MORE ON PARTITIONING TRIPLES OF COUNTABLE ORDINALS

ALBIN L. JONES

(Communicated by Julia Knight)

Abstract. Consider an arbitrary partition of the triples of all countable ordinals into two classes. We show that either for each finite ordinal \( m \) the first class of the partition contains all triples from a set of type \( \omega + m \), or for each finite ordinal \( n \) the second class of the partition contains all triples of an \( n \)-element set. That is, we prove that \( \omega_1 \rightarrow (\omega + m, n)^3 \) for each pair of finite ordinals \( m \) and \( n \).

1. BACKGROUND AND MOTIVATION

If \( A \) is a set of ordinals and \( \beta \) is an ordinal, then \( [A]^\beta \) denotes the set of all subsets of \( A \) which are order isomorphic to \( \beta \). If \( \alpha, \nu \), and \( \{ \beta_i | i < \nu \} \) are ordinals and \( m < \omega \), then the ordinary Ramsey relation \( \alpha \rightarrow (\beta_i)^m_{i<\nu} \) means that for each partition \( f : [\alpha]^m \rightarrow \nu \) there are \( i < \nu \) and \( X \in [\alpha]^\beta_i \) with \( f[X]^m = \{ i \} \). As usual, if \( \nu < \omega \), then this relation might be written \( \alpha \rightarrow (\omega_1^0, \ldots , \omega_1^{\nu-1})^m \); if \( \beta_i = \beta \) for all \( i < \nu \), then it might be written \( \alpha \rightarrow (\beta)^m_{\nu} \); and the negation of any such relation is indicated by replacing the \( \rightarrow \) with \( \nrightarrow \). The study of these relations (as well as all of the notation defined above) was introduced by P. Erdős and R. Rado in [2].

The ordinary Ramsey theory of the countable ordinals is the theory of such partition relations with \( \alpha \leq \omega_1 \), those relations which describe the ordinary Ramsey-theoretic properties of individual countable ordinals or of the totality of all countable ordinals.

This theory has been studied thoroughly and quite successfully. Our understanding of it, as witnessed by the results below, is almost complete. In each of the relations listed below, \( \alpha \) is an arbitrary countable ordinal, and \( m \) and \( n \) are arbitrary finite ordinals, unless otherwise indicated.

1. \( \omega \rightarrow (\omega)^m_n \) (F. P. Ramsey in [10]).
2. (a) \( \alpha \nrightarrow (\omega + 1, \omega)^2 \) (P. Erdős and R. Rado in [2]).
   (b) If \( m \geq 3 \), then \( \alpha \rightarrow (\omega + 1, m + 1)^m \) (P. Erdős and R. Rado in [2]).
3. (a) \( \omega_1 \rightarrow (\omega_1)^1_\omega \).
   (b) If \( m \geq 2 \), then \( \omega_1 \nrightarrow (m + 1)_n^m \) (P. Erdős and R. Rado in [2]).
4. \( \omega_1 \rightarrow (\omega)^2_n \) (J. Baumgartner and A. Hajnal in [1]).
5. (a) \( \omega_1 \nrightarrow (\omega_1)^2_2 \) (W. Sierpiński in [11]).

Received by the editors March 1, 2005 and, in revised form, October 25, 2005.
2000 Mathematics Subject Classification. Primary 03E05, 04A20; Secondary 05A18, 05D10.
Key words and phrases. Countable ordinals, Ramsey theory, transfinite numbers, triples.
The author would like to thank the University of Kansas for its support of this research.

©2006 by Albin L. Jones

1197
(b) $\omega_1 \to (\omega_1, \omega + 1)^2$ (P. Erdős and R. Rado in [2]).
(c) CH implies that $\omega_1 \to (\omega_1, \omega + 2)^2$ (A. Hajnal in [3]).
(d) It is consistent with Martin’s Axiom that $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$
(S. Todorčević in [13]).
(e) $\omega_1 \to (\omega_1, 4)^3$ (P. Erdős and R. Rado in [2]).
(f) $\omega_1 \to (\omega + 2, \omega)^3$ (P. Erdős and R. Rado in [2]).
(g) $\omega_1 \to (\omega + 1)^m$ (F. Galvin in [3]).
(d) If $m \geq 4$, then $\omega_1 \to (\omega + 2, m + 1)^m$ (P. Erdős and R. Rado in [2]).

Taken together these results leave little of the story of the ordinary Ramsey theory of countable ordinals untold. Essentially only two open problems remain:

(1) For which $\alpha, \beta < \omega_1$ and $n < \omega$ does $\alpha \to (\beta, n)^2$?
(2) For which $\alpha < \omega_1$ and $n < \omega$ does $\omega_1 \to (\alpha, n)^3$?

Each of these questions seems very difficult to resolve, though some progress has been made on each of them.

Regarding the latter problem, E. C. Milner and K. Prikry (among others) have conjectured that the answer is “all of them”. We definitely agree with their conjecture, though we have not been able to confirm it. In [2], they were able to prove the conjecture for $\alpha = \omega + \omega + 1$ and $n = 4$. Thus, the simplest unresolved cases of the question are whether $\omega_1 \to (\omega + 2, 5)^3$ and whether $\omega_1 \to (\omega + \omega + 2, 4)^3$.

Below, we resolve the first case (and a little more) affirmatively. Unfortunately, we leave the second unresolved.

2. Our notation

Our basic notation and all terms which we leave undefined are explained in any standard text on set theory (for example, [4] or [7]).

We will code countable ordinals as finite sets of finite ordinals in various ways and will then use these codings to “lift” Ramsey properties of sets of finite ordinals to appropriate sets of countable ordinals. To that end, we will need to be able to quickly express certain relationships between sets of ordinals.

Suppose that $x$ and $y$ are sets of ordinals. If each element of $x$ is less than each element of $y$, then we will say that $x$ is supremely less than $y$ and will write $x \ll y$. Note that if $x$ is supremely less than $y$, then $x$ and $y$ are necessarily disjoint. Also, we will put $x \ll \emptyset$ for any nonempty set of ordinals $x$. We will write $x \subseteq y$ and say that $x$ is an initial segment of $y$, if $x \subseteq y$ and $x \ll y \setminus x$. This is the proper analog of the usual notion of initial segments if we identify sets of ordinals with strictly increasing sequences of ordinals in the natural way.

We will call the maximal common initial segment of $x$ and $y$ the stem of $x$ and $y$; we will denote it by $\sigma(x, y)$. What remains in $x$ after the removal of its common stem with $y$ we will call the branch of $x$ from $y$; we will denote it by $\beta(x, y)$. We will denote the order-type of the stem of $x$ and $y$ by $\delta(x, y)$. In summary,

$$\sigma(x, y) = \bigcup \{z \mid z \subseteq x \land z \subseteq y\},$$
$$\beta(x, y) = x \setminus \sigma(x, y),$$
$$\delta(x, y) = \mathrm{ot}(\sigma(x, y)).$$

Note that $\sigma(x, y) = \sigma(y, x)$ always, but $\beta(x, y) \neq \beta(y, x)$ unless $x = y$.

The lexicographic ordering is defined in the usual way by putting $x \leq_{\text{lex}} y$ if $x \subset y$ or $\min \beta(x, y) < \min \beta(y, x)$. We also define the superlexicographic ordering.
by putting \( x \lessdot \text{lex} \ y \) if \( x \subseteq y \) or \( \beta(x, y) \lessdot \beta(y, x) \). Note that \( x \lessdot \text{lex} \ y \) if and only if \( x \neq y \) and there are sets of ordinals \( a, b, \) and \( c \) with \( a \lessdot b \lessdot c \), and such that \( x = a \cup b \) and \( y = a \cup c \).

For each \( k < \omega \) and \( A \in [\omega_1]^\omega_k \) there is a unique order isomorphism \( \pi_A : \langle [\omega]_k, \lessdot_{\text{lex}} \rangle \cong (A, \lessdot) \). For \( x \in [\omega]_k \), let \( A(x) = \pi_A(x) \) be “the \( x^{th} \) element of \( A \).

A family of sets is a filter base if the intersection of any finitely many of its elements is infinite. In particular, any subfamily of a nonprincipal filter over \( \omega \) is a filter base. An infinite set \( X \in [\omega]^\omega \) is a pseudo-intersection of the filter base \( \mathcal{F} \subseteq [\omega]^\omega \) if \( X \subseteq^* F \) for every \( F \in \mathcal{F} \). (Here, \( X \subseteq^* F \) if \( |X \setminus F| < \omega \).)

The cardinal \( p \) is the pseudo-intersection number, the minimal cardinality of a filter base for which there is no pseudo-intersection. In particular, and most importantly, every filter base of cardinality less than \( p \) must have a pseudo-intersection. As usual, the cardinal \( c \) is the cardinality of the continuum. Note that \( \omega_1 \leq p \leq c \).

A filter \( \mathcal{F} \subseteq [\omega]^\omega \) is a Ramsey filter if for every pair of finite ordinals \( m \) and \( n \) and every partition \( f : [\omega]^m \to n \) there are \( X \in \mathcal{F} \) and \( i < n \) such that \( f^i[X]^m = \{i\} \).

If we assume either CH or Martin’s Axiom (or more generally that \( p = c \)), then such filters are easily constructed.

3. The proof

This section is devoted to a proof of Theorem 1 below. The proof is given in a series of lemmas.

**Theorem 1.** \( \omega_1 \to (\omega + m, n)^3 \) for all \( m, n < \omega \).

The general outline of the proof is based on an idea of J. Baumgartner and A. Hajnal from [1]. We will first prove the theorem with the aid of an additional combinatorial assumption, then demonstrate that we can find a suitable generic extension of the universe in which this combinatorial assumption (and hence the theorem) is true, and finally argue that because we can force the theorem to be true, the theorem must already be true. This method has also been used by S. Todorčević in [13] and E. C. Milner and K. Prikry in [8] and [9].

First, we prove Theorem 1 with the additional assumption that \( p = c > \omega_1 \).

**Lemma 1.** If \( p = c > \omega_1 \), then \( \omega_1 \to (\omega + m, n)^3 \) for each \( m, n < \omega \).

**Proof.** Fix \( m < \omega \). We will prove the lemma by induction on \( n < \omega \). Since the lemma is trivial for \( n \leq 3 \), we assume that it holds for an arbitrary (but hereafter fixed) finite ordinal \( n \geq 3 \) and will deduce that it remains true with \( n \) replaced by \( n + 1 \).

Let \( f : [\omega_1]^3 \to 2 \) be an arbitrary (but hereafter fixed) partition of the triples of \( \omega_1 \) into two classes. We need to show that either

(a) there is \( A \in [\omega_1]^\omega + m \) with \( f^0[A]^3 = \{0\} \), or
(b) there is \( B \in [\omega_1]^{n+1} \) with \( f^0[B]^3 = \{1\} \).

To do that, we will analyze the behavior of \( f \) on sets of type \( \omega^k \) for \( k < \omega \).

For each \( k < \omega \), each \( j < k \), each \( A \in [\omega_1]^\omega \), and each \( \beta \in \omega_1 \) with \( A \lessdot \beta \) we define a partition of \( [\omega]^{2k-j} \) into two classes \( f_{j,k,A,\beta} : [\omega]^{2k-j} \to \{0, 1\} \) in the

---

1This result and the proof presented here first appeared in [6], albeit in a more terse form and with a few typographical errors.
Assume that $\Phi$ holds this. Define a partition of $[\omega]^{2k-j}$ into three sets $x = a \cup b \cup c$ with $|a| = j$, $|b| = |c| = k - j$, and $a \ll b \ll c$, and let

$$f_{j,k,A,\beta}(x) = f_{j,k,A,\beta}(a \cup b \cup c) = f\{A(a \cup b), A(a \cup c), \beta\}. $$

Note that this is basically a partition of all pairs $x, y \in [\omega]^k$ with $x \ll_{lex} y$ and $\delta(x, y) = j$.

Fix a Ramsey filter $F$. As $F$ is Ramsey, for each such $j, k, A$, and $\beta$ there are a set $X_{j,k,A,\beta} \in F$ and an index $i_{j,k,A,\beta} \in \{0, 1\}$ with $f_{j,k,A,\beta}^i[X_{j,k,A,\beta}]^{2k-j} = \{i_{j,k,A,\beta}\}$. Let $X_{k,A,\beta} = \bigcap \{X_{j,k,A,\beta} \mid j < k\}$. Note that $X_{k,A,\beta} \in F$.

Consider the following statement for each $k < \omega$:

$$\Phi_k: \text{There are } j < k, A \in [\omega_1]^\omega, \text{ and } B \in [\omega_1]^\omega \text{ with } A \ll B$$

such that $i_{j,k,A,\beta} = 0$ for every $\beta \in B$.

The next lemma shows that any one of the $\Phi_k$’s is strong enough to establish Lemma [1] on its own.

**Lemma 1.1.** If $\Phi_k$ is true for some $k < \omega$, then either (a) or (b) holds.

**Proof.** Assume that $\Phi_k$ holds for some $k < \omega$. Let $j < k, A \in [\omega_1]^\omega$, and $B \in [\omega_1]^\omega$ witness this. Define a partition of $[B]^2$ into two pieces $g: [B]^2 \to \{0, 1\}$ by letting $g(\beta, \gamma) = i$ if and only if there is a set $X_{k,A,\beta,\gamma} \in F$ with $f\{A(x), \beta, \gamma\} = i$ for every $x \in [X_{k,A,\beta,\gamma}]^k$. That $g$ is a well-defined partition of $[B]^2$ follows from the fact that $F$ is a Ramsey filter.

We know that $\omega_1 \to (\omega_1, \omega)^2$, so either there is $D \in [B]^\omega$ with $f\{A(x), \beta, \gamma\} = 1$ for each $\beta, \gamma \in D^2$ and $x \in [X_{k,A,\beta,\gamma}]^k$ or there is $C \in [B]^\omega$ with $f\{A(x), \beta, \gamma\} = 0$ for each $\beta, \gamma \in C^2$ and $x \in [X_{k,A,\beta,\gamma}]^k$.

In the first case, we apply our inductive hypothesis that $\omega_1 \to (\omega + m, n)^3$ to $D$ to conclude that either there is $E \in [D]^{\omega+m}$ with $f^i[E]^3 = \{0\}$ (in which case (a) holds, and we are done), or there is $F \in [D]^n$ with $f^i[F]^3 = \{1\}$. In this latter case, let

$$X = \bigcap \{X_{k,A,\beta,\gamma} \mid \beta, \gamma \in [F]^2\},$$

and choose any $x \in [X]^k$. Then let $G = \{A(x)\} \cup F$, so that $G \in [\omega_1]^n+1$ and $f^i[G]^3 = \{1\}$. Hence (b) holds, and we are done.

In the second case, we choose a finite ordinal $r$ such that $r \to (m, n + 1) \to$ and then choose $C' \in [C]^r$. Let

$$X_0 = \bigcap \{X_{j,k,A,\beta} \mid \beta \in C'\},$$

$$X_1 = \bigcap \{X_{k,A,\beta,\gamma} \mid \beta, \gamma \in [C']^2\},$$

$$X = X_0 \cap X_1.$$  

Fix $a \in [X]^j$. For each $i < \omega$, choose $b_i \in [X]^{k-j}$ with $a \ll b_0 \ll b_1 \ll b_2 \ll \ldots$. Let $A' = \{A(a \cup b_i) \mid i < \omega\}$. Note that $f^i[A', C']^2.1 \cup [A', C']^1.2 = \{0\}$. If (b) does not hold, then since $A' \to (\omega, n + 3)^3$ we can find $A'' \in [A']^\omega$ with $f^i[A'']^3 = \{0\}$; and since $C' \to (m, n + 3)^3$ we can find $C'' \in [C']^m$ with $f^i[C'']^3 = \{0\}$; hence $A'' \cup C'' \in [\omega_1]^{\omega+m}$ and $f^i[A'' \cup C'']^3 = \{0\}$. Thus (a) holds, and the proof of the lemma is complete.  


Let us set aside the previous lemma for a moment and consider the following statement for each $k < \omega$:

$\Psi_k$: There is $A \in [\omega_1]^{\omega_1}$ such that $f\{A(x), A(y), A(z)\} = 1$ for each trio of sets $x, y, z \in [\omega]^k$ with $x \ll_{\text{lex}} y \ll_{\text{lex}} z$ and $\delta(x, z) < \delta(x, y)$.

The next lemma demonstrates that $\Psi_k$ is strong enough to establish Lemma 1 on its own. This is important because (as we will see below) the $\Phi_k$’s and the $\Psi_k$’s are complementary: If enough of the $\Phi_k$’s fail, then many of the $\Psi_k$’s will hold.

**Lemma 1.2.** For each $k < \omega$, if $\Psi_k$ is true, then there is $B \in [\omega_1]^{k+1}$ with $f^3[B] = 1$. (In particular, if $\Psi_n$ holds, then (b) holds.)

**Proof.** Assume that $k < \omega$ and the set $A \in [\omega_1]^{\omega_1}$ witnesses that $\Psi_k$ holds. Define $k + 1$ sets of finite ordinals $\{x_{j,k} \mid j \leq k\} \subseteq [\omega]^k$ by the following formula:

\[
x_{0,k} = \{0, 1, \ldots, k - 1\}
\]

\[
x_{1,k} = \{0, 1, \ldots, k - 2\} \cup \{k\}
\]

\[
\vdots
\]

\[
x_{j,k} = \{0, 1, \ldots, k - j - 1\} \cup \{k + t_j, k + t_j + 1, \ldots, k + t_j + 1\}
\]

\[
\vdots
\]

\[
x_{k-1,k} = \{0\} \cup \{k + t_{k-1}, k + t_{k-1} + 1, \ldots, k + t_{k-1} + (k - 2)\}
\]

\[
x_{k,k} = \{k + t_k, k + t_k + 2, \ldots, k + t_k + k\},
\]

where $t_j = \sum_{r = 1}^{j} j$ for each $j < \omega$ (so that $t_0 = t_1 = 0$, $t_2 = 1$, $t_3 = 3$, $t_4 = 6$, $t_5 = 10$, and so forth). For example, if $k = 6$, then the seven sets are:

\[
x_{0,6} = \{0, 1, 2, 3, 4, 5\}
\]

\[
x_{1,6} = \{0, 1, 2, 3, 4, 6\}
\]

\[
x_{2,6} = \{0, 1, 2, 3, 7, 8\}
\]

\[
x_{3,6} = \{0, 1, 2, 9, 10, 11\}
\]

\[
x_{4,6} = \{0, 1, 12, 13, 14, 15\}
\]

\[
x_{5,6} = \{0, 16, 17, 18, 19, 20\}
\]

\[
x_{6,6} = \{21, 22, 23, 24, 25, 26\}
\]

It is relatively easy to check for each triple $j_0 < j_1 < j_2 \leq k$ that $x_{j_0,k} \ll_{\text{lex}} x_{j_1,k} \ll_{\text{lex}} x_{j_2,k}$ and $\delta(x_{j_0,k}, x_{j_2,k}) < \delta(x_{j_0,k}, x_{j_1,k})$. It follows then from our assumption that $\Psi_k$ holds for $A$ with $f\{A(x_{j_0,k}), A(x_{j_1,k}), A(x_{j_2,k})\} = 1$ for each triple $j_0 < j_1 < j_2 \leq k$. In other words, $f$ is identically one on the triples of the $(k+1)$-element set $\{A(x_{j,k}) \mid j \leq k\}$. \(\square\)

With the two previous lemmas in hand, our proof of Lemma 1 is almost complete. We need only establish that either $\Phi_k$ is true for some $k < \omega$ or $\Psi_n$ is true. We satisfy this need in Lemma 1.4 below. Before we can do that, however, we need to analyze the $\Psi_k$’s a little more.

For each $k < \omega$ and each $D \in [\omega_1]^{\omega_1}$, let $W_k(D)$ be the set of all $A \in [D]^{\omega_k}$ such that $f\{A(x), A(y), A(z)\} = 1$ for each $x, y, z \in [\omega]^k$ with $x \ll_{\text{lex}} y \ll_{\text{lex}} z$ and $\delta(x, z) < \delta(x, y)$. \(\square\)
$\delta(x, z) < \delta(x, y)$, so that, for example, $\Psi_k$ is simply the assertion that $W_k(\omega_1)$ is nonempty.

We will need the following facts in our proof of Lemma 1.3 below.

**Lemma 1.3.** Assume that $k < \omega$ and $D \in [\omega]^{\omega_1}$.

1. Suppose that $A \in W_k(D)$ and $X \in [\omega]^\omega$. Let $A \upharpoonright X = \{ A(x) \mid x \in [X]^k \}$. Then $A \upharpoonright X$ is also in $W_k(D)$.

2. Suppose $\{ A_i \mid i < \omega \} \subseteq W_k(D)$ with $A_i \ll A_j$ and $f \{ A_i(x), A_i(y), A_j(z) \} = 1$ for each pair $i < j < \omega$ and each triple $x, y, z \in [\omega]^k$ with $x \ll_{\text{lex}} y$. Let $A = \bigcup \{ A_i \mid i < \omega \}$. Then $A \in W_{k+1}(D)$.

**Proof.** The first follows from a trivial reindexing argument, which we leave to the reader. The second is a little more devious, but not too much so. To help us in its proof we define the “downshift” map $d : [\omega]^{k+1} \to [\omega]^k$ as follows. For each $x \in [\omega]^{k+1}$ let

$$d(x) = \{ \ell - \min(x) - 1 \mid \ell \in \omega \land \ell \neq \min(x) \}.$$

Note that $A(x) = A_{\min(x)}(d(x))$ for each $x \in [\omega]^{k+1}$ and that if $\min(x) = \min(y)$, then $x \ll_{\text{lex}} y$ if and only if $d(x) \ll_{\text{lex}} d(y)$.

Clearly $A \in [D]^{\omega^{k+1}}$. As for the rest, let $x, y, z \in [\omega]^{k+1}$ with $x \ll_{\text{lex}} y \ll_{\text{lex}} z$ and $\delta(x, z) < \delta(x, y)$ be given. We need to show that $f \{ A(x), A(y), A(z) \} = 1$.

There are two cases to consider.

**Case 1.** Suppose $\delta(x, z) = 0$, so that $\min(x) = \min(y) < \min(z)$. Let $i = \min(x) = \min(y)$ and $j = \min(z)$. Then

$$f \{ A(x), A(y), A(z) \} = f \{ A_i(d(x)), A_i(d(y)), A_j(d(z)) \} = 1,$$

by hypothesis since $i < j$ and $d(x) \ll_{\text{lex}} d(y)$.

**Case 2.** Suppose that $\delta(x, z) > 0$, so that $\min(x) = \min(y) = \min(z)$. Let $i = \min(x) = \min(y) = \min(z)$. Then

$$f \{ A(x), A(y), A(z) \} = f \{ A_i(d(x)), A_i(d(y)), A_i(d(z)) \} = 1,$$

by hypothesis since $A_i \in W_k(D)$.

This final lemma will complete our proof of Lemma 1 by demonstrating that either $\Psi_n$ holds or $\Phi_k$ holds for some $k < \omega$. This, together with Lemmas 1.3 and 1.4, means that either (a) or (b) is obtained, and thus that Lemma 1 is true.

**Lemma 1.4.** For each finite ordinal $k$, if $\Phi_j$ fails for every $j \leq k$, then $\Psi_k$ holds. (In particular, either $\Phi_k$ holds for some $k < \omega$ and hence (a) or (b) holds by Lemma 1.1 or $\Psi_n$ holds and hence (b) holds by Lemma 1.2.)

**Proof.** Fix a finite ordinal $k \geq 1$. Assume that $\Phi_j$ fails for every $j \leq k$. We will prove that $\Psi_k$ holds. Well, actually, we will prove by induction on $\ell \leq k$ that the stronger statement $\Psi'_\ell$ holds:

$$\Psi'_\ell: \text{For each } D \in [\omega_1]^{\omega_1} \text{ there are sets } A \in W_\ell(D) \text{ and } B \in [D]^{\omega_1} \text{ with } A \ll B \text{ such that } f \{ A(x), A(y), \beta \} = 1 \text{ for each } x, y \in [\omega]^\ell \text{ with } x \ll_{\text{lex}} y \text{ and each } \beta \in B.$$

The sets $A$ and $B$ in the statement above are said to witness $\Psi'_\ell$ for $D$.

Note that $\Psi'_0$ is trivially true. Thus, we need only prove for each $\ell < k$ that $\Psi'_{\ell+1}$ follows from $\Psi'_\ell$ (and the failure of $\Phi_{\ell+1}$).
Suppose that $D \in [\omega_1]^{\omega_1}$. Recursively construct sequences $\{A_i \mid i < \omega\} \subseteq W_\ell(D)$ and $\{B_i \mid i < \omega\} \subseteq [D]^{\omega_1}$ as follows. First choose $A_0 \in W_\ell(D)$ and $B_0 \in [D]^{\omega_1}$ which witness $\Psi'_\ell$ for $D$. Next, for each $i < \omega$, assuming that $A_i$ and $B_i$ have been chosen, pick $A_{i+1} \in W_\ell(B_i)$ and $B_{i+1} \in [B_i]^{\omega_1}$ which witness $\Psi'_i$ for $B_i$. This completes the recursive construction.

Let $A = \bigcup\{A_i \mid i < \omega\}$. By Lemma 1.3 we know that $A \in W_{\ell+1}(D)$. As $\Phi_{\ell+1}$ fails, all but countably many $\beta \in D$ have $A \preccurlyeq \{\beta\}$ and $i_{j,\ell+1,A,\beta} = 1$ for all $j \leq \ell$.

Let 

$$B = \{\beta \in D \mid A \preccurlyeq \{\beta\} \land \forall j \leq \ell \ [i_{j,\ell+1,A,\beta} = 1]\}.$$ 

As $p > \omega_1$, we can find $Y \in [\omega]^{\omega_1}$ such that $Y \subseteq^* X_{\ell+1,A,\beta}$ for all $\beta \in B$. (Recall that $X_{\ell+1,A,\beta} = \bigcap\{X_{j,\ell+1,A,\beta} \mid j \leq \ell\}$.) For each $\beta \in B$, let $N_\beta$ be the least $N < \omega$ such that $Y \setminus N \subseteq X_{\ell+1,A,\beta}$. As $B$ is uncountable, there must be $N < \omega$ and $B' \in [B]^{\omega_1}$ with $N_\beta = N$ for all $\beta \in B'$. Let $A' = A \upharpoonright (Y \setminus N) = \{A(x) \mid x \in [Y \setminus N]^{\ell+1}\}$. By Lemma 1.3 $A' \in W_{\ell+1}(D)$. Moreover, these $A'$ and $B'$ satisfy $\Psi'_{\ell+1}$ for $D$.

The following two lemmas finish our proof of Theorem 1 as outlined at the start of this section. Lemma 2 indicates that we can find a conservative generic extension of the universe in which for each $m, n < \omega$ the partition relation $\omega_1 \to (\omega + m, n)^3$ holds. Lemma 3 then shows that this relation is absolute. From this, Theorem 1 then follows.

**Lemma 2.** There is a notion of forcing $\mathbb{P}$ such that

$$\mathbb{P} \vdash \omega_1 = \omega_1 \land \omega_1 \to (\omega + m, n)^3 \text{ for each } m, n < \omega.$$ 

**Proof.** By Lemma 1, any notion of forcing $\mathbb{P}$ for which $\mathbb{P} \vdash \omega_1 = \omega_1 \land \omega_1 \not\prec p$ will do. Constructions of such notions of forcing appear in [7], [9], [14], etc.

**Lemma 3** (J. Silver). Let $m$ and $n$ be finite ordinals, and $\{\alpha_i \mid i < n\}$ be countable ordinals. If there is a notion of forcing $\mathbb{P}$ such that

$$\mathbb{P} \vdash \omega_1 = \omega_1 \land \omega_1 \to (\alpha_i)^m_{i < n},$$

then, indeed, $\omega_1 \to (\alpha_i)^m_{i < n}$. Thus if one can conservatively force such a partition relation to be true, then one did not need to force in the first place.

**Proof.** See the presentation in either [11] or [12].

4. **Conclusion**

The following question still remains largely unanswered:

For which $\alpha < \omega_1$ and $n < \omega$ does $\omega_1 \to (\alpha, n)^3$?

By the main result of [9] and Theorem 1 above, the simplest unresolved cases of this question are now:

**Problem 1.** Does $\omega_1 \to (\omega + \omega + 2, 4)^3$?

**Problem 2.** Does $\omega_1 \to (\omega + \omega, 5)^3$?
References


Department of Mathematics, University of Kansas, Lawrence, Kansas 66045-2142
Current address: 2153 Oakdale Rd., Pasadena, Maryland 21122-5715
E-mail address: alj@mojumi.net