ON MRA RIESZ WAVELETS

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Abstract. We investigate the properties of univariate MRA Riesz wavelets. In particular we obtain a generalization to semiorthogonal MRA wavelets of a well-known representation theorem for orthonormal MRA wavelets.

In what follows \( Z \) will denote the integers and \( \mathbb{R} \) the real numbers; \( t \) and \( x \) will always denote real variables. The Fourier transform of a function \( f \) will be denoted by \( \hat{f} \). If \( f \in L^2(\mathbb{R}) \),

\[
\hat{f}(x) := \int_{\mathbb{R}} e^{-txi} f(t) \, dt.
\]

Let \( \mathbb{H} \) be a (separable) Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| := \langle \cdot, \cdot \rangle^{1/2} \). A sequence \( \{f_k, k \in \mathbb{Z}\} \subset \mathbb{H} \) is called a frame if there are constants \( 0 < A \leq B \) such that for every \( f \in \mathbb{H} \)

\[
A \|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \|f\|^2.
\]

The constants \( A \) and \( B \) are called (lower and upper) bounds of the frame. If only the right-hand inequality in the preceding displayed formula is satisfied for all \( f \in \mathbb{H} \), then \( \{f_k, k \in \mathbb{Z}\} \) is called a Bessel sequence with bound \( B \). A sequence \( \{f_k, k \in \mathbb{Z}\} \subset \mathbb{H} \) is called a Riesz basis if its linear span is dense in \( \mathbb{H} \) and there are constants \( A \) and \( B, A > 0 \), such that for every sequence \( \{c_k, k \in \mathbb{Z}\} \subset \ell^2 \),

\[
A \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \left\| \sum_{k \in \mathbb{Z}} c_k f_k \right\|^2 \leq B \sum_{k \in \mathbb{Z}} |c_k|^2.
\]

The constants \( A \) and \( B \) are called (lower and upper) bounds of the Riesz basis. Every Riesz basis is a frame, every orthonormal basis is a Riesz basis with bounds \( A = B = 1 \), and Riesz bounds and frame bounds coincide. A sequence \( \{f_k, k \in \mathbb{Z}\} \subset \mathbb{H} \) is a Riesz basis if and only if it is a frame having the additional property that upon the removal of any element from the sequence, it ceases to be a frame.

In this paper the underlying Hilbert space will be \( L^2(\mathbb{R}) \) with the usual inner product and norm, and we will study binary affine sequences generated by a single function, i.e. sequences of the form \( \{\psi_{j,k}; j, k \in \mathbb{Z}\} \), where \( \psi \in L^2(\mathbb{R}) \) and

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The function \( \psi \) is called the low pass filter associated with \( \varphi \) if and only if there is a measurable unimodular and \( 2\pi \)-periodic function \( \nu(x) \), such that

\[
\hat{h}(2x) = e^{ix} \nu(2x) \overline{\hat{\varphi}(x)} \quad \text{a.e.}
\]

(1)

**Theorem A** ([6] p. 57). If \( \varphi \) is a scaling function for an MRA \( \{V_j; j \in \mathbb{Z}\} \) and \( p \) is the associated low pass filter, then \( h \) is an orthonormal wavelet in \( W_0 \) if and only if there is a measurable unimodular and \( 2\pi \)-periodic function \( \nu(x) \), such that

\[
A \leq \sum_{k \in \mathbb{Z}} |\hat{\varphi}(x + 2k\pi)|^2 \leq B \quad \text{a.e.}
\]

In particular, \( \{\varphi(x - k); k \in \mathbb{Z}\} \) is an orthonormal basis of the closure of its linear span if and only if

\[
\sum_{k \in \mathbb{Z}} |\hat{\varphi}(x + 2k\pi)|^2 = 1 \quad \text{a.e.}
\]

Let \( \psi \) be a frame wavelet in \( L^2(\mathbb{R}) \); for \( j \in \mathbb{Z} \), let \( P_j \) denote the closure of the linear span of \( \{\psi_{j,k}; k \in \mathbb{Z}\} \), and let \( V_j := \sum_{r \leq j} P_r \). Note that \( \psi \in V_1 \). We say that \( \psi \) is associated with an MRA, or that \( \psi \) is an MRA wavelet, if \( M := \{V_j; j \in \mathbb{Z}\} \) is a multiresolution analysis; we also say that \( \psi \) is associated with \( M \).

We begin the statement and proof of our results with the following proposition, of some independent interest.
**Proof.** We will prove (a) only. The proof of (b) will then reduce to finding the lower bound, which is done in a similar way.

**Theorem 1.** Let \( \psi \) be a Riesz wavelet with bound \( C \) and let \( \varphi \) be a measurable \( 2\pi \)-periodic function. Then:

(a) If there are constants \( B, D > 0 \) such that \( |\mu(x)|^2 \leq B \) a.e., and \( \varphi \) is a Bessel wavelet with bound \( D \), then \( \psi \) is a Bessel wavelet with bound \( BD \).

(b) If there are constants \( 0 < A \leq B, 0 < C \leq D \) such that \( A \leq |\mu(x)|^2 \leq B \) a.e., and \( \varphi \) is a frame wavelet with bounds \( C \) and \( D \), then \( \psi \) is a frame wavelet with bounds \( AC \) and \( BD \).

**Proposition 1.** Let \( v \in L^2(\mathbb{R}) \) and let \( \mu(x) \) be a measurable \( 2\pi \)-periodic function. Let \( \psi \) be a function such that

\[
\hat{\psi}(x) = \mu(x)\hat{\varphi}(x) \quad \text{a.e.}
\]

Then:

(a) If there are constants \( B, D > 0 \) such that \( |\mu(x)|^2 \leq B \) a.e., and \( \varphi \) is a Bessel wavelet with bound \( D \), then \( \psi \) is a Bessel wavelet with bound \( CD \).

(b) If there are constants \( 0 < A \leq B, 0 < C \leq D \) such that \( A \leq |\mu(x)|^2 \leq B \) a.e., and \( \varphi \) is a frame wavelet with bounds \( C \) and \( D \), then \( \psi \) is a frame wavelet with bounds \( AC \) and \( BD \).

**Proof.** We will prove (a) only. The proof of (b) will then reduce to finding the lower bound, which is done in a similar way.

Let \( f \in L^2(\mathbb{R}) \). By periodization, as in e.g. [3] p. 270, (2.2), we have

\[
\sum_{j,k \in \mathbb{Z}} |\langle f, v_{j,k} \rangle|^2 = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} 2^j \int_0^{2\pi} \left| \sum_{r \in \mathbb{Z}} \hat{f}(2^j(x + 2r\pi))\hat{\varphi}(x + 2r\pi) \right|^2 dx.
\]

Applying (3), we also have

\[
\sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} 2^j \int_0^{2\pi} \left| \sum_{r \in \mathbb{Z}} \hat{f}(2^j(x + 2r\pi))\overline{\hat{\varphi}(x + 2r\pi)} \right|^2 dx
\]

\[
= \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} 2^j \int_0^{2\pi} |\mu(x)| \sum_{r \in \mathbb{Z}} \hat{f}(2^j(x + 2r\pi))\overline{\hat{\varphi}(x + 2r\pi)} |^2 dx
\]

\[
\leq B \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} 2^j \int_0^{2\pi} \left| \sum_{r \in \mathbb{Z}} \hat{f}(2^j(x + 2r\pi))\overline{\hat{\varphi}(x + 2r\pi)} \right|^2 dx.
\]

Since

\[
\sum_{j,k \in \mathbb{Z}} |\langle f, v_{j,k} \rangle|^2 \leq D\|f\|^2,
\]

the assertion follows from (3). \( \square \)

We now turn to Riesz wavelets.

**Proposition 2.** Let \( \mu(x) \) be a measurable \( 2\pi \)-periodic function, and let \( v \in L^2(\mathbb{R}) \) be a Riesz wavelet with bounds \( 0 < A \leq B \). If \( \psi \) is defined by (2) and there are constants \( C \) and \( D \), \( 0 < C < |D|\sqrt{A/B} \) such that

\[
|\mu(x) - D| \leq C \quad \text{a.e.,}
\]

then \( \psi \) is a Riesz wavelet.

**Proof.** Let \( u := \psi - Dv \). From (2), \( \hat{u}(x) = (\mu(x) - D)\hat{\varphi}(x) \). Since \( v \) is a Riesz wavelet with upper bound \( B \), applying Proposition 1, we conclude that \( u \) is a Bessel wavelet with bound \( C^2B < |D|^2A \). Since \( Dv \) is a Riesz wavelet with lower bound \( |D|^2A \), the assertion follows from [5] Theorem 5. \( \square \)

**Theorem 1.** Let \( M := \{V_j : j \in \mathbb{Z}\} \) be an MRA, and assume that \( \psi \in V_j \) for some \( j \leq 0 \). Then \( \psi \) cannot be a Riesz wavelet.
Proof. We proceed by contradiction. Assume $\psi$ is a Riesz wavelet in $V_j$ with bounds $0 < A \leq B$, and let $\varphi$ be a scaling function for $M$ with associated low pass filter $p$.

Since $\psi \in V_0$, there is a $2\pi$-periodic function $s(x)$ such that

$$\widehat{\psi}(x) = s(x)\widehat{\varphi}(x) \quad \text{a.e.}$$

This implies that

$$\sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 = |s(x)|^2 \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(x + 2k\pi)|^2 \quad \text{a.e.,}$$

whence from Theorem B we conclude that

$$A \leq |s(x)|^2 \leq B \quad \text{a.e.}$$

Moreover, the well-known identity

$$|p(x)|^2 + |p(x + \pi)|^2 = 1$$

implies that $|p(x)| \leq 1$. Thus, if

$$m(x) := \frac{s(2x)p(x)}{s(x)},$$

we infer that $|m(x)|^2 \leq B/A$. Since

$$\widehat{\psi}(2x) = s(2x)p(x)\widehat{\varphi}(x) = m(x)\widehat{\psi}(x)$$

and $m \in L^2(0, 2\pi)$ and is $2\pi$–periodic, we conclude that $\psi(t/2)$ is in the closure of the linear span of the sequence $\{\psi(t - k); k \in \mathbb{Z}\}$. This is incompatible with the assumption that $\{\psi_{j,k}; j, k \in \mathbb{Z}\}$ is a Riesz basis, and we have reached a contradiction. \[\square\]

**Corollary 1.** Let $M := \{V_j; j \in \mathbb{Z}\}$ be an MRA. If $j \neq 0$ there is no function $u$ such that $\{u(t - k); k \in \mathbb{Z}\}$ is a Riesz basis of $V_j$.

Proof. If $\{u(t - k); k \in \mathbb{Z}\}$ is a Riesz basis of $V_j$, then there is a function $v$ such that $\{v(t - k); k \in \mathbb{Z}\}$ is an orthonormal basis of $V_j$ [pp. 48–51]. Setting $U_r := V_{r+j}$ we see that $\{U_r; r \in \mathbb{Z}\}$ is an MRA. If $W'_0$ denotes the orthogonal complement of $U_0$ in $U_1$, then there is an orthonormal (and therefore Riesz) wavelet $h'$ in $W'_0 \subset U_1 = V_{j+1}$. If $j < 0$ this implies that $V_0$ contains an orthonormal wavelet, which is a contradiction. On the other hand if $j > 0$, let $h$ be an orthonormal wavelet in $W_0$. Since $W_0 \subset V_1 = U_{1-j} \subset U_0$, we see that $U_0$ contains an orthonormal wavelet, and we have a contradiction in this case as well. \[\square\]

Theorem 1 is supplemented by the following.

**Theorem 2.** Let $M := \{V_j; j \in \mathbb{Z}\}$ be an MRA, and assume that $\psi \in W_r$ for some $r \neq 0$. Then $\psi$ cannot be a Riesz wavelet.

Proof. Since

$$V_0 = \bigoplus_{j < 0} W_j,$$

the assertion for $r < 0$ follows from Theorem 1. Assume now that $r > 0$. Let $h \in W_0$ be an orthonormal wavelet, and let $\psi$ be a Riesz wavelet in $W_r$ with bounds $A$ and $B$. Then $\psi_{-r,0} \in W_0$. From, e.g., [p. 57, Lemma 2.11], we know that $W_0$ is closed under integral translations (this can also be easily obtained from the definition of $W_0$). Since $r > 0$, we deduce that $\psi_{-r,k} \in W_0$ for every $k \in \mathbb{Z}$. Thus $\psi_{j,k} \in W_{j+r}$ for all $j, k \in \mathbb{Z}$.
Let \( f \in W_r \). The hypotheses imply there is a sequence \( \{d_{j,k}; j, k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z}^2) \) such that \( f = \sum_{j,k} d_{j,k} \psi_{j,k} \) (in the \( L^2 \) sense). Define \( f_1 := \sum_{k} d_{0,k} \psi_{0,k} \) and \( f_2 = f - f_1 \). Since \( f - f_1 \in W_r \) and \( f_2 \in W_r^\perp \), we conclude that \( f_2 = 0 \); thus, \( \text{span}\{\psi(\cdot-k); k \in \mathbb{Z}\} = W_r \). Since \( h_{r,0} \in W_r \), this implies that there is a \( 2\pi \)-periodic function \( m(x) \) such that
\[
\hat{h}_{r,0}(x) = m(x)\hat{\psi}(x) \quad \text{a.e.}
\]

Thus Theorem B implies that
\[
2^{-r} = 2^{-r} \sum_{k \in \mathbb{Z}} \left| \hat{h}(2^{-r}x + 2k\pi) \right|^2
= |m(x)|^2 \sum_{k \in \mathbb{Z}} \left| \hat{\psi}(x + 2^{r+1}k\pi) \right|^2 \leq |m(x)|^2 B \quad \text{a.e.}
\]

On the other hand, since
\[
\sum_{k \in \mathbb{Z}} \left| \hat{h}(x + 2^{-r}k\pi) \right|^2 = \sum_{\ell=0}^{2^r-1} \sum_{n \in \mathbb{Z}} \left| \hat{h}(x + 2^{-r}\ell\pi + 2n\pi) \right|^2 = 2^r \quad \text{a.e.,}
\]
we see that
\[
1 = 2^{-r} \sum_{k \in \mathbb{Z}} \left| \hat{h}(2^{-r}x + 2^{-r}k\pi) \right|^2
= |m(x)|^2 \sum_{k \in \mathbb{Z}} \left| \hat{\psi}(x + 2k\pi) \right|^2 \geq |m(x)|^2 A \quad \text{a.e.}
\]

We therefore conclude that
\[
2^{-r}B^{-1} \leq |m(x)|^2 \leq A^{-1} \quad \text{a.e.,}
\]
and Proposition \( \square \) implies that \( h_{r,0} \) is a frame wavelet. This implies that
\[
\{2^j/2 h(2^j t - 2^r k); j, k \in \mathbb{Z}\}
\]
is a frame wavelet. Since
\[
\{2^j/2 h(2^j t - k); j, k \in \mathbb{Z}\}
\]
is a Riesz basis by hypothesis and \( \square \) is a subsequence of \( \square \), this is incompatible with the definition of Riesz basis, and we have obtained a contradiction. \( \square \)

We are now ready to prove Theorem 3. Let \( \varphi \) be a scaling function for an MRA \( M = \{V_j; j \in \mathbb{Z}\} \), and let \( p \) be the associated low pass filter. The following propositions are equivalent:

(a) \( \psi \) is a semiorthogonal Riesz wavelet associated with \( M \) with bounds \( 0 < A \leq B \).

(b) There is a measurable \( 2\pi \)-periodic function \( \mu(x) \) such that
\[
\hat{\psi}(2x) = e^{ix} \mu(2x)\overline{\varphi}(x) \quad \text{a.e.}
\]
and
\[
A \leq |\mu(x)|^2 \leq B \quad \text{a.e.}
\]
Proof: The definitions imply that \( \psi \) is a semiorthogonal wavelet if and only if \( \psi \in W_0 \). Let
\[
 r(x) := \left( \sum_{k \in \mathbb{Z}} |\hat{\psi}(x + 2k\pi)|^2 \right)^{1/2}
\]
and
\[
 \hat{h}(x) := \frac{\hat{\psi}(x)}{r(x)}.
\]
Then, as remarked in [3, p. 78], \( h \) is an orthonormal wavelet. Since \( 1/r(x) \) is \( 2\pi \)-periodic and bounded, it follows that \( h \in W_0 \). Thus \( h \) has a representation of the form (I). Setting \( \mu(x) := r(x)\nu(x) \), we readily see that (6) and (7) are satisfied.

Conversely, assume there is a \( 2\pi \)-periodic function \( \mu(x) \) such that (6) and (7) are satisfied. Let
\[
 \text{sign} \mu(x) := \begin{cases} \mu(x)/|\mu(x)| & \text{if } \mu(x) \neq 0, \\ 1 & \text{if } \mu(x) = 0. \end{cases}
\]
Setting
\[
 \hat{h}_1(2x) := e^{ix} \text{sign} (\mu(x)) |\mu(x)| p(x + \pi) \hat{\phi}(x),
\]
we conclude from Theorem A that \( h_1 \) is an orthonormal wavelet. Moreover, (6) implies that
\[
 \hat{\psi}(x) = |\mu(x)| \hat{h}_1(x) \quad \text{a.e.}
\]
Since
\[
 |||\mu(x)|| - \sqrt{B}|| \leq \sqrt{B} - \sqrt{A} < \sqrt{B} \quad \text{a.e.,}
\]
we deduce from Proposition [2] that \( \psi \) is a Riesz wavelet. Indeed, since \( A' = 1 \) and \( B' = 1 \) are Riesz bounds for \( h_1 \), setting \( C = \sqrt{B} - \sqrt{A} \) and \( D = \sqrt{B} \), we see that
\[
 0 < C < |D| \sqrt{A'/B'} 
\]
and
\[
 |||\mu(x)|| - D|| \leq C \quad \text{a.e.}
\]
Proposition [2] does not yield sufficiently accurate information about the Riesz bounds of \( \psi \). However, from Proposition [1] we readily see that \( \psi \) is a frame wavelet with lower bound \( A \) and upper bound \( B \). Since frame bounds and Riesz bounds coincide, we conclude that \( A \) and \( B \) are a lower and an upper Riesz bound of \( \psi \), respectively. \( \square \)

Clearly Theorem A is a particular case of Theorem [3]. From Theorem A and Theorem [3] we obtain

**Corollary 2.** A function \( \psi \) is a semiorthogonal MRA Riesz wavelet with bounds \( 0 < A \leq B \) if and only if there is a measurable \( 2\pi \)-periodic function \( \mu(x) \) that satisfies (7), and an MRA orthonormal wavelet \( h \), such that
\[
 \hat{\psi}(x) = \mu(x) \hat{h}(x) \quad \text{a.e.}
\]

**References**


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