STRONG UNIQUE CONTINUATION FOR $m$-TH POWERS OF A LAPLACIAN OPERATOR WITH SINGULAR COEFFICIENTS

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Abstract. In this paper we prove strong unique continuation for $u$ satisfying an inequality of the form $|\Delta^m u| \leq f(x, u, Du, \cdots, D^k u)$, where $k$ is up to $[3m/2]$. This result gives an improvement of a work by Colombini and Grammatico (1999) in some sense. The proof of the main theorem is based on Carleman estimates with three-parameter weights $|x|^{2\sigma_1} (\log |x|)^{2\sigma_2} \exp(\beta^2 (\log |x|)^2)$.

1. Introduction

Let $\Omega$ be a connected open subset of $\mathbb{R}^n$ ($n \geq 2$) containing 0. In [6], Colombini and Grammatico prove some results of strong unique continuation for $u$ satisfying an inequality of the form

$$|\Delta^m u| \leq f(x, u, Du, \cdots, D^k u)$$

for $k = m$.

In this paper we are interested in $u$ satisfying the following forms:

$$|\Delta^m u| \leq C_0 \sum_{|\alpha| \leq m-1} |x|^{-2m+|\alpha|} |D^\alpha u| + C_0 \sum_{|\alpha| = m} |x|^{-2m+|\alpha|+\epsilon} |D^\alpha u|,$$

where $0 < \epsilon < 1/2$ and the orders of lower order terms are up to $[3m/2]$.

Theorem 1.1. Let $u \in H^{2m}_0(\Omega)$ be a solution of (1.2), and for all $N > 0$ let

$$\int_{|x| \leq R} |u|^2 dx = O(R^N), \quad R \to 0.$$

Then $u$ is identically zero in $\Omega$.

This problem for $m = 1$ has drawn a lot of attention in partial differential equations and mathematical physics. For the development of this problem, the author refers the readers to [15]. In particular, their work was focused on second order equations in which the coefficients of the lower order terms are allowed to be singular (see [2], [3], [11] and [5]). Here we mention two articles which are closely related to our work in this paper.

Hörmander [8] proved that if $u \in H^1_{0, loc}(\Omega)$ satisfies

$$|\Delta u| \leq C_1 |x|^{-2+\epsilon} |u| + C_2 |x|^{-1+\epsilon} |\nabla u|, \quad \epsilon > 0,$$

and $u$ vanishes of infinite order at 0, then $u$ is identically zero in $\Omega$.
Regbaoui [13] extended Hörmander’s result to the sharp case \( \epsilon = 0 \) with small \( C_2 \). His proof was also based on suitable Carleman estimates, but he worked with the strictly convex weights \( \varphi_\beta(x) = \exp\left(\frac{\beta}{2}(\log|x|)^2\right) \) rather than the usual polynomial weights. As for negative results, some counterexamples were given when \( C_2 \) is not small (such as [1] and [16]).

For the case \( m = 2 \), Borgne [4] got some results about strong unique continuation when \( k = 3 \) in [11]. On the other hand, Watanabe [14] established some uniqueness results for \( m = 3 \) with \( k = 5 \). For \( m = 4 \), Ōkaji [10] solved unique continuation when \( k = 7 \). For the higher orders, Protter [12] got the unique continuation for inequalities of the form (1.1) in which \( k \) is up to \([3m/2]\). According to the counterexamples for \( n = 2 \) in [7], Colombini and Grammatico [6] proved the sharp results about the coefficients of the \( m \)-th lower order terms for strong unique continuation with some cases when \( k = m \). In this paper, we consider the cases (1.2).

We will follow Regbaoui’s approach in [13] and organize this paper as follows. In Section 2, we shall study the asymptotic behavior of the function \( u \) near 0 which guarantees the use of the singular weights \(|x|^{2\sigma}(\log|x|)^{2\sigma_2}\varphi_\beta^2\). Some of the key Carleman estimates with the weights \(|x|^{2\sigma}(\log|x|)^{2\sigma_2}\varphi_\beta^2\) will be derived in Section 3. Using these Carleman estimates, we prove Theorem 1.1 in Section 4.

2. Carleman estimates with polynomial weights

**Lemma 2.1.** There exists a positive constant \( C_m \) such that for any \( u \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \) and for any \( \tau \in \{k + 1/2, k \in \mathbb{N}\} \), we have the estimate

\[
(2.1) \quad \sum_{|\alpha| \leq 2m} \int |x|^{-2\tau+2|\alpha|} |x|^{-\alpha} |D^\alpha u|^2 dx \leq C_m \int |x|^{-2\tau+4m-\alpha} |\Delta^m u|^2 dx.
\]

**Proof.** In [13], Regbaoui proved in Lemma 2.1 for any \( u \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \) and for any \( \tau \in \{k + 1/2, k \in \mathbb{N}\} \) that

\[
(2.2) \quad \sum_{|\alpha| \leq 2} \int |x|^{-2\tau+2|\alpha|} |x|^{-\alpha} |D^\alpha u|^2 dx \leq C \int |x|^{-2\tau+4-n} |\Delta u|^2 dx.
\]

By the repeated use of (2.2), we can get (2.1).

**Remark 2.2.** The estimate (2.1) in Lemma 2.1 remains valid if we assume \( u \in H^{2m}_loc(\Omega) \) with compact support and satisfying for all \( |\alpha| \leq 2m \) and all \( N > 0 \),

\[
\int_{|x| \leq R} |D^\alpha u|^2 dx = O(R^N) \quad \text{as} \quad R \to 0.
\]

This can be easily obtained by cutting \( u \) off for small \( |x| \) and regularizing.

Motivated by [8] and [13], we get the following theorem. The proof of this theorem is based on Lemma 2.1. From now on, \( c \) stands for a generic constant and its value may vary from line to line.

**Theorem 2.3.** Let \( u \in H^{2m}_loc(\Omega) \) be a solution to (1.2). If \( |\alpha| \leq 2m \), then

\[
(2.3) \quad \int_{|x| \leq R} |D^\alpha u|^2 dx = O(e^{-BR^{-\epsilon/m}}), \quad R \to 0,
\]

for some positive constant \( B \).
Proof. Following Hörmander’s argument in [9] (Corollary 17.1.4., p. 8), we show that if \( u \in H^2_{\text{loc}}(\Omega) \) is a solution of (1.2), then for all \( |\alpha| \leq 2m \) and for all \( N > 0 \)
\begin{equation}
\int_{|x| \leq R} |D^\alpha u|^2 \, dx = O(R^N), \quad R \to 0.
\end{equation}

In view of Remark 2.2 we can apply (2.1) to the function \( \xi u \), where \( \xi(x) \in C^\infty_0(\mathbb{R}^n) \) such that \( \xi(x) = 1 \) for \( |x| \leq R \) and \( \xi(x) = 0 \) for \( |x| \geq 2R \) (\( R > 0 \) sufficiently small). Here the number \( R \) is not yet fixed and is given by \( R = (\gamma \tau)^{-m/\epsilon} \), where \( \gamma > 0 \) is a large constant which will be chosen later. Using the estimate (2.1) and the equation (1.2), we can derive that
\begin{align}
&\sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \, dx \\
&+ \sum_{|\alpha| = m} \gamma^{2m} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n+2\epsilon} |D^\alpha u|^2 \, dx \\
&\quad = \sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \, dx \\
&+ \sum_{|\alpha| = m} R^{-2\tau+2m} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n+2\epsilon} |D^\alpha u|^2 \, dx \\
&\quad \leq \sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \, dx \\
&+ \sum_{|\alpha| = m} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \, dx \\
&\quad \leq \sum_{|\alpha| \leq 2m} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau+4m-n} |D^\alpha u|^2 \, dx \\
&\quad \leq C_m \int_{|x| \leq R} |x|^{-2\tau+4m-n} |\Delta^m (\chi u)|^2 \, dx \\
&\quad \leq cC_m \int_{|x| \leq R} |x|^{-2\tau+4m-n} |\Delta^m u|^2 \, dx \\
&\quad + cC_m \int_{|x| > R} |x|^{-2\tau+4m-n} |[\chi, \Delta^m] u|^2 \, dx \\
&\quad \leq cC_mC_0 \sum_{|\alpha| \leq m-1} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \, dx \\
&\quad + cC_mC_0 \sum_{|\alpha| = m} \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n+2\epsilon} |D^\alpha u|^2 \, dx \\
&\quad + cC_m \int_{|x| > R} |x|^{-2\tau+4m-n} |[\chi, \Delta^m] u|^2 \, dx,
\end{align}
where \([\cdot, \cdot]\) denotes the commutator.

Therefore, carefully checking terms on both sides of (2.4), we can choose \( \gamma \) and let \( \tau \) be large enough such that all terms with \( \int_{|x| \leq R} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \, dx \) and
\[ \int_{|x| \leq R} |x|^{-2\tau + 2|\alpha| - n + 2\epsilon} |D^\alpha u|^2 \, dx \] on the right side of (2.5) are absorbed by the left-hand side. We now fix such \( \gamma \). By construction of \( \chi \) we have \( |D^\alpha \chi| \leq c_1 R^{-|\alpha|} \), where \( c_1 \) is a positive constant. Consequently, it follows for \( R < 1/2 \) and (2.5) that

\[
\begin{align*}
\tau^{-2m}(R/2)^{-2\tau + 4m - n + 2\epsilon} \sum_{|\alpha| \leq 2m} \int_{|x| \leq R/2} |D^\alpha u|^2 \, dx \\
\leq \sum_{|\alpha| \leq 2m} \tau^{2m-2|\alpha|}(R/2)^{-2\tau + 2|\alpha| - n + 2\epsilon} \int_{|x| \leq R/2} |D^\alpha u|^2 \, dx \\
\leq \sum_{|\alpha| \leq 2m} \tau^{2m-2|\alpha|} \int_{|x| \leq R/2} |x|^{-2\tau + 2|\alpha| - n + 2\epsilon} |D^\alpha u|^2 \, dx \\
\leq \sum_{|\alpha| \leq m-1} \tau^{2m-2|\alpha|} \int_{|x| \leq R} |x|^{-2\tau + 2|\alpha| - n} |D^\alpha u|^2 \, dx \\
+ \sum_{|\alpha| = m} \gamma^{2m} \int_{|x| \leq R} |x|^{-2\tau + 2|\alpha| - n + 2\epsilon} |D^\alpha u|^2 \, dx \\
\leq c \int_{|x| > R} |x|^{-2\tau + 4m - n} |[\chi, \Delta^m] u|^2 \, dx \\
\leq c R^{-4m} R^{-2\tau + 4m - n} \|u\|_{H^m}^2 \\
= c \|u\|_{H^{2m}}^2 R^{-2\tau - n},
\end{align*}
\] (2.6)

where \( \|u\|_{H^{2m}}^2 \) is the \( H^{2m} \) norm of \( u \) in the ball \( B(0, 2R) \).

Recall that for \( R = (\gamma \tau)^{-m/\epsilon} \), we have

\[
\begin{align*}
\sum_{|\alpha| \leq 2m} \int_{|x| \leq R/2} |D^\alpha u|^2 \, dx \\
\leq c R^{-4m-4\epsilon} 2^{-2\gamma^{-1} R^{-\epsilon}/m} \|u\|_{H^m}^2 \\
\leq c e^{-BR^{-\epsilon/m}}.
\end{align*}
\] (2.7)

It should be noted that (2.7) is valid for \( \tau \in \mathbb{N} + \frac{1}{2} \) and \( R = (\gamma \tau)^{-m/\epsilon} \). Therefore, if we choose \( \tau \in \{ j + \frac{1}{2} : j \in \mathbb{N} \} \), then (2.7) only holds for \( R_j = (\gamma(j + \frac{1}{2}))^{-m/\epsilon} \). Nevertheless, we can see that

\[ R_{j+1} < R_j < 2R_{j+1} \quad \text{for} \quad j \quad \text{large enough and} \quad R_j \to 0 \quad \text{as} \quad j \to \infty. \]

Thus, we can conclude that

\[
\sum_{|\alpha| \leq 2m} \int_{|x| \leq R} |D^\alpha u|^2 \, dx \leq c e^{-BR^{-\epsilon/m}},
\]

for all sufficiently small \( R > 0 \) with \( B = 2^{-2\epsilon/m} \).

3. Carleman Estimates with More Singular Weights

To prove the following Carleman estimates, we introduce polar coordinates in \( \mathbb{R}^n \setminus \{0\} \) by setting \( x = rw \), with \( r = |x| \), \( \omega = (\omega_1, \cdots, \omega_n) \in S^{n-1} \) when \( x \neq 0 \). Furthermore, setting \( t = \log r \), we can substitute a new coordinate \( t \) for \( r \) such that

\[ \frac{\partial}{\partial x_j} = e^{-t}(\omega_j \partial_t + \Omega_j), \]
where $\Omega_j$ is a vector field in $S^{n-1}$. Then the Laplacian becomes
\begin{equation}
(3.1) \quad e^{2t} \Delta = \partial_t^2 + (n-2)\partial_t + \Delta_\omega,
\end{equation}
where $\Delta_\omega = \sum_{j=1}^n \Omega_j^2$ is the Laplace-Beltrami operator in $S^{n-1}$. The vector field $\Omega_j$ has the properties
\begin{equation}
(3.2) \quad \sum_{j=1}^n \omega_j \Omega_j = 0, \quad \sum_{j=1}^n \Omega_j \omega_j = n - 1.
\end{equation}
The adjoint of $\Omega_j$ as an operator in $L^2(S^2)$ is
\begin{equation}
(3.3) \quad \Omega_j^* = (n-1)\omega_j - \Omega_j
\end{equation}
and
\begin{equation}
(3.4) \quad \sum_{j=1}^n \Omega_j^* \Omega_j = -\Delta_\omega.
\end{equation}
Hereafter we shall use the following notation:
\begin{equation}
(3.5) \quad D_0 = (1/i)\partial_t; \quad D_k = (1/i)\Omega_k, \quad k = 1, \ldots, n; \quad D^\alpha = D_0^{\alpha_0} \cdots D_n^{\alpha_n}, \quad \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}.
\end{equation}
When $r \to 0$, $t \to -\infty$, we will be interested in values of $t$ in a neighborhood of $-\infty$. Motivated by [13], we derive the following Carleman estimate with weights $\varphi_\beta = \varphi_\beta(x) = \exp(\frac{\beta}{2}(\log |x|)^2)$.

**Theorem 3.1.** Given $\sigma_1 \in \mathbb{Z}$ and $\sigma_2 \in \mathbb{Z}$, there exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on $n$, $\sigma_1$ and $\sigma_2$ such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta_0 \geq \beta_0$, we have that
\begin{align}
(3.6) \quad C \sum_{|\alpha| \leq 2} \beta^{3-2|\alpha|} & \int \varphi_\beta^2 |x|^{2\sigma_1+2|\alpha|-n}(\log |x|)^{2\sigma_2+2-2|\alpha|} |D^\alpha u|^2 dx \\
& \leq \int \varphi_\beta^2 |x|^{2\sigma_1+4-n}(\log |x|)^{2\sigma_2} |\Delta u|^2 dx,
\end{align}
where $U_{r_0} = \{ u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \text{supp}(u) \subset B_{r_0} \}$ and $C$ is a positive constant.

**Proof.** By the polar coordinate system, we have
\begin{align}
(3.7) \quad \int \varphi_\beta^2 |x|^{2\sigma_1+4-n}(\log |x|)^{2\sigma_2} |\Delta u|^2 dx \\
& = \int \int e^{\beta t^2} e^{2\sigma_1 t+4t} |\Delta u|^2 dtd\omega \\
& = \int \int e^{\beta t^2/2} e^{\sigma_1 t} e^{2t} |\Delta u|^2 dtd\omega.
\end{align}
If we set $u = e^{-\beta t^2/2} e^{-\sigma_1 t} e^{-\sigma_2} v$ and use (3.1), then
\begin{equation}
(3.8) \quad e^{\beta t^2/2} e^{\sigma_1 t} e^{\sigma_2} \Delta u = (\partial_t - \beta t)^2 v + (n - 2)(\partial_t - \beta t)v + \Delta_\omega v + a\partial_t v + bv,
\end{equation}
where $b = 2\beta\sigma_1 + 2\beta\sigma_2 + \sigma_1^2 + 2\sigma_1\sigma_2 t^{-1} - \sigma_2 t^{-2} + \sigma_2^2 t^{-2} - (n-2)(\sigma_1 + \sigma_2 t^{-1})$ and $a = -2\sigma_1 - 2\sigma_2 t^{-1}$. 

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Similarly, we can get

\[ \sum_{|\alpha| \leq 2} \beta^{3-2|\alpha|} \int \int t^{2-2|\alpha|}|D^\alpha v|^2 dt d\omega \leq C \int \int |\Delta_\beta v|^2 dt d\omega, \tag{3.8} \]

where \( \Delta_\beta = (\partial_t - \beta \partial_t)^2 + (n-2)(\partial_t - \beta \partial_t) + \Delta_\omega v \) and \( C \) is a positive constant.

Note that

\[ \Delta_\beta = \partial_t^2 v + (n-2)\partial_t v - 2\beta t \partial_t v - (n-2)\beta tv - \beta v + \beta^2 t^2 v + \Delta_\omega v. \]

Denote

\[ \Delta_\beta^* = \partial_t^2 v - (n-2)\partial_t v + 2\beta t \partial_t v - (n-2)\beta tv - \beta v + \beta^2 t^2 v + \Delta_\omega v. \]

Using integration by parts, we get

\[ I := \int \int |\Delta_\beta v|^2 dt d\omega - \int \int |\Delta_\beta^* v|^2 dt d\omega \]

\[ = \int \int (12\beta^2 t^2 + \beta^2 O(t)) |v|^2 dt d\omega + 4\beta \int \int |\partial_t v|^2 dt d\omega \]

\[ - 4\beta \int \sum_j |\Omega_j v|^2 dt d\omega. \tag{3.9} \]

Similarly, we can get

\[ J := \int \int t^{-2} |\Delta_\beta v|^2 dt d\omega + \int \int t^{-2} |\Delta_\beta^* v|^2 dt d\omega \]

\[ = \int \int (2\beta^4 t^2 + \beta^3 O(t)) |v|^2 dt d\omega + \int (8\beta^2 + \beta O(t^{-1})) |\partial_t v|^2 dt d\omega \]

\[ + 2 \int \int t^{-2} |\partial_t v|^2 dt d\omega - \int \sum_j (4\beta^2 + \beta O(t^{-1})) |\Omega_j v|^2 dt d\omega \]

\[ + 2 \int \int t^{-2} |\Delta_\omega v|^2 dt d\omega + 4 \int \int t^{-2} \sum_j |\partial_t \Omega_j v|^2 dt d\omega. \tag{3.10} \]

If \( v \in C_{0}^\infty(B(0,e^{T_0})) \) and \( \beta >> |T_0| \) is large enough, then it follows from (3.9) and (3.10) that

\[ \beta I + J \geq 13\beta^4 \int \int t^2 |v|^2 dt d\omega + 2 \int \int t^{-2} |\Delta_\omega v|^2 dt d\omega \]

\[ - 9\beta^2 \int \sum_j |\Omega_j v|^2 dt d\omega + 11\beta^2 \int |\partial_t v|^2 dt d\omega \]

\[ + 2 \int \int t^{-2} |\partial_t v|^2 dt d\omega + 4 \int \int t^{-2} \sum_j |\partial_t \Omega_j v|^2 dt d\omega \geq K + \frac{13}{39} \beta^4 \int \int t^2 |v|^2 dt d\omega + \frac{1}{20} \int \int t^{-2} |\Delta_\omega v|^2 dt d\omega \]

\[ + \beta^2 \int \sum_j |\Omega_j v|^2 dt d\omega + 11\beta^2 \int |\partial_t v|^2 dt d\omega \]

\[ + 2 \int \int t^{-2} |\partial_t v|^2 dt d\omega + 4 \int \int t^{-2} \sum_j |\partial_t \Omega_j v|^2 dt d\omega, \tag{3.11} \]
where
\[ K = \frac{500}{39} \beta^4 \int t^2 |v|^2 dt \omega - 10 \beta^2 \int \sum_j |\Omega_j v|^2 dt \omega + \frac{39}{20} \int t^{-2} |\Delta_\omega v|^2 dt \omega. \]

Now we shall get a lower bound of \( K \). Then, by (3.3), we have
\[
(3.12) \quad -\beta^2 \int \sum_j |\Omega_j v|^2 dt \omega = \beta^2 \int v \Delta_\omega v dt \omega.
\]

Since \( 10 |ab| \leq \frac{500}{39} \beta^4 |v|^2 + \frac{39}{20} \beta^2 |\Delta_\omega v|^2 \), it follows that
\[
(3.13) \quad 10 |\beta^2 \Delta_\omega v| \leq \frac{500}{39} \beta^4 |v|^2 + \frac{39}{20} \beta^2 |\Delta_\omega v|^2.
\]

Combining (3.12) and (3.13), we get a lower bound of \( K \):
\[
(3.14) \quad K = \frac{500}{39} \beta^4 \int t^2 |v|^2 dt \omega - 10 \beta^2 \int \sum_j |\Omega_j v|^2 dt \omega + \frac{39}{20} \int t^{-2} |\Delta_\omega v|^2 dt \omega \geq 0.
\]

In addition, the ellipticity of \( \Delta_\omega \) implies that there exists a new positive constant \( C \) such that
\[
(3.15) \quad \sum_{|\alpha|=2} \int t^{-2} |\Omega^\alpha v|^2 dt \omega \leq C \int t^{-2} |\Delta_\omega v|^2 dt \omega.
\]

Thus, it follows from (3.11), (3.14) and (3.15) that
\[
C(\beta I + J) \geq \beta^4 \int t^2 |v|^2 dt \omega + \int t^{-2} |\Delta_\omega v|^2 dt \omega
+ \beta^2 \int \sum_j |\Omega_j v|^2 dt \omega + \beta^2 \int |\partial v|^2 dt \omega
+ \int t^{-2} |\partial^2 v|^2 dt \omega + \int t^{-2} \sum_j |\partial \Omega_j v|^2 dt \omega.
\]
Together with
\[
\beta I + J \leq (\beta + 1) \int |\Delta_\omega v|^2 dt \omega,
\]
(3.8) holds. So we have the result.

**Corollary 3.1.** Given \( \sigma_1 \in \mathbb{Z} \) and \( \sigma_2 \in \mathbb{Z} \), there exist a sufficiently large number \( \beta_0 > 0 \) and a sufficiently small number \( r_0 > 0 \) depending on \( n, m, \sigma_1 \) and \( \sigma_2 \) such that for all \( u \in U_{r_0} \) with \( 0 < r_0 < e^{-1} \), \( \beta \geq \beta_0 \), we have that
\[
C \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2\sigma_1+2|\alpha|+n} (\log |x|)^{2\sigma_2+2m-2|\alpha|} |D^n u|^2 dx
\leq \int \varphi_\beta^2 |x|^{2\sigma_1+4m-n} (\log |x|)^{2\sigma_2} |\Delta^m u|^2 dx,
\]
where \( C \) is a positive constant.
There exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$ depending on $n$ and $m$ such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta \geq \beta_0$, we have that

$$C \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx$$

(3.16)

$$\leq \int \varphi_\beta^2 |x|^{4m-n} |\Delta^m u|^2 dx,$$

where $C$ is a positive constant.

Remark 3.2. The estimate (3.16) in Corollary 3.2 remains valid if we assume $u \in H^{2m}_{210}(\Omega)$ with compact support and satisfies for all $|\alpha| \leq 2m$, $\int |x| \leq R |D^\alpha u|^2 dx = O(e^{-BR^{-r/m}})$ as $R \to 0$, $B > 0$.

4. Proof of Theorem 1.1

Let $u \in H^{2m}_{210}(\Omega)$ be a solution of (1.2). By Lemma 2.1 and Theorem 2.3, $u$ is in $H^{2m}_{100}(\Omega)$ and satisfies (2.3). Thus by Remark 3.2 we can apply (3.16) with the function $\xi u$, where $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \leq R$ and $\xi(x) = 0$ for $|x| \geq 2R$ ($R > 0$ small enough). Then

$$\sum_{|\alpha| \leq (2m/3)} \beta^{3m-2|\alpha|} \int |x| < R \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx$$

$$\leq \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int |x| < R \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha u|^2 dx$$

$$\leq \sum_{|\alpha| \leq 2m} \beta^{3m-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log |x|)^{2m-2|\alpha|} |D^\alpha (\xi u)|^2 dx$$

$$\leq c \int \varphi_\beta^2 |x|^{4m-n} |\Delta^m (\xi u)|^2 dx$$

(4.1)

$$\leq c \int |x| \leq R \varphi_\beta^2 |x|^{4m-n} |\Delta^m u|^2 dx + c \int |x| > R \varphi_\beta^2 |x|^{4m-n} |\xi, \Delta^m u|^2 dx$$

$$\leq c \sum_{|\alpha| \leq m-1} \int |x| \leq R \varphi_\beta^2 |x|^{2|\alpha|-n} |D^\alpha u|^2 dx$$

$$+ c \sum_{|\alpha| = m} \int |x| \leq R \varphi_\beta^2 |x|^{2|\alpha|-n+2} |D^\alpha u|^2 dx$$

$$+ c \int |x| > R \varphi_\beta^2 |x|^{4m-n} |\xi, \Delta^m u|^2 dx,$$

because $\lim_{|x| = R} r^2 (\log r)^{-2m} = 0$ which implies $c|x|^{2\epsilon} \leq (\log |x|)^{2m}$ as $R \leq R_0$ for some small $R_0 > 0$. Let $\beta > \beta_0$ ($\beta_0$ large enough); the first two terms on the right-hand side will be swallowed by the left-hand side. Recall that $\varphi_\beta = \varphi_\beta(x) = \ldots$
\[ \exp\left( \frac{\beta}{2} (\log |x|)^2 \right) . \] Therefore, we obtain that
\[
\beta^{3m} \exp\left( \beta (\log R_0)^2 \right) \int_{|x| < R_0} |x|^{-n} (\log |x|)^{2m} |u|^2 dx 
\leq \sum_{|\alpha| \leq \frac{1}{2}} \beta^{3m-2|\alpha|} \int_{|x| < R_0} \varphi^2 |x|^{-|\alpha|} (\log |x|)^{2m-2|\alpha|} D^\alpha |u|^2 dx 
\leq \sum_{|\alpha| \leq \frac{1}{2}} \beta^{3m-2|\alpha|} \int_{|x| < R_0} \varphi^2 |x|^{-|\alpha|} (\log |x|)^{2m-2|\alpha|} D^\alpha |u|^2 dx 
\leq c \int_{|x| > R_0} \varphi^2 |x|^{4m-n} [\xi, \Delta^m] |u|^2 dx 
\leq c \exp(\beta (\log R_0)^2) \int_{|x| > R_0} |x|^{4m-n} [\xi, \Delta^m] |u|^2 dx.
\] (4.2)

Divide \( \exp(\beta (\log R_0)^2) \) into both sides of (4.2) and then let \( \beta \to \infty \); this implies that \( u = 0 \) in \( B(0, R_0) \). By standard arguments, we can get \( u = 0 \) in \( \Omega \). This completes the proof.

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\textbf{References}


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