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DUHAMEL SOLUTIONS OF NON-HOMOGENEOUS q^2-ANALOGUE WAVE EQUATIONS

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Abstract. q-analogue non-homogeneous wave equations are solved by a Duhamel solution strategy using constructions with q-analogue Fourier multipliers to compensate for the dependence of the analogue differential Leibnitz rule on the parity of the functions involved.

1. Introduction

Classically, Duhamel's Principle exploits interconnections between commutative algebraic and differential structures to construct an elegant solution to the non-homogeneous wave equation. The goal of this paper is to provide a similar construction for a q-analogue context. In particular, our development, made with the aid of q-analogue Fourier multipliers, realizes the Duhamel strategy for q-analogue non-homogeneous wave equations. A key challenge these constructions must overcome is the sensitivity of the analogue differential Leibnitz rule to the parities of the functions involved, which would appear to destroy symmetries essential to the classical solution technique.

The analogue transform we employ to make our constructions is based on analogue trigonometric functions and orthogonality results from [6] which have important applications to q-deformed quantum mechanics. Our wave operator is defined in terms of the q^2-analogue differential operator, ∂_q, introduced in [7]. This operator has correct eigenvalue relationships for analogue exponential Fourier analysis using the functions and orthogonalities of [6].

Our approach has the pleasant feature that many preliminary calculations follow familiar patterns of commutative analysis as given in, say, [8] or [5]. The main result, Theorem 9, presents the delicate constructions used to counteract the effect of deficiencies in the analogue differential structure.

We begin by renormalizing the principal operator used in [7], making certain analogue calculations better reflect their classical counterparts. We then turn to implementing the D'Alembert and Duhamel techniques for studying wave equations, using the Fourier multiplier tools to aid in our constructions. In our opinion, the Duhamel calculation demonstrates the effectiveness and elegance of this analogue Fourier multiplier approach via the nature of the symmetries it imposes on this fundamental problem.

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The renormalization of the fundamental q^2-analogue differential operator of [7] we use is

$$\partial_q f(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}. \tag{1}$$

(This amounts to multiplying the operator (11) in [7] by $\frac{2}{1-q}$.) We will also use the related analogue partial differential operators $\partial_{q,z}y(z,t), \partial_{q,t}y(z,t)$, where, e.g.,

$$\partial_{q,z}y(z,t) = \frac{y(q^{-1}z,t) + y(-q^{-1}z,t) - y(qz,t) + y(-qz,t) - 2y(-z,t)}{2(1-q)z}. \tag{2}$$

Note that if f is differentiable at z, then $\lim_{q\to 1} \partial_q f(z) = \frac{df}{dz}(z)$ and $\lim_{q\to 1} \partial_{q,z}y(z,t) = \frac{\partial y}{\partial z}(z,t)$. ∂_q is closely related to the classical q-derivative operators.

Property 1. *If f is odd, $\partial_q f(z) = \frac{f(z)-f(qz)}{(1-q)z}$, while if f is even, $\partial_q f(z) = \frac{f(q^{-1}z)-f(z)}{(1-q)z}$.*

We adopt the notational conventions of [6] and of [3]. In addition, we set $f_w(z) \equiv f(w^{-1}z)$, where w represents a complex constant. Unless otherwise specified, always assume that functions are defined (if necessary, in each variable) on sets, S, which are symmetric in the following sense: if $z \in S$, then $-z \in S$ and $\pm q^{\pm 1}z \in S$. $\mathcal{X}_S$ is the indicator function of S.

Throughout this paper, always take $q \in (0,1)$. $\mathcal{Q} \equiv \{\pm q^k : k \in \mathbb{Z}\}$ and $\mathcal{Q}^+ \equiv \{q^k : k \in \mathbb{Z}\}$. The notation $t > 0$, for $t \in \mathcal{Q}$, means $t \in \mathcal{Q}^+$. For $x \in \mathcal{Q}$, $\operatorname{sgn}(x) \equiv \frac{|x|}{x}$. We use the q-integral introduced by Jackson, which we consider as an integral on $\mathcal{Q}$, i.e., writing $d_q x$ implies $x \in \mathcal{Q}$.

$$\int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} \{f(q^n) + f(-q^n)\} q^n.$$

$$L^p d_q \equiv \{f : \int_{-\infty}^{\infty} |f(x)|^p d_q t < \infty\} \text{ and } L^\infty d_q \equiv \{f : \sup\left[|f(\pm q^k)| : k \in \mathbb{Z}\right] < \infty\}.$$

As we saw in [7] (cf. Lemma 1b) in the next section), ∂_q possesses important eigenvalue properties with respect to the q^2-analogue trigonometric functions studied by Koornwinder and Swarttouw in [6] (here with z replaced by $(1-q)z$ in the power series):

$$\cos(z;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}((1-q)z)^{2k}}{(q;q)_{2k}},$$

$$\sin(z;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}((1-q)z)^{2k+1}}{(q;q)_{2k+1}}.$$

These functions induce a ∂_q-adapted q^2-analogue exponential by

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2). \tag{3}$$

$e(z;q^2)$ is absolutely convergent for all z in the plane, $0 < q < 1$, since both of its component functions are. $\lim_{q\uparrow 1} e(z;q^2) = exp(z)$ pointwise and uniformly on compacta, because both of its component functions satisfy corresponding limits (cf. [6, p. 459]).

Because the orthogonalities we use reside on $\mathcal{Q} = \{\pm q^k : k \in \mathbb{Z}\}$, and also to get convergence of our analogue functions to their classical counterparts as $q \uparrow 1$,

as in [6] and [7], we impose the condition that $1-q=q^{2m}$ for some integer m. Therefore, in the remainder of this paper, assume that

$$q \in \{q \in (0,1) : 1-q = q^{2m} \text{ for some integer } m\}. \tag{4}$$

2. The operator ∂_q and the related q^2-analogue Fourier Transform

This section contains the basic tools we apply in the remainder of the paper. For the reader's convenience, we provide some statements (Lemmas 1a)-c),e); 2a),b); Theorem 3) which are renormalized versions or minor extensions of material in [7].

We list some useful facts. e) and f) reflect the non-classical q-nature of this analogue differentiation. Verifying the formulas, except for c), is straightforward. c) is contained in Lemma 6 of [7]. d) follows from the orthogonalities [6, (2.10), p. 450].

Lemma 1. a) $\alpha \in \mathbb{C} \Rightarrow \alpha[\partial_q(f_\alpha)(z)] = (\partial_q f)_\alpha(z)$.
b) $\partial_q \sin(z;q^2) = \cos(z;q^2)$, $\partial_q \cos(z;q^2) = -\sin(z;q^2)$, $\partial_q e(z;q^2) = e(z;q^2)$.
c) $|\cos(x;q^2)|$, $|\sin(x;q^2)|$, *and* $|e(ix;q^2)|$ *are bounded for* $x \in \mathcal{Q}$.
d) *On* $\mathcal{Q}$, $\lim_{x\to\pm\infty} x^{\frac{1}{2}} \sin(x;q^2) = \lim_{x\to\pm\infty} x^{\frac{1}{2}} \cos(x;q^2) = 0$.
e) *If* f *and* g *are both odd,* $\partial_q(fg)(z) = q^{-1}(\partial_q f)(\frac{z}{q})g(z) + q^{-1}f(\frac{z}{q})(\partial_q g)(\frac{z}{q})$.
f) *If* f *is odd and* g *is even,* $\partial_q(fg)(z) = (\partial_q f)(z)g(z) + qf(qz)(\partial_q g)(qz)$.

Explicit bounds for Lemma 1c) were discussed in [7]. We will use the following.

Corollary 1. *For* $x \in \mathcal{Q}$, $\exists B_q > 0$ *such that either* $|\cos(q^{-\frac{1}{2}}x;q^2)| \geq \frac{1}{B_q}$ *or* $|\sin(q^{\frac{1}{2}}x;q^2)| \geq \frac{1}{B_q}$.

Proof. In [1, (4.8) p. 139], it is shown that if $C_q(z) = \cos(q^{-\frac{3}{4}}z;q^2)$ and $S_q(z) = \sin(q^{-\frac{1}{4}}z;q^2)$, then $S_q(q^{-\frac{1}{2}}z)S_q(z) + C_q(q^{-\frac{1}{2}}z)C_q(z) = 1$. Translated to our setup, $\sin(q^{-\frac{1}{2}}z;q^2)\sin(z;q^2) + \cos(\frac{z}{q};q^2)\cos(q^{-\frac{1}{2}}z;q^2) = 1$. Thus $|\sin(q^{-\frac{1}{2}}z;q^2)\sin(z;q^2)| + |\cos(\frac{z}{q};q^2)\cos(q^{-\frac{1}{2}}z;q^2)| \geq 1$. Take $q^{-\frac{1}{2}}z \equiv x \in \mathcal{Q}$. By Lemma 1c), $\exists B_q \in \mathbb{R}$ such that $|\sin(x;q^2)|$ and $|\cos(x;q^2)| \leq \frac{B_q}{2}$. Use the pigeonhole principle. □

([1], cited above, gives another interesting approach to analogue expansions using different operators and orthogonalities.)

The following simple properties, reflecting the discrete singular nature of q-integrals on $\mathcal{Q}$ and the finite nature of $\partial_{q,z}$, can be verified by direct calculation.

Lemma 2. *If the stated integrals exist, then:*
a) $s \in \mathbb{Z} \Rightarrow \int_{-\infty}^{\infty} f(q^s x)d_q x = \int_{-\infty}^{\infty} f(x)q^{-s}d_q x$.
b) *If* $\int_{-\infty}^{\infty} (\partial_q f)(x)g(x)d_q x$ *exists,*

$$\int_{-\infty}^{\infty} (\partial_q f)(x)g(x)d_q x = -\int_{-\infty}^{\infty} f(x)(\partial_q g)(x)d_q x.$$

c) $\int_{-\infty}^{\infty} |f(x)|^p d_q x = 0 \Leftrightarrow f(x) = 0 \ \forall x \in \{\pm q^k : k \in \mathbb{Z}\}$.
d) $\partial_{q,t}[\int_{-\infty}^{\infty} y(x,t)d_q x] = \int_{-\infty}^{\infty} [\partial_{q,t} y](x,t)d_q x$.

Remark. Lemma 2c) says that, on $\mathcal{Q}$, an identity in $L^p d_q$ is a pointwise identity.

As in [7], with $\Gamma_q(x) \equiv \frac{(q;q)_\infty}{(q^2;q)_\infty}(1-q)^{(1-x)}$ (cf. [3]), we define the q^2-analogue Fourier Transform to be

$$\widehat{f}(x;q^2) \equiv K\int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt, \quad \text{where } K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}. \tag{5}$$

Letting $q \uparrow 1$ subject to condition (4), gives, formally, the classical Fourier Transform on the line. (See [6] and [2] for related results.)

Since $|e(\pm iq^k;q^2)|$ is bounded for all integers k, the q^2-Fourier Transform defines a bounded linear operator from L^1d_q to $L^\infty d_q$. Also, $(L^1 \cap L^2)d_q$ is dense in L^2d_q. (Consider functions with finite support.) Since the q^2-analogue Fourier Transform is defined and bounded on $(L^1 \cap L^2)d_q$ for such functions, it defines a bounded extension to all of L^2d_q.

Several q^2-analogue Fourier Transform results we will need are contained in

Theorem 3. a) *If $f, g \in L^1d_q$, then $\int_{-\infty}^{\infty} \widehat{f}(x;q^2)g(x)d_qx = \int_{-\infty}^{\infty} f(x)\widehat{g}(x;q^2)d_qt$.*

b) *If $f \in L^1d_q$ and s is an integer, $(f_{q^{-s}})\widehat{\ }(x) = q^{-s}(\widehat{f})_{q^s}(x)$.*

c) *If $f(u), uf(u) \in L^1d_q, \partial_q\widehat{f}(x;q^2) = (-iuf(u))\widehat{\ }(x)$.*

d) *If $f, \partial_q f \in L^1d_q : (\partial_q f)\widehat{\ }(x;q^2) = ix\widehat{f}(x;q^2)$.*

e) *$f \in L^2d_q$ implies $f(q^n) = K\int_{-\infty}^{\infty} \widehat{f}(x;q^2)e(ixq^n;q^2)d_qx$.*

f) *$f \in L^2d_q$ implies $\|f\|_2 = \|\widehat{f}(\cdot;q^2)\|_2$.*

Proof. a), b), c) are routine calculations. d) follows by expanding the left side using (5) and then applying Lemmas 1b),c) and 2b). e) is essentially Theorem 1 of [7], where we use two results of Koornwinder and Swarttouw and (2.11) and (5.5) of [6] to obtain Fourier inversion in $(L^1 \cap L^2)d_q$. The extension to all of L^2d_q follows by standard arguments from the fact that $(L^1 \cap L^2)d_q$ is dense in L^2d_q via, e.g., functions with finite support. f) is Corollary 3 in [7]. □

An example we will use later is the function defined on $\mathcal{Q}$ by

$$\delta_y(x) = \begin{cases} [(1-q)y]^{-1} & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Using Theorem 3, it is not hard to establish:

Property 2. *For $y \in \mathcal{Q}$:*

a) $\int_{-\infty}^{\infty} \delta_y(x)g(x)d_qx = g(y)$.

b) $\|\delta_y(x)\|_2 = |[(1-q)y]^{-1}|$ *and* $\|\delta_y(x)\|_1 = 1$.

c) $\widehat{\delta}_y(x;q^2) = Ke(-iyx;q^2) \in L^2d_qx$.

We also require an inverse operator to ∂_q. We adapt the finite Jackson q-integral, [3], to our operators by making the following definition:

$$\partial_q^{-1}f(x) \equiv \frac{(1-q)x}{2}\sum_{n=0}^{\infty} q^n\{f(xq^n) - f(-xq^n) + qf(xq^{n+1}) + qf(-xq^{n+1})\}. \tag{6}$$

We list some easily verified properties of (6). c) is Theorem 3c) reformulated.

Lemma 4. a) *Assuming $\partial_q^{-1}f(x)$ exists, for any constant E, $\partial_q[\partial_q^{-1}f(x) + E] = f(x)$.*

b) *$f \in L^1d_q \Rightarrow \partial_q^{-1}f(x)$ exists.*

c) *$\widehat{f}(x;q^2), [\widehat{\partial_q^{-1}f}](x;q^2) \in L^1d_q \Rightarrow [\widehat{\partial_q^{-1}f}](x;q^2) = -ix^{-1}\widehat{f}(x;q^2)$.*

3. The Fourier multipliers

For $x, y \in \mathcal{Q}$, the Fourier multiplier operator corresponding to translation by y is

$$f(x \ominus y) \equiv [T_y f](x) \equiv K \int_{-\infty}^{\infty} e(-ity; q^2)\widehat{f}(t; q^2)e(itx; q^2)d_q t, \tag{7}$$

whenever the integral makes sense. We write $f(x \oplus y) \equiv f(x \ominus (-y))$.

The notation $\lim_{h\to 0} f(h) = L$, on $\mathcal{Q}$, will mean $\lim_{n\to\infty} f(\pm q^n) = L$.

Lemma 5. *If $f \in L^2 d_q$:*
a) *$f(x \ominus y) \in L^2 d_q x$, with $\|T_y\|_{2,2} \le \|e(-i[\cdot]; q^2)\|_\infty$,*
b) *$\lim_{y\to 0} f(x \ominus y) = f(x)$.*

Proof. a): If $f \in L^2 d_q$, then the fact that $|e(-i[\cdot]; q^2)|$ is bounded on $\mathcal{Q}$ (by Lemma 1c)) and the Plancherel Theorem show that $e(-iyt; q^2)\widehat{f}(t) \in L^2 d_q t$ independently of y, so $\|T_y f\|_{2,t} = \|\widehat{T_y f}\|_{2,t} \le \|e(-i[\cdot]; q^2)\|_\infty \|\widehat{f}\|_2 = \|e(-i[\cdot]; q^2)\|_\infty \|f\|_2$.

b): Observing that using (3), we get $\lim_{h\to 0} e(-ihy; q^2) = 1$, the result follows by (7), the boundedness of e, dominated convergence, and Theorem 3. □

Note that $f(x \oplus y) = f(y \oplus x)$ and that $f(x \oplus (y \oplus z)) = f((x \oplus y) \oplus z)$. Also, $\lim_{q\uparrow 1} f(x \ominus y) = f(x - y)$ due to the approximating behavior of the Fourier Transform and the exponential as $q \uparrow 1$.

We present some formulas that will be used in the next section and which follow by standard calculations using Theorem 3d), and Lemmas 1a) and 2.

Lemma 6. *If $f, \partial_q f \in (L^1 \cap L^2)d_q$, then $\partial_q[T_y f](x) = [T_y(\partial_q f)](x)$ and $\partial_{q,y}[T_{cy} f](x) = -c[T_{cy}(\partial_q f)](x)$.*

For $f \in L^2 d_q$, $g \in L^1 d_q$, define the multiplier corresponding to Fourier convolution of f with g to be

$$f * g(z) \equiv \int_{-\infty}^{\infty} [T_a f](z) g(a) d_q a \equiv \int_{-\infty}^{\infty} f(z \ominus a) g(a) d_q a. \tag{8}$$

Lemma 7. *$f \in L^2 d_q, g \in L^1 d_q \Rightarrow f * g \in L^2 d_q$, and $\|f * g\|_2 \le \|e(-i[\cdot]; q^2)\|_\infty \|g\|_1 \|f\|_2$.*

Proof. By Lemma 5, $\|T_a\|_{2,2} \le \|e(-i[\cdot]; q^2)\|_\infty$, independent of a. Using Minkowski's integral inequality (cf. [4]),

$$\begin{aligned}\|f * g\|_2 &\le \int_{-\infty}^{\infty} \|[T_a f](z) g(a)\|_{2,z} d_q a \\ &\le \int_{-\infty}^{\infty} \|[T_a]\|_{2,2} \|f\|_2 |g(a)| d_q a \le \|e(-i[\cdot]; q^2)\|_\infty \|f\|_2 \|g\|_1.\end{aligned}$$

□

The convolution multiplier behaves classically on $(L^1 \cap L^2)d_q$.

Theorem 8. *$f, g \in (L^1 \cap L^2)d_q \Rightarrow \widehat{(f * g)} = \widehat{f}\widehat{g}$.*

Proof. $f, g \in (L^1 \cap L^2)d_q \Rightarrow \widehat{f}, \widehat{g} \in (L^\infty \cap L^2)d_q \Rightarrow \widehat{f}\widehat{g} \in L^2 d_q$. Using Theorem 3, (5), and Fubini's theorem, we get:

$$\begin{aligned}
\widehat{g}(z;q^2)\widehat{f}(z;q^2) &= K\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\widehat{g}(t;q^2)\widehat{f}(t;q^2)e(itx;q^2)d_qt]e(-ixz;q^2)d_qz \\
&= K\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}[K\int_{-\infty}^{\infty}e(-ity;q^2)g(y)d_qy]\widehat{f}(t;q^2)e(itx;q^2)d_qt]e(-ixz;q^2)d_qz \\
&= K\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}[K\int_{-\infty}^{\infty}e(-ity;q^2)\widehat{f}(t;q^2)e(itx;q^2)d_qt]g(y)d_qy]e(-ixz;q^2)d_qz \\
&= \widehat{f*g}(z).
\end{aligned}$$

□

Observe that $f * g = g * f$ and $(f * g) * h = f * (g * h)$ follow immediately from Theorem 8.

4. Applications to ∂_q wave equations

We begin with a D'Alembert solution and a Fourier solution to the q-analogue homogeneous wave equation in ∂_q. Define

$$\Box_{c,q}y(x,t) \equiv \{c^{-2}\partial_{q,t}\circ\partial_{q,t} - \partial_{q,x}\circ\partial_{q,x}\}y(x,t).$$

Also define $\Box_q y(x,t) \equiv \Box_{1,q}y(x,t)$.

Corollary 2.
a) $f, g \in (L^1 \cap L^2)d_q$ *and* $y(x,t) = f(x \ominus ct) + g(x \oplus ct) \Rightarrow \Box_{c,q}y(x,t) = 0$.
b) $f, \partial_q^{-1}g \in (L^1 \cap L^2)d_q$ *and*

$$\begin{aligned}
y(x,t) &= \frac{1}{2}\{f(x\oplus ct) + f(x\ominus ct)\} + \frac{1}{2c}\{\partial_q^{-1}g(x\oplus ct) - \partial_q^{-1}g(x\ominus ct)\} \\
&= K\int_{-\infty}^{\infty}\{\cos(ctu;q^2)\widehat{f}(u;q^2) + \frac{i}{c}\sin(ctu;q^2)\widehat{[\partial_q^{-1}g]}(u;q^2)\}e(iux;q^2)d_qu \\
&\Rightarrow \Box_{c,q}y(x,t) = 0, \ \textit{with} \ \lim_{t\to 0}y(x,t) = f(x), \ \textit{and} \ \lim_{t\to 0}\partial_{q,t}y(x,t) = g(x).
\end{aligned}$$

Proof. The D'Alembert formulas in a) and b) follow from direct computation using Lemma 6. The initial conditions follow from Lemma 5b). The Fourier formula in b) also follows by direct computation using Lemma 1 and Theorem 3. □

(If one prefers convolution notation, Theorems 3, 8, and Property 2 give $f(x \ominus y) = \frac{1}{K}(\delta_y * f)(x)$. So, e.g., for $f \in (L^1 \cap L^2)d_q$, a solution to $\Box_{c,q}y(x,t) = 0$, $\lim_{t\to 0}y(x,t) = f(x)$, $\lim_{t\to 0}\partial_{q,t}y(x,t) = 0$ can be written as $y(x,t) = [\frac{1}{2K}(\delta_{ct} + \delta_{-ct}) * f](x)$.)

Next, we study a non-homogeneous q-analogue wave equation on $\mathcal{Q} \times \mathcal{Q}^+$. For simplicity set $c = 1$. We adapt a Duhamel-type strategy for our construction.

Since we will be working on $\mathcal{Q} \times \mathcal{Q}^+$, we need to clarify the meaning of some of our operators on $\mathcal{Q}^+$.

For $f : \mathcal{Q}^+ \to \mathbb{C}$, we use the ordinary Jackson finite $q-$integral:

$$\int_0^t f(s)d_qs \equiv (1-q)t\sum_{n=0}^{\infty}q^n f(tq^n), \tag{9}$$

i.e., $\int_0^{q^M} f(s)d_qs = (1-q)\sum_{n=M}^{\infty} q^n f(q^n)$. Let $f : \mathcal{Q}^+ \to \mathbb{C}$; then if $\int_0^t f(s)d_qs$ exists for some $t > 0$, it also exists for all $t > 0$. Define $L^p d_q^+ \equiv \{f : \mathcal{Q}^+ \to \mathbb{C} : \exists t \ni \int_0^t |f(s)|^p d_qs < \infty\} = \{f : \mathcal{Q}^+ \to \mathbb{C} : \forall t,\ \int_0^t |f(s)|^p d_qs < \infty\}$. To motivate part of the construction in the next paragraph, extend the finite integral on $\mathcal{Q}^+$ to $\mathcal{Q}$ using the odd extension: for $t > 0$, $[\int_0^{\pm t} f(s)d_qs]_o \equiv \pm\int_0^t f(s)d_qs$. Then, by Property 1, the classical Jackson q-calculus yields that for $t > 0, \partial_q[\int_0^t f(s)d_qs]_o = f(t)$.

To define ∂_q on $\mathcal{Q}^+$, we employ a method based on the observation that, on $\mathcal{Q}$, for linear combinations of functions of known parity, ∂_q is a linear operator determined by applying the formulas given in Property 1, which restrict nicely to $\mathcal{Q}^+$. We use this idea to define ∂_q on $\mathcal{Q}^+$ for absolutely convergent series of products of 'specified parity-type' functions. For our purposes, the only 'specified parity-type' functions we need to consider are restrictions to $\mathcal{Q}^+$ of functions from the set: $\{\sin(\star; q^2)$, $\{\ [\int_0^\star f(s)d_qs]_o : f \in L^1 d_q^+ s\}, \cos(\star; q^2)$, constant functions$\}$. On $\mathcal{Q}^+$, let $\sin(\star; q^2)$ and $[\int_0^\star f(s)d_qs]_o$ be of odd parity-type, while $\cos(\star; q^2)$ and constants are of even parity-type. The parity-type of a product follows the usual rules, e.g., even times odd is odd. Dilations f_w have the same parity-type as f. Finally, in this $\mathcal{Q}^+$ setting, ∂_q is defined to act termwise on absolutely convergent series using the formulas of Property 1. This construction preserves the results of Lemma 1.

It is easy to verify

Property 3. *For $f : \mathcal{Q}^+ \to \mathbb{C}$, if $[\int_0^t f(s)d_qs]_o$ exists for some t, then on $\mathcal{Q}^+$:*

a) $\partial_q[\int_0^t f(s)d_qs]_o = f(t)$.

b) $\lim_{t\to 0}[\int_0^t f(s)d_qs]_o = 0$.

Define $g(\widehat{x}, t) \equiv K\int_{-\infty}^{\infty} g(u,t)e(-iux; q^2)d_qu$. (So $g(\widehat{\star}, \star)$ is evaluated at (x,t).) $f(x,t) \in L^p d_q^B x$ means $f \in L^p d_q x$, and supp$f \subseteq B \equiv \{(x,t) : -q^m \le x \le q^m$ some $m \in \mathbb{Z}\}$.

$f \in L^p d_q^+ t d_q^B x$ means that $\forall s > 0$, $\int_{-\infty}^{\infty}[\int_0^s |f(x,t)|^p d_qt]d_qx < \infty$, and supp$f \subseteq B$.

Next, we list some applications of these spaces to Theorem 9. Although for $a, b \in \mathbb{R}$, $a < 0 < b$, $[a,b] \cap \mathcal{Q}$ is not compact in the relative (discrete) topology on $\mathcal{Q}$, the restriction of a continuous function on $\mathbb{R}$ to $\mathcal{Q}$ (e.g., $\cos(x; q^2)$, $\sin(x; q^2)$) is bounded on $[a,b]\cap\mathcal{Q}$. Thus, $x^{-1}f(\widehat{x}, t) \in (L^1 \cap L^2)d_q^B x$ and h continuous on $\mathbb{R}$ gives, for each $t > 0$, $x^{-1}h(|x|t)f(\widehat{x}, t) \in (L^1 \cap L^2)d_qx$. Also, when $f(x,s) \in L^1 d_q^B x$ and $|g(x,s)| \le G(t)$ for $(x,s) \in$ supp f with $s \le t$, we have $\|[\int_0^t\{g(x,s)f(x,s)\}d_qs]_o\|_{1,d_qx} \le G(t)\|f\|_{1,d_q^+ t d_q^B x}$.

In Theorem 9, abbreviate $\cos(z; q^2)$ to $\cos(z)$, $\sin(z; q^2)$ to $\sin(z)$, and $e(z; q^2)$ to $e(z)$.

Theorem 9. *Let $g(x,t) : \mathcal{Q} \times \mathcal{Q}^+ \to \mathbb{C}$. Suppose $x^{-1}g(\widehat{x}, t) \in (L^1 \cap L^2)d_q^+ t d_q^B x$. On $\mathcal{Q} \times \mathcal{Q}^+$, let $y(x,t) \equiv$*

$$q^{\frac{3}{2}}\int_{-\infty}^{\infty}\sin(q^{\frac{1}{2}}t|u|)[\int_0^t\{\frac{\cos(q^{\frac{1}{2}}s|u|)}{[\cos(q^{\frac{1}{2}}s|u|)]^2 + q[\sin(q^{\frac{3}{2}}s|u|)]^2}g(\widehat{u}, qs)\}d_qs]_o e(iux)\frac{d_qu}{u}$$

$$-q^{\frac{3}{2}}\int_{-\infty}^{\infty}\cos(q^{-\frac{1}{2}}t|u|)[\int_0^t\{\frac{\sin(q^{\frac{3}{2}}s|u|)}{[\cos(q^{\frac{1}{2}}s|u|)]^2 + q[\sin(q^{\frac{3}{2}}s|u|)]^2}g(\widehat{u}, qs)\}d_qs]_o(\frac{t}{q})e(iux)\frac{d_qu}{u}.$$

Then $\Box_q y(x,t) = g(x,t)$ with $\lim_{t\to 0} y(x,t) = 0$ and $\lim_{t\to 0}\partial_{q,t}y(x,t) = 0$.

Proof. Initially assume

$$\exists f : \mathcal{Q} \times \mathcal{Q}^+ \to \mathbb{C} \text{ such that: } x^{-1} f(\widehat{x}, t) \in L^1 d_q^+ t d_q^B x. \tag{10}$$

On $\mathcal{Q} \times \mathcal{Q}^+$ define $y(x,t) = y_1(x,t) - y_2(x,t)$, where

$$y_1(x,t) = \int_{-\infty}^{\infty} [\int_0^t \{\cos(q^{\frac{1}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o \sin(q^{\frac{1}{2}} t|u|) e(iux) \frac{d_q u}{u},$$

$$y_2(x,t) = \int_{-\infty}^{\infty} [\int_0^t \{\sin(q^{\frac{3}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o (\frac{t}{q}) \cos(q^{-\frac{1}{2}} t|u|) e(iux) \frac{d_q u}{u}.$$

The t parity-types of the functions in the integrands, Lemma 1, and Property 3 yield:

$$\partial_{q,t} y_1(x,t) = q^{-1} \int_{-\infty}^{\infty} \cos(q^{-\frac{1}{2}} t|u|) f(\widehat{u}, t) \sin(q^{\frac{1}{2}} t|u|) e(iux) \frac{d_q u}{u}$$

$$+ q^{-\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) [\int_0^t \{\cos(q^{\frac{1}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o (q^{-1} t) \cos(q^{-\frac{1}{2}} t|u|) e(iux) d_q u,$$

$$\partial_{q,t} y_2(x,t) = q^{-1} \int_{-\infty}^{\infty} \sin(q^{\frac{1}{2}} t|u|) f(\widehat{u}, t) \cos(q^{-\frac{1}{2}} t|u|) e(iux) \frac{d_q u}{u}$$

$$- q^{\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) [\int_0^t \{\sin(q^{\frac{3}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o \sin(q^{\frac{1}{2}} t|u|) e(iux) d_q u.$$

On $\mathcal{Q} \times \mathcal{Q}^+$, we have $\partial_{q,t} y(x,t) =$

$$q^{-\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) [\int_0^t \{\cos(q^{\frac{1}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o (q^{-1} t) \cos(q^{-\frac{1}{2}} t|u|) e(iux) d_q u$$

$$+ q^{\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) [\int_0^t \{\sin(q^{\frac{3}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o \sin(q^{\frac{1}{2}} t|u|) e(iux) d_q u.$$

Similarly,

$$(\partial_{q,t} \circ \partial_{q,t}) y(x,t) = q^{-\frac{3}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) \cos(q^{-\frac{1}{2}} t|u|) f(\widehat{u}, t) \cos(q^{-\frac{1}{2}} t|u|) e(iux) d_q u$$

$$- \int_{-\infty}^{\infty} u [\int_0^t \{\cos(q^{\frac{1}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o \sin(q^{\frac{1}{2}} t|u|)] (t) e(iux) d_q u$$

$$+ q^{-\frac{1}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) \sin(q^{\frac{1}{2}} t|u|) f(\widehat{u}, t) \sin(q^{\frac{1}{2}} t|u|) e(iux) d_q u$$

$$+ \int_{-\infty}^{\infty} u [\int_0^t \{\sin(q^{\frac{3}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o (-q^{-1} t) \cos(q^{-\frac{1}{2}} t|u|) e(iux) d_q u.$$

Also

$$(\partial_{q,x} \circ \partial_{q,x}) y(x,t) = - \int_{-\infty}^{\infty} u [\int_0^t \{\cos(q^{\frac{1}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o \sin(q^{\frac{1}{2}} t|u|) e(iux) d_q u$$

$$+ \int_{-\infty}^{\infty} u [\int_0^t \{\sin(q^{\frac{3}{2}} s|u|) f(\widehat{u}, qs)\} d_q s]_o (q^{-1} t) \cos(q^{-\frac{1}{2}} t|u|) e(iux) d_q u.$$

Combining the above shows that on $\mathcal{Q} \times \mathcal{Q}^+$:

$$\square_q y(x,t) = q^{-\frac{3}{2}} \int_{-\infty}^{\infty} \operatorname{sgn}(u) \{[\cos(q^{-\frac{1}{2}} t|u|)]^2 + q[\sin(q^{\frac{1}{2}} t|u|)]^2\} f(\widehat{u}, t) e(iux) d_q u. \tag{11}$$

The expression in brackets is $\geq \frac{(1+q)}{B_q^2} > 0$ by Corollary 1, so we can now define

$$f(\widehat{u},t) \equiv \frac{\operatorname{sgn}(u)q^{\frac{3}{2}}}{[\cos(q^{-\frac{1}{2}}t|u|)]^2 + q[\sin(q^{\frac{1}{2}}t|u|)]^2} g(\widehat{u},t) \in (L^1 \cap L^2) d_q^+ t d_q^B x. \tag{12}$$

Thus, using the hypotheses, $f(\widehat{u},t)$ as defined by (12) satisfies conditions (10). Now (11), (12), Theorem 3 and Lemma 2c) yield for $t > 0$,

$$\Box_q y(x,t) = \int_{-\infty}^{\infty} g(\widehat{u},t) e(iux) d_q u = g(x,t).$$

Finally, using (12), the initial conditions follow from Property 3b) and the expressions given above for y and $\partial_{q,t} y$, applying dominated convergence. $\Box$

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