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BOUNDS AND A MAJORIZATION FOR THE REAL PARTS OF THE ZEROS OF POLYNOMIALS

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Abstract. We apply some eigenvalue inequalities to the real parts of the Frobenius companion matrices of monic polynomials to establish new bounds and a majorization for the real parts of the zeros of these polynomials.

1. Introduction

Matrix analysis methods have been used by several mathematicians to obtain new proofs of classical bounds for the zeros of polynomials and to derive new bounds for these zeros. These methods include eigenvalue locations, matrix norms computations, eigenvalue-singular value majorization relations, and numerical radii estimations. See, e.g., [1]–[2], [4]–[7], [9]–[14], [16], and the references therein.

Let

$$p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1 \tag{1}$$

be a monic polynomial of degree $n \geq 2$, with complex coefficients.

Then the Frobenius companion matrix of p is given by

$$C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{2}$$

It is well known that the zeros of p are exactly the eigenvalues of $C(p)$. See, e.g., [7, p. 316].

Using a numerical radius estimation of $C(p)$, it has been shown in [5] (see also [11] for a different proof) that if z is any zero of p, then

$$|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|a_n| + \sqrt{\sum_{j=1}^{n} |a_j|^2}\right). \tag{3}$$

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In this paper we apply some eigenvalue inequalities to the real part of $C(p)$ to establish new bounds and a majorization for the real parts of the zeros of p that are related to the bound (3).

2. Preliminary results

Let $z_1, z_2, \cdots, z_n$ be the zeros of p (or the eigenvalues of $C(p)$). To obtain our new bounds and majorization for $\operatorname{Re} z_1, \cdots, \operatorname{Re} z_n$, we need several lemmas involving inequalities and majorization relations for eigenvalues.

Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, the eigenvalues of A are denoted by $\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A)$. If A is Hermitian, then the eigenvalues of A are arranged in such a way that $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

For two sequences of real numbers arranged in decreasing order,

$$x = (x_1, x_2, \cdots, x_n) \quad \text{and} \quad y = (y_1, y_2, \cdots, y_n),$$

we say that x is majorized by y if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j \quad \text{for} \quad k = 1, 2, \cdots, n-1$$

and

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

For the theory of majorization, the reader is referred to [3], [8], and [15].

Lemma 1. *Let $A \in M_n(\mathbb{C})$ with real part $\operatorname{Re} A = \frac{A+A^*}{2}$. Then*

$$\lambda_n(\operatorname{Re} A) \leq \operatorname{Re} \lambda_j(A) \leq \lambda_1(\operatorname{Re} A) \quad \textit{for} \quad j = 1, 2, \cdots, n. \tag{4}$$

Lemma 2. *Let B, $C \in M_n(\mathbb{C})$ be Hermitian. Then*

$$\lambda_j(B) + \lambda_n(C) \leq \lambda_j(B+C) \leq \lambda_j(B) + \lambda_1(C) \quad \textit{for} \quad j = 1, 2, \cdots, n. \tag{5}$$

In particular,

$$\lambda_1(B+C) \leq \lambda_1(B) + \lambda_1(C) \tag{6}$$

and

$$\lambda_n(B) + \lambda_n(C) \leq \lambda_n(B+C). \tag{7}$$

Lemma 3. *Let $A \in M_n(\mathbb{C})$ with eigenvalues arranged in such a way that $\operatorname{Re} \lambda_1(A) \geq \operatorname{Re} \lambda_2(A) \geq \cdots \geq \operatorname{Re} \lambda_n(A)$. Then*

$$\sum_{j=1}^{k} \operatorname{Re} \lambda_j(A) \leq \sum_{j=1}^{k} \lambda_j(\operatorname{Re} A) \quad \textit{for} \quad k = 1, 2, \cdots, n-1 \tag{8}$$

and

$$\sum_{j=1}^{n} \operatorname{Re} \lambda_j(A) = \sum_{j=1}^{n} \lambda_j(\operatorname{Re} A). \tag{9}$$

Lemma 4. *Let B, $C \in M_n(\mathbb{C})$ be Hermitian. Then*

$$\sum_{j=1}^{k} \lambda_j(B+C) \le \sum_{j=1}^{k} \lambda_j(B) + \sum_{j=1}^{k} \lambda_j(C) \quad \text{for} \quad k = 1, 2, \cdots, n-1 \tag{10}$$

and

$$\sum_{j=1}^{n} \lambda_j(B+C) = \sum_{j=1}^{n} \lambda_j(B) + \sum_{j=1}^{n} \lambda_j(C). \tag{11}$$

Lemmas 1 and 2 are immediate consequences of the Courant-Fischer-Weyl minimax principle, and Lemmas 3 and 4 follow from Ky Fan's maximum principle. See, e.g., [3, Chapter III], [7, Chapter 4], [8, Chapter 3], and [15, Chapter 9]. It should be mentioned here that the real parts in Lemmas 1 and 3 can be replaced by the imaginary parts.

3. Main results

Our new bounds for the real parts of the zeros of p can be stated as follows.

Theorem 1. *For $j = 1, 2, \cdots, n$, we have*

$$\begin{aligned} \frac{1}{2}\left(-\operatorname{Re} a_n - \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2}\right) + \cos\frac{n\pi}{n+1} &\le \operatorname{Re} z_j \\ &\le \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2}\right) + \cos\frac{\pi}{n+1}. \end{aligned} \tag{12}$$

Proof. It follows from (2) that

$$\operatorname{Re} C(p) = \begin{bmatrix} -\operatorname{Re} a_n & (1-a_{n-1})/2 & -a_{n-2}/2 & \cdots & -a_1/2 \\ (1-\bar{a}_{n-1})/2 & 0 & 1/2 & \cdots & 0 \\ -\bar{a}_{n-2}/2 & 1/2 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1/2 \\ & & & \ddots & \\ -\bar{a}_{1/2} & 0 & 0 \cdots & 1/2 & 0 \end{bmatrix}.$$

Thus,

$$\operatorname{Re} C(p) = S_n + T_n,$$

where S_n is the partitioned matrix

$$S_n = \begin{bmatrix} -\operatorname{Re} a_n & x^* \\ x & 0 \end{bmatrix}$$

with $x = [-\frac{1}{2}\bar{a}_{n-1}, -\frac{1}{2}\bar{a}_{n-2}, \cdots, -\frac{1}{2}\bar{a}_1]^t$, and T_n is the $n \times n$ tridiagonal matrix

$$T_n = \begin{bmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 \\ & & & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1/2 \\ 0 & 0 & 0 \cdots & 1/2 & 0 \end{bmatrix}.$$

It can easily be shown that the eigenvalues of S_n are

$$\lambda_1(S_n) = \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2}\right), \tag{13}$$

$$\lambda_n(S_n) = \frac{1}{2}\left(-\operatorname{Re} a_n - \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2}\right), \tag{14}$$

and

$$\lambda_j(S_n) = 0 \quad \text{for} \quad j = 2, \cdots, n-1. \tag{15}$$

It is well known that the eigenvalues of T_n are

$$\lambda_j(T_n) = \cos\frac{j\pi}{n+1} \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{16}$$

Applying Lemmas 1 and 2 to $C(p)$ and the Hermitian matrices S_n, T_n, respectively, we obtain

$$\lambda_n(S_n) + \lambda_n(T_n) \le \lambda_n(\operatorname{Re} C(p)) \le \operatorname{Re} z_j \le \lambda_1(\operatorname{Re} C(p)) \le \lambda_1(S_n) + \lambda_1(T_n) \tag{17}$$

for $j = 1, 2, \cdots, n$. Now, the desired bounds (12) follow from (13), (14), (16), and (17).

By considering the imaginary part of $C(p)$, an analysis similar to that used in deriving the bounds (12) enables us to obtain analogous bounds for the imaginary parts of the zeros of p. These bounds, when combined with (12), describe a rectangle that contains all the zeros of p. Related rectangles have been described in [11, Theorem 3].

Our second result gives a majorization for the real parts of the zeros of p.

Theorem 2. *If the zeros of p are arranged in such a way that $\operatorname{Re} z_1 \ge \operatorname{Re} z_2 \ge \cdots \ge \operatorname{Re} z_n$, then*

$$\sum_{j=1}^{k} \operatorname{Re} z_j \le \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2}\right) + \frac{1}{2}\left(\frac{\sin\left(\frac{2k+1}{2n+2}\right)\pi}{\sin\frac{\pi}{2n+2}}\right) - \frac{1}{2} \tag{18}$$

for $k = 1, 2, \cdots, n-1$ and

$$\sum_{j=1}^{n} \operatorname{Re} z_j = -\operatorname{Re} a_n. \tag{19}$$

Proof. Applying Lemmas 3 and 4 to $C(p)$ and the Hermitian matrices S_n, T_n, respectively, we obtain, in view of our analysis in the proof of Theorem 1, that

$$\begin{aligned}\sum_{j=1}^{k} \operatorname{Re} z_j &= \sum_{j=1}^{k} \operatorname{Re} \lambda_j(C(p)) \\ &\le \sum_{j=1}^{k} \lambda_j(\operatorname{Re}(C(p))) \\ &\le \sum_{j=1}^{k} \lambda_j(S_n) + \sum_{j=1}^{k} \lambda_j(T_n) \\ &= \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2}\right) + \sum_{j=1}^{k} \cos\frac{j\pi}{n+1} \qquad (20)\end{aligned}$$

for $k = 1, 2, \cdots, n-1$. But, for every real number $t \neq 2m\pi$ (m is an integer), we have

$$\sum_{j=1}^{k} \cos jt = \frac{1}{2}\left(\frac{\sin(2k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right) - \frac{1}{2}. \qquad (21)$$

Thus,

$$\sum_{j=1}^{k} \cos\frac{j\pi}{n+1} = \frac{1}{2}\left(\frac{\sin\left(\frac{2k+1}{2n+2}\right)\pi}{\sin\frac{\pi}{2n+2}}\right) - \frac{1}{2}, \qquad (22)$$

which, together with (20), yields the desired majorization (18). The relation (19) follows from the fact that $\sum_{j=1}^{n} z_j = -a_n$. This completes the proof of the theorem.

Finally, we remark that by similar arguments, we can establish analogous majorization for the imaginary parts of the zeros of p. It should be mentioned here that inequalities for the absolute values of the imaginary parts of the zeros of polynomials have been given in [16].

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