STRONG UNIQUE CONTINUATION FOR $m$-TH POWERS OF A LAPLACIAN OPERATOR WITH SINGULAR COEFFICIENTS

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ABSTRACT. In this paper we prove strong unique continuation for $u$ satisfying an inequality of the form $|\Delta^m u| \leq f(x, u, Du, \cdots, D^k u)$, where $k$ is up to $[3m/2]$. This result gives an improvement of a work by Colombini and Grammatico (1999) in some sense. The proof of the main theorem is based on Carleman estimates with three-parameter weights $|x|^{2\sigma_1} (\log |x|)^{2\sigma_2} \exp(\frac{\beta}{2} (\log |x|)^2)$.

1. Introduction

Let $\Omega$ be a connected open subset of $\mathbb{R}^n (n \geq 2)$ containing 0. In [6], Colombini and Grammatico prove some results of strong unique continuation for $u$ satisfying an inequality of the form

$$|\Delta^m u| \leq f(x, u, Du, \cdots, D^k u)$$

for $k = m$.

In this paper we are interested in $u$ satisfying the following forms:

$$|\Delta^m u| \leq C_0 \sum_{|\alpha| \leq m-1} |x|^{-2m+|\alpha|}|D^\alpha u| + C_0 \sum_{|\alpha| = m}^{[3m/2]} |x|^{-2m+|\alpha|+\epsilon}|D^\alpha u|,$$

where $0 < \epsilon < 1/2$ and the orders of lower order terms are up to $[3m/2]$.

Theorem 1.1. Let $u \in H^{2m}_{\text{loc}}(\Omega)$ be a solution of (1.2), and for all $N > 0$ let

$$\int_{|x| \leq R} |u|^2 dx = O(R^N), \quad R \to 0.$$ 

Then $u$ is identically zero in $\Omega$.

This problem for $m = 1$ has drawn a lot of attention in partial differential equations and mathematical physics. For the development of this problem, the author refers the readers to [15]. In particular, their work was focused on second order equations in which the coefficients of the lower order terms are allowed to be singular (see [2], [9], [11] and [15]). Here we mention two articles which are closely related to our work in this paper.

Hörmander [8] proved that if $u \in H^1_{\text{loc}}(\Omega)$ satisfies

$$|\Delta u| \leq C_1 |x|^{-2+\epsilon} |u| + C_2 |x|^{-1+\epsilon} |\nabla u|, \quad \epsilon > 0,$$

and $u$ vanishes of infinite order at 0, then $u$ is identically zero in $\Omega$.

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Regbaoui [13] extended Hörmander’s result to the sharp case \( \epsilon = 0 \) with small \( C_2 \). His proof was also based on suitable Carleman estimates, but he worked with the strictly convex weights \( \varphi_\beta(x) = \exp\left(\frac{1+3}{2}(\log |x|)^2\right) \) rather than the usual polynomial weights. As for negative results, some counterexamples were given when \( C_2 \) is not small (such as [1] and [16]).

For the case \( m = 2 \), Borgne [4] got some results about strong unique continuation when \( k = 3 \) in \([13]\). On the other hand, Watanabe [14] established some uniqueness results for \( m = 3 \) with \( k = 5 \). For \( m = 4 \), Okaji [10] solved unique continuation when \( k = 7 \). For the higher orders, Protter [12] got the unique continuation for inequalities of the form \([1.1]\) in which \( k \) is up to \([3m/2]\). According to the counterexamples for \( n = 2 \) in [7], Colombini and Grammatico [6] proved the sharp results about the coefficients of the \( m \)-th lower order terms for strong unique continuation with some cases when \( k = m \). In this paper, we consider the cases \([1.2]\).

We will follow Regbaoui’s approach in [13] and organize this paper as follows. In Section 2, we shall study the asymptotic behavior of the function \( u \) near 0 which guarantees the use of the singular weights \( |x|^{2\tau_1}(\log |x|)^{2\tau_2}\varphi_\beta^2 \). Some of the key Carleman estimates with the weights \( |x|^{2\tau_1}(\log |x|)^{2\tau_2}\varphi_\beta^2 \) will be derived in Section 3. Using these Carleman estimates, we prove Theorem 1.1 in Section 4.

## 2. Carleman estimates with polynomial weights

### Lemma 2.1
There exists a positive constant \( C_m \) such that for any \( u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) and for any \( \tau \in \{k + 1/2, k \in \mathbb{N}\} \), we have the estimate

\[
(2.1) \quad \sum_{|\alpha| \leq 2m} \int \tau^{2m-2|\alpha|} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \leq C_m \int |x|^{-2\tau+4m-n} |\Delta^m u|^2 dx.
\]

**Proof.** In [13], Regbaoui proved in Lemma 2.1 for any \( u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) and for any \( \tau \in \{k + 1/2, k \in \mathbb{N}\} \) that

\[
(2.2) \quad \sum_{|\alpha| \leq 2} \int \tau^{2-2|\alpha|} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 dx \leq C \int |x|^{-2\tau+4-n} |\Delta u|^2 dx.
\]

By the repeated use of \( (2.2) \), we can get \( (2.1) \).

**Remark 2.2.** The estimate \( (2.1) \) in Lemma 2.1 remains valid if we assume \( u \in H^{2m}_0(\Omega) \) with compact support and satisfying for all \( |\alpha| \leq 2m \) and all \( N > 0 \), \( \int_{|x| \leq R} |D^\alpha u|^2 dx = O(R^N) \) as \( R \to 0 \). This can be easily obtained by cutting \( u \) off for small \( |x| \) and regularizing.

Motivated by [8] and [13], we get the following theorem. The proof of this theorem is based on Lemma 2.1. From now on, \( c \) stands for a generic constant and its value may vary from line to line.

### Theorem 2.3
Let \( u \in H^{2m}_0(\Omega) \) be a solution to \( (1.2) \). If \( |\alpha| \leq 2m \), then

\[
(2.3) \quad \int_{|x| \leq R} |D^\alpha u|^2 dx = O(e^{-BR^{-\epsilon/m}}), \quad R \to 0,
\]

for some positive constant \( B \).
Proof. Following Hörmander’s argument in [9] (Corollary 17.1.4., p. 8), we show that if $u \in H^{2m}_{\text{loc}}(\Omega)$ is a solution of (2.2), then for all $|\alpha| \leq 2m$ and for all $N > 0$

\begin{equation}
(2.4) \quad \int_{|x| \leq R} |D^\alpha u|^2 \, dx = O(R^N), \quad R \to 0.
\end{equation}

In view of Remark 2.2 we can apply (2.1) to the function $\xi u$, where $\xi(x) \in C^\infty_0(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \leq R$ and $\xi(x) = 0$ for $|x| \geq 2R$ ($R > 0$ sufficiently small). Here the number $R$ is not yet fixed and is given by $R = (\gamma \tau)^{-m/\epsilon}$, where $\gamma > 0$ is a large constant which will be chosen later. Using the estimate (2.1) and the equation (1.2), we can derive that

\begin{align*}
&\sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau}\alpha(-n)|D^\alpha u|^2 \, dx \\
+ &\sum_{|\alpha| = m} \gamma^{2m} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&= \sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&\quad + \sum_{|\alpha| = m} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&\quad + \sum_{|\alpha| = m} R^{-2\epsilon}\tau^{-2m} \int_{|x| \leq R} |x|^{-2\tau\alpha}|D^\alpha u|^2 \, dx \\
&\quad \leq \sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&\quad + \sum_{|\alpha| = m} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&\quad \leq \sum_{|\alpha| \leq 2m} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+4m-n\epsilon}\Delta^m(\chi u)|^2 \, dx \\
&\quad \leq C_m \int_{|x| \leq R} |x|^{-2\tau+4m-n\epsilon}\Delta^m u|^2 \, dx \\
&\quad + \epsilon C_m \int_{|x| > R} |x|^{-2\tau+4m-n\epsilon} |\chi, \Delta^m u|^2 \, dx \\
&\quad \leq \epsilon C_m C_0 \sum_{|\alpha| \leq m-1} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&\quad + \epsilon C_m C_0 \sum_{|\alpha| = m} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx \\
&\quad + \epsilon C_m \int_{|x| > R} |x|^{-2\tau+4m-n\epsilon} |\chi, \Delta^m u|^2 \, dx,
\end{align*}

where $[\cdot, \cdot]$ denotes the commutator.

Therefore, carefully checking terms on both sides of (2.4), we can choose $\gamma$ and let $\tau$ be large enough such that all terms with $\int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n\epsilon}|D^\alpha u|^2 \, dx$ and
\[ \int_{|x| \leq R} |x|^{-2\tau + 2\alpha - n + 2\epsilon} |D^\alpha u|^2 \, dx \text{ on the right side of (2.5)} \] are absorbed by the left-hand side. We now fix such \( \gamma \). By construction of \( \chi \) we have \(|D^\alpha \chi| \leq c_1 R^{-|\alpha|}\), where \( c_1 \) is a positive constant. Consequently, it follows for \( R < 1/2 \) and (2.5) that

\[
\int_{|x| \leq R} |x|^{-2\tau + 2\alpha - n + 2\epsilon} |D^\alpha u|^2 \, dx \leq c \int_{|x| \leq R} |x|^{-2\tau + 2\alpha - n + 2\epsilon} |D^\alpha u|^2 \, dx
\]

where \( c \) is a positive constant. Consequently, it follows for \( R > 0 \) with \( B = 2^{-2\epsilon/m} \).

Thus, we can conclude that

\[
\int_{|x| \leq R} |x|^{-2\tau + 2\alpha - n + 2\epsilon} |D^\alpha u|^2 \, dx \leq c e^{-BR^{-\epsilon/m}},
\]

for all sufficiently small \( R > 0 \) with \( B = 2^{-2\epsilon/m} \).

3. Carleman estimates with more singular weights

To prove the following Carleman estimates, we introduce polar coordinates in \( \mathbb{R}^n \setminus \{0\} \) by setting \( x = rw \), with \( r = |x| \), \( w = (\omega_1, \cdots, \omega_n) \in S^{n-1} \) when \( x \neq 0 \). Furthermore, setting \( t = \log r \), we can substitute a new coordinate \( t \) for \( r \) such that

\[
\frac{\partial}{\partial x_j} = e^{-t} (\omega_j \partial_t + \Omega_j),
\]
where $\Omega_j$ is a vector field in $S^{n-1}$. Then the Laplacian becomes

$$e^{2t} \Delta = \partial_t^2 + (n - 2) \partial_t + \Delta_\omega,$$

where $\Delta_\omega = \sum_{j=1}^{n} \Omega_j^2$ is the Laplace-Beltrami operator in $S^{n-1}$. The vector field $\Omega_j$ has the properties

$$\sum_{j=1}^{n} \omega_j \Omega_j = 0, \quad \sum_{j=1}^{n} \Omega_j \omega_j = n - 1.$$

The adjoint of $\Omega_j$ as an operator in $L^2(S^2)$ is

$$\Omega_j^* = (n - 1) \omega_j - \Omega_j$$

and

$$\sum_{j=1}^{n} \Omega_j^* \Omega_j = -\Delta_\omega.$$

Hereafter we shall use the following notation:

\begin{align*}
D_0 &= (1/i) \partial_t; \\
D_k &= (1/i) \Omega_k, \quad k = 1, \ldots, n; \\
D^\alpha &= D_0^{\alpha_1} \cdots D_n^{\alpha_n}, \quad \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}.
\end{align*}

When $r \to 0$, $t \to -\infty$, we will be interested in values of $t$ in a neighborhood of $-\infty$. Motivated by [13], we derive the following Carleman estimate with weights $\varphi_\beta(x) = \exp(\frac{\beta}{2} \log |x|^2)$.

**Theorem 3.1.** Given $\sigma_1 \in \mathbb{Z}$ and $\sigma_2 \in \mathbb{Z}$, there exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on $n$, $\sigma_1$ and $\sigma_2$ such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta \geq \beta_0$, we have that

$$\int \big| \varphi_\beta^2 |x|^{2\sigma_1 + 4 - \beta \sigma_2} |\Delta u|^2 \big| dx \leq C \sum_{|\alpha| \leq 2} \beta^{3 - 2|\alpha|} \int \varphi_\beta^2 |x|^{2\sigma_1 + 2|\alpha| - n} (\log |x|)^{2\sigma_2 + 2 - 2|\alpha|} |D^\alpha u|^2 dx,$$

where $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \text{supp}(u) \subset B_{r_0}\}$ and $C$ is a positive constant.

**Proof.** By the polar coordinate system, we have

\begin{align*}
\int \varphi_\beta^2 |x|^{2\sigma_1 + 4 - \beta \sigma_2} |\Delta u|^2 dx &= \int \int e^{\beta t^2} e^{2\sigma_1 t + 4t \sigma_2} |\Delta u|^2 dt d\omega \\
&= \int \int e^{\beta t^2 / 2} e^{\sigma_1 t} e^{\sigma_2^2} |\Delta u|^2 dt d\omega.
\end{align*}

If we set $u = e^{-\beta t^2 / 2} e^{-\sigma_1 t} e^{-\sigma_2^2 v}$ and use (3.1), then

$$e^{\beta t^2 / 2} e^{\sigma_1 t} e^{\sigma_2^2} \Delta u = (\partial_t - \beta t)^2 v + (n - 2)(\partial_t - \beta t) v + \Delta_\omega v + a \partial_t v + bv,$$

where $b = 2\beta \sigma_1 t + 2\beta \sigma_2 + \sigma_1^2 + 2\sigma_1 \sigma_2 t^{-1} - \sigma_2 t^{-2} + \sigma_2^2 t^{-2} - (n - 2)(\sigma_1 + \sigma_2 t^{-1})$ and $a = -2\sigma_1 - 2\sigma_2 t^{-1}$. 

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By (3.6), (3.7) and \(\beta\) large enough, (3.5) holds if we have

\[
\sum_{|\alpha| \leq 2} \beta^{3-2|\alpha|} \int \int t^{2-2|\alpha|} |D^{\alpha}v|^2 dt d\omega \leq C \int \int |\Delta_{\beta}v|^2 dt d\omega,
\]

where \(\Delta_{\beta}v = (\partial_t - \beta t)^2 v + (n-2)(\partial_t - \beta t)v + \Delta_{\omega}v\) and \(C\) is a positive constant. Note that

\[
\Delta_{\beta}v = \partial_t^2 v + (n-2)\partial_t v - 2\beta t \partial_t v - (n-2)\beta tv - \beta v + \beta^2 t^2 v + \Delta_{\omega}v.
\]

Denote

\[
\Delta_{\beta}^2 v = \partial_t^2 v - (n-2)\partial_t v + 2\beta t \partial_t v - (n-2)\beta tv - \beta v + \beta^2 t^2 v + \Delta_{\omega}v.
\]

Using integration by parts, we get

\[
I := \int \int |\Delta_{\beta}v|^2 dt d\omega - \int \int |\Delta_{\beta}^2 v|^2 dt d\omega
= \int \left(12 \beta^3 t^2 + \beta^2 O(t)\right) |v|^2 dt d\omega + 4\beta \int \int |\partial_t v|^2 dt d\omega
- 4\beta \int \int \sum_j \Omega_j |v|^2 dt d\omega.
\]

Similarly, we can get

\[
J := \int \int t^{-2} |\Delta_{\beta}v|^2 dt d\omega + \int \int t^{-2} |\Delta_{\beta}^2 v|^2 dt d\omega
= \int \left(2\beta^4 t^2 + \beta^3 O(t)\right) |v|^2 dt d\omega + \int \left(8\beta^2 + \beta O(t^{-1})\right) |\partial_t v|^2 dt d\omega
+ 2 \int \int t^{-2} |\partial_t^2 v|^2 dt d\omega - \int \int \sum_j \left(4\beta^2 + \beta O(t^{-1})\right) |\Omega_j v|^2 dt d\omega
+ 2 \int \int t^{-2} |\Delta_{\omega}v|^2 dt d\omega + 4 \int \int t^{-2} \sum_j |\partial_t \Omega_j v|^2 dt d\omega.
\]

If \(v \in C_0^\infty(B(0,e^{T_0}))\) and \(\beta >> |T_0|\) is large enough, then it follows from (3.9) and (3.10) that

\[
\beta I + J \geq 13 \beta^4 \int \int t^2 |v|^2 dt d\omega + 2 \int \int t^{-2} |\Delta_{\omega}v|^2 dt d\omega
- 9\beta^2 \int \int \sum_j \Omega_j |v|^2 dt d\omega + 11\beta^2 \int \int |\partial_t v|^2 dt d\omega
+ 2 \int \int t^{-2} |\partial_t^2 v|^2 dt d\omega + 4 \int \int t^{-2} \sum_j |\partial_t \Omega_j v|^2 dt d\omega
\geq K + \frac{13}{39} \beta^4 \int \int t^2 |v|^2 dt d\omega + \frac{1}{20} \int \int t^{-2} |\Delta_{\omega}v|^2 dt d\omega
+ \beta^2 \int \int \sum_j \Omega_j |v|^2 dt d\omega + 11\beta^2 \int \int |\partial_t v|^2 dt d\omega
+ 2 \int \int t^{-2} |\partial_t^2 v|^2 dt d\omega + 4 \int \int t^{-2} \sum_j |\partial_t \Omega_j v|^2 dt d\omega,
\]
where
\[ K = \frac{500}{39} \beta^4 \int t^2|v|^2 dt d\omega - 10\beta^2 \int \sum_j |\Omega_j v|^2 dt d\omega + \frac{39}{20} \int t^{-2} |\Delta v|^2 dt d\omega. \]

Now we shall get a lower bound of \( K \). Then, by (3.3), we have
\[ (3.12) \quad -\beta^2 \int \sum_j |\Omega_j v|^2 dt d\omega = \beta^2 \int v_\omega v dt d\omega. \]

Since \( 10|ab| \leq \frac{500}{39} \beta^4 |v|^2 + \frac{39}{20} t^{-2} |\Delta v|^2 \), it follows that
\[ (3.13) \quad 10|\beta^2 v_\omega v| \leq \frac{500}{39} \beta^4 |v|^2 + \frac{39}{20} t^{-2} |\Delta v|^2. \]

Combining (3.12) and (3.13), we get a lower bound of \( K \):
\[ (3.14) \quad K = \frac{500}{39} \beta^4 \int t^2|v|^2 dt d\omega - 10\beta^2 \int \sum_j |\Omega_j v|^2 dt d\omega + \frac{39}{20} \int t^{-2} |\Delta v|^2 dt d\omega \geq 0. \]

In addition, the ellipticity of \( \Delta v \) implies that there exists a new positive constant \( C \) such that
\[ (3.15) \quad \sum_{|\alpha|=2} \int t^{-2} |\Omega^\alpha v|^2 dt d\omega \leq C \int t^{-2} |\Delta v|^2 dt d\omega. \]

Thus, it follows from (3.11), (3.14) and (3.15) that
\[ C(\beta I + J) \geq \beta^4 \int t^2|v|^2 dt d\omega + \int t^{-2} |\Delta v|^2 dt d\omega + \beta^2 \int \sum_j |\Omega_j v|^2 dt d\omega + \beta^2 \int |v|^2 dt d\omega \]
\[ + \int t^{-2} |\partial^2 v|^2 dt d\omega + \int t^{-2} \sum_j |\partial v|^2 dt d\omega. \]

Together with
\[ \beta I + J \leq (\beta + 1) \int |\Delta v|^2 dt d\omega, \]
(3.8) holds. So we have the result.

**Corollary 3.1.** Given \( \sigma_1 \in \mathbb{Z} \) and \( \sigma_2 \in \mathbb{Z} \), there exist a sufficiently large number \( \beta_0 > 0 \) and a sufficiently small number \( r_0 > 0 \) depending on \( n, m, \sigma_1 \) and \( \sigma_2 \) such that for all \( u \in \mathcal{U}_{r_0} \) with \( 0 < r_0 < e^{-1}, \beta \geq \beta_0 \), we have that
\[ C \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi^{2}_{3} |x|^{2m_1+2|\alpha|-n} (\log |x|)^{2\sigma_2+2m-2|\alpha|} |D^\alpha u|^2 dx \]
\[ \leq \int \varphi^{2}_{3} |x|^{2m_1+4m-n} (\log |x|)^{2\sigma_2} |\Delta^n u|^2 dx, \]
where \( C \) is a positive constant.
Corollary 3.2. There exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on $n$ and $m$ such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}, \beta \geq \beta_0$, we have that

$$C \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx$$

(3.16)

$$\leq \int \varphi_\beta^2 |x|^{4m-n} |\Delta^m u|^2 dx,$$

where $C$ is a positive constant.

Remark 3.2. The estimate (3.16) in Corollary 3.2 remains valid if we assume $u \in H^2_{\text{loc}}(\Omega)$ with compact support and satisfies for all $|\alpha| \leq 2m$, $\int_{|x| \leq R} |D^\alpha u|^2 dx = O(e^{-BR^{-1/m}})$ as $R \to 0$, $B > 0$.

4. Proof of Theorem 1.1

Let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of (1.2). By Lemma 2.1 and Theorem 2.3, $u$ is in $H^2_{\text{loc}}(\Omega)$ and satisfies (2.3). Thus by Remark 3.2 we can apply (3.16) with the function $\xi u$, where $\xi(x) \in C^\infty_0(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \leq R$ and $\xi(x) = 0$ for $|x| \geq 2R (R > 0$ small enough). Then

$$\sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int_{|x| < R} \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx$$

$$\leq \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int_{|x| < R} \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx$$

$$\leq \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int_{|x| < R} \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha (\xi u)|^2 dx$$

$$\leq c \int \varphi_\beta^2 |x|^{4m-n} |\Delta^m (\xi u)|^2 dx$$

(4.1)

$$\leq c \int_{|x| \leq R} \varphi_\beta^2 |x|^{4m-n} |\Delta^m u|^2 dx + c \int_{|x| > R} \varphi_\beta^2 |x|^{4m-n} |\xi, \Delta^m u|^2 dx$$

$$\leq c \sum_{|\alpha| \leq m-1} \int_{|x| \leq R} \varphi_\beta^2 |x|^{2|\alpha|-n} |D^\alpha u|^2 dx$$

$$+ c \sum_{|\alpha| = m} \int_{|x| \leq R} \varphi_\beta^2 |x|^{2|\alpha|-n+2\epsilon} |D^\alpha u|^2 dx$$

$$+ c \int_{|x| > R} \varphi_\beta^2 |x|^{4m-n} |\xi, \Delta^m u|^2 dx,$$

because $\lim_{|x| = -\infty} r^2 (\log r)^{-2m} = 0$ which implies $c|x|^{2\epsilon} \leq (\log |x|)^{2m}$ as $R \leq R_0$ for some small $R_0 > 0$. Let $\beta > \beta_0$ ($\beta_0$ large enough); the first two terms on the right-hand side will be swallowed by the left-hand side. Recall that $\varphi_\beta = \varphi_{\beta}(x) =$
exp(\(\alpha^2 \log |x|^2\)). Therefore, we obtain that
\[
\beta^{3m} \exp(\beta (\log R_0)^2) \int_{|x| < R_0} |x|^{-n} (\log |x|)^{2m} |u|^2 dx
\]
\[
\leq \sum_{|\alpha| \leq \frac{m}{2}} \beta^{3m-2|\alpha|} \int_{|x| < R_0} \phi_{\beta}^2 |x|^{2|\alpha|} |D^\alpha u|^2 dx
\]
\[
(4.2)
\]
\[
\leq c \int_{|x| > R_0} \phi_{\beta}^2 |x|^{4m-n} ||\xi, \triangle^m u||^2 dx
\]
\[
\leq c \exp(\beta (\log R_0)^2) \int_{|x| > R_0} |x|^{4m-n} ||\xi, \triangle^m u||^2 dx.
\]
Divide \(\exp(\beta (\log R_0)^2)\) into both sides of (4.2) and then let \(\beta \to \infty\); this implies that \(u = 0\) in \(B(0, R_0)\). By standard arguments, we can get \(u = 0\) in \(\Omega\). This completes the proof.

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