \cdots \lambda_n) [a, 1] \leq Z_p (\gamma_0 \cdots \gamma_n) [a, 1] + 4\varepsilon n + 4.
$$

Dividing by $n$ and letting $n \to \infty$ gives

$$
\limsup_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [a, 1] \leq \limsup_{n \to \infty} \frac{1}{n} Z_p (\gamma_0 \cdots \gamma_n) [a, 1] + 4\varepsilon.
$$

As $\varepsilon > 0$ is arbitrary, (2.7) follows. Similarly, (2.8) follows.

Next, we study the zero distribution for the comparison sequence $\{\alpha_j\}_{j=0}^\infty$.

Lemma 2.4. Let $\alpha \in (0, \infty)$ and

$$
\gamma_j = \alpha j, \quad j \geq 0.
$$

Then for $0 \leq a < b \leq 1$,

$$
\lim_{n \to \infty} \frac{1}{n} Z_p (\gamma_0 \cdots \gamma_n) [a, b] = \mu_\alpha ([a, b]).
$$

Proof. Suppose first that $p < \infty$. Let $T_{n,p}^*$ denote the monic (ordinary) polynomial of degree $n$ satisfying

$$
\int_0^1 |T_{n,p}^* (x)|^p \frac{1}{\alpha} x^{1/\alpha - 1} dx = \min_{\deg(P) \leq n-1} \int_0^1 |x^n - P (x)|^p \frac{1}{\alpha} x^{1/\alpha - 1} dx.
$$

The substitution $x = t^\alpha$ gives

$$
\int_0^1 |T_{n,p}^* (t^\alpha)|^p dt = \min_{\deg(P) \leq n-1} \int_0^1 |t^{\alpha n} - P (t^\alpha)|^p dt.
$$

It follows from uniqueness that

$$
T_{n,p}^* (t^\alpha) = T_{n,p} (\gamma_0 \cdots \gamma_n) (t).
$$

We then see that the total multiplicity of zeros of $T_{n,p} (\gamma_0 \cdots \gamma_n)$ in $[a, b]$ is the total multiplicity of zeros of $T_{n,p}^*$ in $[a^\alpha, b^\alpha]$. Since the weight $\frac{1}{\alpha} x^{1/\alpha - 1}$ is positive
a.e. in \([0,1]\), classical results assert that the limiting zero distribution of \(\{T_{n,p}^*\}_{n=0}^\infty\) is the arcsine distribution \([1, \text{Cor. 5.7, p. 261}]\). Hence as \(n \to \infty\),

\[
\lim_{n \to \infty} \frac{1}{n} \times \text{number of zeros of } T_{n,p}^* \text{ in } [a,b] = \frac{1}{\pi} \int_a^b \frac{dx}{\pi x (1-x)} = \frac{\alpha}{\pi} \int_a^b \frac{t^{\alpha-1}}{\sqrt{t^\alpha (1-t^\alpha)}} dt = \int_a^b d\mu_\alpha(t).
\]

\(\square\)

**Proof of Theorem 1.2** Our hypothesis is

\[\lim_{j \to \infty} \frac{\lambda_j}{j} = \alpha.\]

Assume first that \(0 < \alpha < \infty\). Let \(\varepsilon \in (0,\alpha)\). We then obtain for large enough \(n\), from (1.10),

\[j \in \{0, 1, 2, \ldots, n\} \setminus S_{n,\varepsilon} \Rightarrow (\alpha - \varepsilon) j \leq \lambda_j \leq (\alpha + \varepsilon) j.\]

Applying Lemma 2.3 with \(\gamma_j = (\alpha + \varepsilon) j\), \(j \geq 0\), we deduce that

\[
\limsup_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] \\
\leq \limsup_{n \to \infty} \frac{1}{n} Z_p(0, (\alpha + \varepsilon), 2(\alpha + \varepsilon), \ldots, n(\alpha + \varepsilon))[a,1],
\]

and similarly applying Lemma 2.3 to \((\alpha - \varepsilon) j\), \(j \geq 0\), and \(\lambda_j\), \(j \geq 0\) (with roles swapped),

\[
\liminf_{n \to \infty} \frac{1}{n} Z_p(0, (\alpha - \varepsilon), 2(\alpha - \varepsilon), \ldots, n(\alpha - \varepsilon))[a,1] \\
\leq \liminf_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1].
\]

Applying Lemma 2.4 with \(\gamma_j = (\alpha \pm \varepsilon) j\), \(j \geq 0\), gives

\[
\int_a^1 d\mu_{\alpha \pm \varepsilon}(t) \leq \liminf_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] \\
\leq \limsup_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] \leq \int_a^1 d\mu_{\alpha \pm \varepsilon}(t).
\]

Letting \(\varepsilon \to 0+\), and using dominated convergence gives

\[
\lim_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] = \int_a^1 d\mu_\alpha(t).
\]

This gives the result when \([a,b] = [a,1]\). For general \([a,b]\), we use

\[
\lim_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,b] \\
= \lim_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] - \lim_{n \to \infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)(b,1) \\
= \int_a^1 d\mu_\alpha(t) - \int_b^1 d\mu_\alpha(t).
\]
Note that because \( \mu_\alpha \) is absolutely continuous, the number of zeros in a neighborhood of the point \( b \) is negligible in the sense of asymptotic distribution. Finally, if \( \alpha = 0 \), the arguments above give for \( 0 < a \leq 1 \),

\[
\limsup_{n \to \infty} \frac{1}{n} Z_p (0, \varepsilon, 2\varepsilon, \ldots, n\varepsilon) [a, 1] = \int_a^1 d\mu_\varepsilon (t).
\]

Letting \( \varepsilon \to 0^+ \) (and using some straightforward estimates) gives

\[
\lim_{n \to \infty} \frac{1}{n} Z_p (0, a, 1) = 0 = \int_0^1 d\mu_0 (t).
\]

Since \( \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [0, 1] = 1 \), we obtain

\[
\lim_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [0, 1] = 1 = \int_0^1 d\mu_0 (t).
\]

The case \( \alpha = \infty \) is similar. \( \square \)

**Proof of Theorem 1.1.** This is a special case of Theorem 1.2. \( \square \)

**Proof of Theorem 1.3.** Let \( 0 < a < b < 1 \). Because of (1.13) and interlacing properties, to the left of each zero of \( T_{n,p} \{ \gamma_0 \cdots \gamma_k(n) \} (x) \) in \([0, a]\), there is a zero of \( T_{n,p} \{ \lambda_0 \cdots \lambda_n \} (x) \). Moreover,

\[
Z_p (\lambda_0 \cdots \lambda_n) [0, a] \geq Z_p (\gamma_0 \cdots \gamma_k(n)) [0, a],
\]

so applying Theorem 1.1 to \( \{ \gamma_j \}_{j=0}^\infty \),

\[
\liminf_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [0, a] \geq \liminf_{n \to \infty} \frac{k(n)}{n} \frac{1}{k(n)} Z_p (\gamma_0 \cdots \gamma_k(n)) [0, a]
\]

(2.12)

\[
= \varepsilon \int_0^a d\mu_0 = \varepsilon.
\]

Similarly,

\[
\liminf_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [b, 1] \geq \liminf_{n \to \infty} \frac{\ell(n)}{n} \frac{1}{\ell(n)} Z_p (\rho_0 \cdots \rho_{\ell(n)}) [b, 1]
\]

(2.13)

\[
= (1 - \varepsilon) \int_b^1 d\mu_\infty = 1 - \varepsilon.
\]

Then it follows that

\[
\limsup_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) (a, b)
\]

\[
\leq 1 - \liminf_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [0, a] - \liminf_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) [b, 1] \leq 0.
\]

So for \( 0 < a < b < 1 \),

\[
\lim_{n \to \infty} \frac{1}{n} Z_p (\lambda_0 \cdots \lambda_n) (a, b) = 0.
\]
Next, by (2.12) and (2.13),
\[ \varepsilon \leq \liminf_{n \to \infty} \frac{1}{n} \mathcal{Z}_p(\lambda_0 \cdots \lambda_n) [0,a] \]
\[ \leq \limsup_{n \to \infty} \frac{1}{n} \mathcal{Z}_p(\lambda_0 \cdots \lambda_n) [0,a] \]
\[ \leq 1 - \liminf_{n \to \infty} \frac{1}{n} \mathcal{Z}_p(\lambda_0 \cdots \lambda_n)(a,1) \leq \varepsilon, \]
so
\[ \lim_{n \to \infty} \frac{1}{n} \mathcal{Z}_p(\lambda_0 \cdots \lambda_n) [0,a] = \varepsilon. \]
Similarly,
\[ \lim_{n \to \infty} \frac{1}{n} \mathcal{Z}_p(\lambda_0 \cdots \lambda_n) [b,1] = 1 - \varepsilon. \]
It follows that as \( n \to \infty \),
\[ d\nu_n \overset{*}{\to} \varepsilon \delta_0 + (1 - \varepsilon) \delta_1. \]

□

References


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