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PSEUDOFREE $\mathbb{Z}/3$-ACTIONS ON $K3$ SURFACES

XIMIN LIU AND NOBUHIRO NAKAMURA

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Abstract. In this paper, we give a weak classification of locally linear pseudofree actions of the cyclic group of order 3 on a $K3$ surface, and prove the existence of such an action which cannot be realized as a smooth action on the standard smooth $K3$ surface.

1. Introduction

Let G be the cyclic group of order 3 ($G = \mathbb{Z}/3$), and suppose that G acts locally linearly and pseudofreely on a $K3$ surface X. (An action on a space is called *pseudofree* if it is free on the compliment of a discrete subset.) The purpose of this paper is to give a weak classification of such G-actions and to prove that there exists such an action on X which cannot be realized by a smooth action for the standard smooth structure on X.

Theorem 1.1. *There exists a locally linear pseudofree G-action on a $K3$ surface X which cannot be realized by a smooth action for the standard smooth structure on X.*

After submitting this paper for publication, the authors found that the G-action in Theorem 1.1 is unsmoothable for infinitely many smooth structures on X. This is proved in Remark 3.4.

To state the result more precisely, we prepare notation. Let b_i be the i-th Betti number of X, and let b_+ (resp. b_-) be the rank of a maximal positive (resp. negative) definite subspace $H^+(X;\mathbb{R})$ (resp. $H^-(X;\mathbb{R})$) of $H^2(X;\mathbb{R})$. For any G-space V, let V^G be the fixed point set of the G-action. Let $b_\bullet^G = \dim H^\bullet(X;\mathbb{R})^G$, where $\bullet = 2, +, -$. The Euler number of X is denoted by $\chi(X)$ and the signature of X by $\mathrm{Sign}(X)$.

When we fix a generator g of G, the representation at a fixed point can be described by a pair of nonzero integers (a, b) modulo 3 which is well defined up to order and changing the sign of both together. Hence, there are two types of fixed points.

- The type $(+)$: $(1, 2) = (2, 1)$.
- The type $(-)$: $(1, 1) = (2, 2)$.

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Let m_+ be the number of fixed points of the type $(+)$, and let m_- be the number of fixed points of the type $(-)$.

Theorem 1.1 immediately follows from the next theorem.

Theorem 1.2. *Let G be the cyclic group of order 3. For locally linear pseudofree G-actions on a $K3$ surface X, we have the following*:

(1) *Every locally linear pseudofree G-action on X belongs to one of four types in Table 1. Furthermore, each of four types can actually be realized by a locally linear pseudofree G-action on X.*

TABLE 1. *The classification of actions*

Type	$\#X^G$	m_+	m_-	b_2^G	b_+^G	b_-^G	$\mathrm{Sign}(X/G)$
A_0	6	6	0	10	3	7	-4
A_1	9	3	6	12	3	9	-6
A_2	12	0	12	14	3	11	-8
B	3	0	3	8	1	7	-6

(2) *The type A_1 cannot be realized by a smooth action on the standard smooth $K3$ surface.*

Remark 1.3. Assertion (1) in Theorem 1.2 is an application of the remarkable result by A. L. Edmonds and J. H. Ewing [4] with Freedman's classification of simply-connected topological 4-manifolds [7].

Remark 1.4. To prove assertion (2), we use the mod p vanishing theorem of Seiberg-Witten invariants by F. Fang [5] with the fact that the Seiberg-Witten invariants for the canonical Spin^c-structure of the standard smooth $K3$ surface is ± 1.

Remark 1.5. The types A_0, A_1 and A_2 are actions which act trivially on $H^+(X;\mathbb{R})$.

Remark 1.6. The type A_0 is realized by a smooth action on the Fermat quartic surface. (See Proposition 2.4.)

Remark 1.7. We do not know whether A_2 and B can be realized by a smooth action for some smooth structure on a $K3$ surface, or not.

Remark 1.8. K. Kiyono proved the existence of unsmoothable locally linear pseudofree actions on the connected sums of $S^2 \times S^2$ [11]. Although he also uses the Seiberg-Witten gauge theory, his method is different from ours. It is interesting that he invokes the "G-invariant 10/8-theorem" instead of Seiberg-Witten invariants. (A related paper is [12].)

2. The proof of assertion (1)

As mentioned in Remark 1.3, the proof of assertion (1) of Theorem 1.2 will rely on the realization theorem by A. L. Edmonds and J. H. Ewing [4]. First, we summarize their result in the very special case when $G = \mathbb{Z}/3$.

Theorem 2.1 ([4])**.** *Let G be the cyclic group of order 3. Suppose that one is given a fixed point data*

$$\mathcal{D} = \{(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})\},$$

where $a_i, b_i \in \mathbb{Z}/3 \setminus \{0\}$, *and a* G*-invariant symmetric unimodular form*

$$\Phi\colon V \times V \to \mathbb{Z},$$

where V *is a finitely generated* $\mathbb{Z}$*-free* $\mathbb{Z}[G]$*-module. Then the data* $\mathcal{D}$ *and the form* (V, Φ) *are realizable by a locally linear pseudofree* G*-action on a closed, simply-connected, topological* 4*-manifold if and only if they satisfy the following two conditions*:

(1) *The condition REP*: *As a* $\mathbb{Z}[G]$*-module,* V *splits into* $F \oplus T$*, where* F *is free and* T *is a trivial* $\mathbb{Z}[G]$*-module with* $\operatorname{rank}_{\mathbb{Z}} T = n$.

(2) *The condition GSF*: *The* G*-Signature Formula is satisfied*:

$$\operatorname{Sign}(g, (V, \Phi)) = \sum_{i=0}^{n+1} \frac{(\zeta^{a_i} + 1)(\zeta^{b_i} + 1)}{(\zeta^{a_i} - 1)(\zeta^{b_i} - 1)},$$

where $\zeta = \exp(2\pi\sqrt{-1}/3)$.

Remark 2.2. In [4], A. L. Edmonds and J. H. Ewing prove the realization theorem for all cyclic groups of prime order p, and for general p, the third condition *TOR* which is related to the Reidemeister torsion should be satisfied. However, when $p = 3$, the condition *TOR* is redundant. This follows from the fact that the class number of $\mathbb{Z}[\zeta]$ is 1, and Corollary 3.2 of [4].

Now, let us begin the proof of assertion (1).

Suppose that a locally linear pseudofree G-action on X is given. First of all, the ordinary Lefschetz formula should hold: $L(g, X) = 2 + \operatorname{tr}(g|_{H^2(X)}) = \#X^G$. Noting that $\#X^G = m_+ + m_-$ and $2 + \operatorname{tr}(g|_{H^2(X)}) \leq 24$, we obtain

$$m_+ + m_- \leq 24.$$

This is compatible with the condition *REP*. Note that

$$\chi(X/G) = \frac{1}{3}\{24 + 2(m_+ + m_-)\}.$$

By Theorem 2.1, the G-Signature Formula should hold:

$$\operatorname{Sign}(g, X) = \operatorname{Sign}(g^2, X) = \frac{1}{3}(m_+ - m_-),$$
$$\operatorname{Sign}(X/G) = \frac{1}{3}\left\{-16 + \frac{2}{3}(m_+ - m_-)\right\}.$$

Since $\operatorname{Sign}(X/G)$ is an integer, $m_+ - m_- \equiv 6 \mod 9$. This, with the inequality $-24 \leq m_+ - m_- \leq 24$, implies that

$$m_+ - m_- = -21, -12, -3, 6, 15, 24. \tag{2.3}$$

We can calculate b_+^G and b_-^G from $\chi(X/G)$ and $\operatorname{Sign}(X/G)$. Since b_+^G is 1 or 3, we obtain the following:

- When $b_+^G = 1$, $2m_+ + m_- = 3$.
- When $b_+^G = 3$, $2m_+ + m_- = 12$.

By these equations, (2.3) and nonnegativity of m_+ and m_-, we obtain Table 1.

Next we will prove the existence of actions. First, we construct a smooth G-action of type A_0 on the Fermat quartic surface.

Proposition 2.4. *There exists a smooth G-action of the type A_0 on the Fermat quartic surface X which is defined by the equation $\sum_{i=0}^{3} z_i^4 = 0$ in $\mathbb{C}\mathrm{P}^3$.*

Proof. By the symmetry of the defining equation, the symmetric group of degree 4 acts on X as permutations of variables. Therefore G acts smoothly on X via this action. We can easily check that the G-action is pseudofree and belongs to the type A_0. □

To prove the existence of actions of other types, we invoke Theorem 2.1. We need to construct G-actions on the intersection form. Let (V_{K3}, Φ_{K3}) be the intersection form of the $K3$ surface, which is even and indefinite. Since an even indefinite form is completely characterized by its rank and signature, (V_{K3}, Φ_{K3}) is isomorphic to $3H \oplus \Gamma_{16}$, where H is the hyperbolic form, and Γ_{16} is a negative definite even form of rank 16. We will construct G-actions on $3H$ and Γ_{16} separately.

Lemma 2.5. *For each integer k which satisfies $0 \le k \le 5$, there is a G-action on Γ_{16} such that*

$$\Gamma_{16} \cong (16-3k)\mathbb{Z} \oplus k\mathbb{Z}[G] \text{ as a } \mathbb{Z}[G]\text{-module.}$$

Proof. When $k = 0$, it suffices to take the trivial G-action. Hence we suppose $k \ge 1$.

Recall that the lattice Γ_{16} is the set of $(x_1, \dots, x_{16}) \in (\frac{1}{2}\mathbb{Z})^{16}$ which satisfy

(1) $x_i \equiv x_j \mod \mathbb{Z}$ for any i, j,
(2) $\sum_{i=1}^{16} x_i \equiv 0 \mod 2\mathbb{Z}$.

The unimodular bilinear form on Γ_{16} is defined by $-\sum_{i=1}^{16} x_i^2$.

Note that the symmetric group of degree 16 acts on Γ_{16} as permutations of components. For a fixed generator g of G, define the G-action on Γ_{16} by

$$g = (1,2,3)(4,5,6)\cdots(3k-2, 3k-1, 3k),$$

where (l, m, n) is the cyclic permutation of (x_l, x_m, x_n).

As a basis for Γ_{16}, we take

$$f_i = \begin{cases} e_i + e_{16} & (i = 1, \dots, 9), \\ e_i - e_{16} & (i = 10, \dots, 15), \\ \dfrac{1}{2}(e_1 + e_2 + \cdots + e_{16}) & (i = 16), \end{cases}$$

where $e_1, \dots, e_{16}$ is the usual orthonormal basis for $\mathbb{R}^{16}$. Then the basis $(f_1, f_2, \dots, f_{16})$ gives the required direct splitting. □

Lemma 2.6. *There is a G-action on $3H$ such that $3H \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G]$ as a $\mathbb{Z}[G]$-module, and G-fixed parts of a maximal positive definite subspace and a negative one of $3H \otimes \mathbb{R}$ both have rank 1.*

Proof. Such a G-action is given as permutations of three H's. □

With Lemma 2.5 and Lemma 2.6 understood, for each of A_1, A_2 and B, the corresponding G-action on (V_{K3}, Φ_{K3}) can be constructed. That is,

- for A_1, $3H \cong 6\mathbb{Z}$ and $\Gamma_{16} \cong \mathbb{Z} \oplus 5\mathbb{Z}[G]$,
- for A_2, $3H \cong 6\mathbb{Z}$ and $\Gamma_{16} \cong 4\mathbb{Z} \oplus 4\mathbb{Z}[G]$,
- for B, $3H \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G]$ and $\Gamma_{16} \cong \mathbb{Z} \oplus 5\mathbb{Z}[G]$.

Now the conditions *REP* and *GSF* are satisfied. Therefore we have a locally linear pseudofree G-action on a closed simply-connected 4-manifold X whose intersection form is just (V_{K3}, Φ_{K3}) by Theorem 2.1. Since X is simply-connected and its intersection form is even, we see that X is homeomorphic to the $K3$ surface by Freedman's theorem [7]. Thus assertion (1) is proved.

Remark 2.7. By using Theorem 1.3 in [3], we can prove that the topological conjugacy class of actions of the type B is unique, that is, any action of the type B is conjugate to the action which we have constructed.

Remark 2.8. We can also construct a locally linear pseudofree action of the type A_0 by Theorem 2.1. For this purpose, we need to construct a G-action on $3H$ such that $3H \cong 3\mathbb{Z} \oplus \mathbb{Z}[G]$ as a $\mathbb{Z}[G]$-module, the rank of a G-fixed maximal positive definite subspace of $3H \otimes \mathbb{R}$ is 3, and the rank of a negative one is 1. Such a G-action on $3H$ is constructed from the cohomology ring of a 4-torus with a G-action as follows:

Let $\zeta = \exp(2\pi\sqrt{-1}/3)$, and consider the lattice $\mathbb{Z} \oplus \zeta\mathbb{Z} \subset \mathbb{C}$. For each $i = 0, 1, 2$, let us consider a 2-torus $T_{\zeta^i} = \mathbb{C}/(\mathbb{Z} \oplus \zeta\mathbb{Z})$ with a G-action, where the G-action is defined by the multiplication by ζ^i. Next, consider the 4-torus $T_{12} = T_\zeta \times T_{\zeta^2}$ with the diagonal G-action. Then we can prove that the induced G-action on $H^2(T_{12}; \mathbb{Z})$ has the required properties.

Using this with a G-action on Γ_{16} such that $\Gamma_{16} \cong \mathbb{Z} \oplus 5\mathbb{Z}[G]$, we obtain a G-action of the type A_0 by Theorem 2.1.

3. The proof of assertion (2)

In this section, we consider X as the smooth $K3$ surface with the standard smooth structure. Suppose now that a smooth action of the type A_1 exists. To obtain a contradiction, we use a Seiberg-Witten invariant of X. Recall that, for a smooth 4-manifold with $b_1 = 0$ and $b_+ \geq 2$, Seiberg-Witten invariants constitute a map from the set of equivalence classes of Spin^c-structures on X to $\mathbb{Z}$. That is, for a Spin^c-structure c, the corresponding Seiberg-Witten invariant $\mathrm{SW}_X(c)$ is given as an integer.

We use the canonical Spin^c-structure c_0 which is characterized as one whose determinant line bundle L is trivial in the case of $K3$ surface X. Note that c_0 is also characterized as the Spin^c-structure which is determined by the Spin-structure.

Since X is simply-connected and L is trivial, we can see that every $G = \mathbb{Z}/3$-action on X lifts to a G-action on the Spin^c-structure c_0. Then, the G-index of the Dirac operator D_X can be written as

$$\mathrm{ind}_G\, D_X = \sum_{j=0}^{2} k_j \mathbb{C}_j \in R(G) \cong \mathbb{Z}[t]/(t^3 = 1),$$

where $\mathbb{C}_j$ is the complex 1-dimensional weight j representation of G and $R(G)$ is the representation ring of G.

F. Fang [5] proved the mod p vanishing theorem under a $\mathbb{Z}/p$-action, where p is a prime.

Theorem 3.1 ([5]). *Let Y be a smooth closed oriented 4-dimensional $\mathbb{Z}/p$-manifold with $b_1 = 0$ and $b_+ \geq 2$, where p is a prime. Suppose that c is a Spin^c-structure on which the $\mathbb{Z}/p$-action lifts, and that $\mathbb{Z}/p$ acts trivially on $H^+(Y; \mathbb{R})$. If $2k_j \leq b_+ - 1$ for $j = 0, \ldots, p-1$, then*

$$\mathrm{SW}_Y(c) \equiv 0 \mod p.$$

Remark 3.2. The second author generalized Theorem 3.1 to the case when $b_1 > 0$ [14].

On the other hand, it is well known that $\mathrm{SW}_X(c_0) = \pm 1$ for the standard $K3$ surface X. (See, e.g., [9] or [16].) Therefore, in the case when G acts on (X, c_0), we have $k_j > 1$ for some j by Theorem 3.1.

Coefficients k_j are calculated by the G-spin theorem. (For the G-spin theorem, we refer the reader to [1, 2, 13, 15].) For the fixed generator $g \in G$, the Lefschetz number $\mathrm{ind}_g D_X$ is calculated by the formula as

$$\mathrm{ind}_g D_X = \sum_{j=0}^{2} \zeta^j k_j = \sum_{P \in X^G} \nu(P),$$

where $\zeta = \exp(2\pi\sqrt{-1}/3)$ and $\nu(P)$ is a complex number associated to each fixed point P given as follows.

Suppose that a fixed point P has the representation type (a, b) with respect to g. Then the number $\nu(P)$ associated to P is given by

$$\nu(P) = \frac{1}{(\zeta^a)^{1/2} - (\zeta^a)^{-1/2}} \frac{1}{(\zeta^b)^{1/2} - (\zeta^b)^{-1/2}}. \tag{3.3}$$

The signs of $(\zeta^a)^{1/2}$ and $(\zeta^b)^{1/2}$ are determined such that

$$\left\{(\zeta^a)^{1/2}\right\}^3 = \left\{(\zeta^b)^{1/2}\right\}^3 = 1.$$

(This is because, in our case, the g-action on the Spin-structure generates a G-action on the Spin-structure. See [2, p.20] or [15, p.175].)

With the above understood, we obtain

$$\begin{aligned} \mathrm{ind}_g D_X &= k_0 + \zeta k_1 + \zeta^2 k_2 = \frac{1}{3}(m_+ - m_-), \\ \mathrm{ind}_{g^2} D_X &= k_0 + \zeta^2 k_1 + \zeta k_2 = \frac{1}{3}(m_+ - m_-), \\ \mathrm{ind}_1 D_X &= k_0 + k_1 + k_2 = 2. \end{aligned}$$

Solving these equations, we have

$$\begin{aligned} k_0 &= \frac{1}{9}\left\{6 + 2(m_+ - m_-)\right\}, \\ k_1 = k_2 &= \frac{1}{9}\left\{6 - (m_+ - m_-)\right\}. \end{aligned}$$

In the case of an action of type A_1, $m_+ = 3$ and $m_- = 6$. Hence, we have $k_0 = 0$ and $k_1 = k_2 = 1$. Therefore there is no j so that $k_j > 1$. This is a contradiction. Thus assertion (2) is proved.

Remark 3.4. It is clear that a proposition similar to (2) of Theorem 1.2 is true for the smooth structure such that the Seiberg-Witten invariant for the Spin^c-structure with trivial determinant line bundle is not congruent to 0 modulo 3. Let us examine elliptic surfaces which are homeomorphic to $K3$. Consider relatively minimal regular elliptic surfaces with at most two multiple fibers whose Euler number is 24.

Let p and q be the multiplicities of multiple fibers, and let us write such elliptic surfaces as $E(2)_{p,q}$. (We assume that p and q may be 1.) The following are known about $E(2)_{p,q}$.

(1) $E(2)_{1,1}$ (no multiple fiber) is diffeomorphic to the standard $K3$.
(2) $E(2)_{p,q}$ is homeomorphic to the $K3$ surface if and only if $\gcd(p,q) = 1$. (See, e.g., [17].)
(3) $E(2)_{p,q}$ is not diffeomorphic to $E(2)_{p',q'}$ if $pq \neq p'q'$ [8].
(4) Let c_0 be the Spin^c-structure with trivial determinant line bundle. If p and q are odd, then $\mathrm{SW}_{E(2)_{p,q}}(c_0) = \pm 1$ [10, 6].

Thus we see that the type A_1 cannot be realized by a smooth action on $E(2)_{p,q}$ such that $\gcd(p,q) = 1$ and p and q are odd. Note that there are infinitely many (p,q) which give different smooth structures.

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Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail address: liudlut@yahoo.com

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan

E-mail address: nakamura@kurims.kyoto-u.ac.jp