ON CHARACTERIZATION AND PERTURBATION
OF LOCAL C-SEMIGROUPS

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Abstract. Let \( S(\cdot) \) be a \((C_0)\)-group with generator \(-B\), and let \( \{T(t); 0 \leq t < \tau\}\) be a local C-semigroup commuting with \( S(\cdot)\). Then the operators \( V(t) := S(-t)T(t); 0 \leq t < \tau \), form a local C-semigroup. It is proved that if \( C \) is injective and \( A \) is the generator of \( T(\cdot) \), then \( A + B \) is closable and \( A + B \) is the generator of \( V(\cdot) \). Also proved are a characterization theorem for local C-semigroups with \( C \) not necessarily injective and a theorem about solvability of the abstract inhomogeneous Cauchy problem: \( u'(t) = Au(t) + Cf(t), 0 < t < \tau; u(0) = Cx. \)

1. Introduction and Result

The aim of this paper is to present some theorems about characterization and perturbation of local C-semigroups and the solvability of the associated inhomogeneous Cauchy problem. Let \( X \) be a complex Banach space and let \( B(X) \) be the Banach algebra of all bounded (linear) operators on \( X \). When \( 0 < \tau < \infty \) (resp. \( \tau = \infty \)), a family \( \{T(t); 0 \leq t < \tau\}\) in \( B(X) \) is called a local C-semigroup (resp. (global) C-semigroup) on \( X \) if

(a) \( T(\cdot)x : [0, \tau) \to X \) is continuous for each \( x \in X \).

(b) \( T(s + t)C = T(s)T(t) \) for all \( 0 \leq s, t, t + s < \tau \) and \( T(0) = C \).

C-semigroups have been studied in many papers, among them are [1], [2], [3], [7], and [10]. Local C-semigroups have been studied in [1], [6], [8], [9], [11], [12], and [13].

When \( \tau = \infty \) and \( C = I \), the identity operator, \( T(\cdot) \) is a classical \((C_0)\)-semigroup. Thus \((C_0)\)-semigroups form a subclass of the class of C-semigroups, and clearly, every C-semigroup can be viewed as a local C-semigroup defined on \([0, \tau)\) for any \( 0 < \tau < \infty \). In general, a local C-semigroup on \([0, \tau)\) for some \( \tau < \infty \) is not necessarily extendible to the whole half line \([0, \infty)\). Results concerning extension of local C-semigroups can be found in [4] and [12].

We first state the following general characterization theorem for local C-semigroups. Its proof will be given in Section 2.

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1097
Theorem 1. A strongly continuous family $\{T(t); 0 \leq t < \tau\}$ is a local $C$-semigroup if and only if $T(\cdot)$ commutes with $C$ and satisfies $T(0) = C$ and

$$
(1.1) \quad (1 \ast T)(s)[T(t) - C] = [T(s) - C](1 \ast T)(t) \quad \text{for all } 0 \leq s, t, s + t < \tau.
$$

We are mainly interested in those local $C$-semigroups which are nondegenerate. A local $C$-semigroup $\{T(t); 0 \leq t < \tau\}$ will be said to be nondegenerate if one of the equivalent conditions in the next lemma is satisfied.

Lemma 2. The following statements are equivalent:

1. $C$ is injective.
2. $\lim_{t \to 0^+} T(t)x = 0$ implies $x = 0$.
3. $T(t)x \equiv 0$ on $(0, s_1)$ for some $s_1 \in (0, \tau/2)$ implies $x = 0$.
4. $T(t)x \equiv 0$ on $(0, \tau/2)$ implies $x = 0$.

Proof. (c1) $\Rightarrow$ (c2). If $T(t)x \to 0$ as $t \to 0^+$, by (a) and (b) we have $Cx = \lim_{t \to 0^+} T(t)x = 0$, and (c1) implies $x = 0$.

(c2) implies (c3) and (c4) are obvious.

(c4) $\Rightarrow$ (c1). If $Cx = 0$, then from (b) we see that $T(s)T(t)x = 0$ for all $0 < s, t < \tau/2$, which implies $x = 0$ by (c4). Hence $C$ is injective.

When $\tau = \infty$, the above definition of nondegeneracy coincides with the usual definition of nondegeneracy for $C$-semigroups, i.e., $T(t)x = 0$ for all $t > 0$ implies $x = 0$. But, when $\tau < \infty$, (c4) is even strictly stronger than the following condition:

(c') $T(t)x \equiv 0$ on $(0, s_2)$ for some $s_2 \in (\tau/2, \tau)$ implies $x = 0$,

because, unlike the case $\tau = \infty$, (c') is not equivalent to the injectivity of $C$ when $\tau < \infty$. Also, unlike $C$-semigroups, a local $C$-semigroup $\{T(t); 0 \leq t < \tau\}$ with $\tau < \infty$ need not be commutative, although $C$ commutes with each $T(t), t \in [0, \tau)$, and $\{T(t); 0 \leq t < \tau/2\}$ is a commutative subfamily, by (b). These interesting phenomena are illustrated by the following example.

Let $U: [\tau/2, \tau) \to B(X)$ be a strongly continuous function such that $U(\tau/2) = 0$ and $U(t)$ is injective for all $t \in (\tau/2, \tau)$. The operator $C := 0 \oplus I \in B(X \times X)$, with $I$ the identity operator on $X$, is not injective. We define $T: [0, \tau) \to B(X \times X)$ by

$$
T(t) := \begin{cases} 
0 \oplus I & \text{if } 0 \leq t < \tau/2, \\
U(t) \oplus I & \text{if } \tau/2 \leq t < \tau.
\end{cases}
$$

Then $T(\cdot)$ is strongly continuous and satisfies

$$
T(s)T(t) = T(t)T(s) = 0 \oplus I = T(s + t)C \quad \text{for all } 0 \leq s, t, s + t < \tau.
$$

Hence this $T(\cdot)$ is a local $C$-semigroup with $C$ not injective. But, it satisfies condition (c'). Indeed, if there is a $t \in (\tau/2, \tau)$ such that $T(t)x = 0$, then since

$$
C(t)x = (U(t) \oplus I)(x_1, x_2) = (U(t)x_1, x_2)
$$

and $U(t)$ is injective, we have $x_1 = x_2 = 0$ and hence $x = 0$. Moreover, since the family $\{U(t); \tau/2 \leq t < \tau\}$ need not be commutative, the above local $C$-semigroup $T(\cdot)$ may not be a commutative family.

When $C$ is injective, $T(\cdot)$ is indeed commutative (see Lemma 1.1).

Now for a nondegenerate local $C$-semigroup $T(\cdot)$, by either (c2) or (c4) one can define the generator $A$ by

(d) $x \in D(A)$ and $Ax = y \Leftrightarrow \int_0^t T(s)yds = T(t)x - Cx$ for all $t \in [0, \tau)$. 

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It is known [8] that this definition is equivalent to the one in the sense of Da Prato [H]:

\[ D(A) = \{ x \in X : \lim_{h \to 0^+} (T(h)x - Cx)/h \in R(C) \}, \]
\[ Ax = C^{-1} \lim_{h \to 0^+} (T(h)x - Cx)/h \text{ for } x \in D(A). \]

The following characterization of a nondegenerate local C-semigroup is proved in [8] Lemma 2.1 and Proposition 2.2.

**Proposition 3.** Let \( C \in B(X) \) be an injection and let \( \{ T(t) ; 0 \leq t < \tau \} \) be a strongly continuous family of operators on \( X \).

(i) If \( T(\cdot) \) is a local C-semigroup with generator \( A \), then \( A \) is closed and satisfies \( C^{-1}AC = A, \ R(\int_0^T T(s)ds) \subset D(A) \) and

\[ T(t)x - Cx = \begin{cases} A \int_0^T T(s)xds & \text{for } x \in X, \\ \int_0^T T(s)Axds & \text{for } x \in D(A) \end{cases} \]

for all \( t \in [0, \tau) \).

(ii) If \( A \) is a closed operator satisfying \( R(\int_0^T T(s)ds) \subset D(A) \) and (1.2), then \( T(\cdot) \) is a local C-semigroup with generator \( C^{-1}AC \).

Consequently, a strongly continuous family \( \{ T(t) ; 0 \leq t < \tau \} \) is a local C-semigroup with generator \( A \) if and only if \( A \) is closed and satisfies \( C^{-1}AC = A, \ R(\int_0^T T(s)ds) \subset D(A) \) and (1.2).

The following characterization of generator in terms of solvability of the abstract Cauchy problem is proved in [8] Corollary 2.6.

**Proposition 4.** An operator \( A \) is the generator of a local C-semigroup \( \{ T(t) ; t \in [0, \tau) \} \) if and only if the abstract Cauchy problem

\[ \text{ACP}(A;Cx + C(1 + g), 0) \]
\[ \begin{cases} u'(t) = Au(t) + Cx + \int_0^t Cg(s)ds, & 0 < t < \tau, \\ u(0) = 0 \end{cases} \]

has a unique solution for every \( x \in X \) and \( g \in L^1_{loc}([0, \tau), X) \). The solution is

\[ u(t) = \int_0^t T(s)xds + \int_0^t \int_0^s T(s-r)g(r)drds. \]

This proposition can be used to deduce assertion (ii) of the next theorem, which is concerned with the solvability of the abstract Cauchy problem

\[ \text{ACP}(A; Cf, Cx) \]
\[ \begin{cases} u'(t) = Au(t) + Cf(t), & 0 < t < \tau, \\ u(0) = Cx \end{cases} \]

for suitable vector \( x \) and vector function \( f \).

**Theorem 5.** Let \( A \) be the generator of a nondegenerate local C-semigroup \( \{ T(t) ; 0 \leq t < \tau \} \) on \( X \), \( x \in X \), and \( f \in C([0, \tau), X) \).

(i) If \( \text{ACP}(A; Cf, Cx) \) has a strong solution \( u \), then \( u \equiv T(\cdot)x + Tf \).

(ii) If \( u := T(\cdot)x + Tf \in C([0, \tau), [D(A)]) \), where \( [D(A)] \) is the Banach space \( D(A) \) equipped with the graph norm, then \( u \) is a strong solution of \( \text{ACP}(A; Cf, Cx) \).

(iii) If either \( f \in C^1([0, \tau), X) \) or \( f \in C([0, \tau), [D(A)]) \), then for every \( x \in D(A) \) \( \text{ACP}(A; Cf, Cx) \) has the unique strong solution \( u(t) := T(t)x + (Tf)(t), 0 \leq t < \tau. \)
Theorem 5 is well known for the case \( \tau = \infty \) (cf. \[7, Theorem 7.1\) and Corollary 7.5]) and the case \( C = I \) (cf. \[5\]).

As an application of Proposition 4, it is deduced in \[9, Theorem 2\] that if \( A \) is the generator of a local \( C \)-semigroup and if \( B \in \mathcal{B}(X) \) satisfies \( R(B) \subset R(C) \) and \( BCx = CBx \) for \( x \in D(A) \), then \( A + B \) also generates a local \( C \)-semigroup. In the following theorem we extend this result to those unbounded perturbation operators \( B \) that generate some commuting \((C_0)\)-groups.

**Theorem 6.** Let \( T(\cdot) \) be a local \( C \)-semigroup on a Banach space \( X \) and let \( S(\cdot) \) be a \((C_0)\)-group with generator \(-B\). Suppose \( S(t)T(s) = T(s)S(t) \) for all \( 0 \leq s, t < \tau \). Let \( V(t) := S(-t)T(t) \) for \( t \in [0, \tau) \). Then

(i) \( \{V(t); t \in [0, \tau)\} \) is the unique local \( C \)-semigroup commuting with \( S(\cdot) \) and \( T(\cdot) \) and satisfying

\[
(1.3) \quad \int_0^t S(u)(1 \ast V)(u)du = \int_0^t S(u)(1 \ast T)(t-u)du.
\]

(ii) If \( C \) is injective and \( A \) is the generator of \( T(\cdot) \), then \( A + B \) is closable and \( A + B \) is the generator of \( V(\cdot) \). In particular, if \( B \in \mathcal{B}(X) \) commutes with \( T(\cdot) \), then \( A + B \) generates the local \( C \)-semigroup \( \{e^{-tB}T(t); 0 \leq t < \tau\} \).

We remark that a multiplicative perturbation theorem for local \( C \)-semigroups has been proved in \[13\].

The proofs of Theorems 1, 5, and 6 will be given in Sections 2 and 3, and a simple example of an application of Proposition 4 and Theorems 5 and 6 will be given in Section 4.

### 2. Proofs of Theorems 1 and 5

**Proof of Theorem 1.** Suppose \( T(\cdot) \) is a local \( C \)-semigroup on \( X \). Then, by the commutativity of \((1 \ast T)(t)\) with \( T(s) \) and \( C \) for \( 0 \leq s, t, s + t < \tau \), we have

\[
T(s)(1 \ast T)(t)x = \int_0^t T(s)T(u)xdu = \int_0^t T(s + u)Cxdu = \int_s^{s+t} T(u)Cxdu
\]

\[
= \int_t^{s+t} T(u)Cxdu + \int_0^t T(u)Cxdu - \int_0^s T(u)Cxdu
\]

\[
= (1 \ast T)(s)T(t)x + C(1 \ast T)(t)x - (1 \ast T)(s)Cx
\]

for \( x \in X \) and \( 0 \leq s, t, s + t < \tau \). This shows (1.1).

Conversely, suppose that \( T(\cdot) \) satisfies (1.1). For fixed \( s, t \in [0, \tau) \) with \( s + t < \tau \), we replace \( s \) and \( t \) in (1.1) by \( s + t - r \) and \( r \), respectively, to obtain

\[
(1 \ast T)(s + t - r)T(r)x - T(s + t - r)(1 \ast T)(r)x = (1 \ast T)(s + t - r)Cx - C(1 \ast T)(r)x
\]

for all \( x \in X \) and \( r \in [0, s + t) \). By integrating the right-hand side with respect to \( r \) from 0 to \( t \), we obtain from \( CT(\cdot) = T(\cdot)C \) that

\[
\int_0^t (1 \ast T)(s + t - r)Cxdr - \int_0^t C(1 \ast T)(r)xdr
\]

\[
= \int_s^{s+t} (1 \ast T)(r)Cxr dr - \int_0^t C(1 \ast T)(r)xdr
\]

\[
= \int_0^{s+t} (1 \ast T)(r)Cxr dr - \int_0^t C(1 \ast T)(r)xdr
\]

\[
= \int_0^{s+t} (1 \ast T)(r)Cxr dr.
\]
On the other hand, from the left-hand side we have
\[
\int_0^t (1 * T)(s + t - r)T(r)xdr - \int_0^t T(s + t - r)(1 * T)(r)xdr
\]
\[
= (1 * T)(s + t - r)(1 * T)(r)x^t_0 + \int_0^t T(s + t - r)(1 * T)(r)xdr
\]
\[
- \int_0^t T(s + t - r)(1 * T)(r)xdr
\]
\[
= (1 * T)(s)(1 * T)(t)x - (1 * T)(s + t)(1 * T)(0)x
\]
\[
= (1 * T)(s)(1 * T)(t)x.
\]
Therefore, \((1 * T)(\cdot)\) satisfies
\[
(1 * T)(s)(1 * T)(t)x = (\int_0^s \int_0^t - \int_0^s - \int_0^t ) (1 * T)(r)xdr
\]
for all \(x \in X\) and \(s, t, s + t \in [0, \tau]\). By differentiation, it is clear that \(T(\cdot)\) is a local \(C\)-semigroup.

**Proof of Theorem 5.** (i) Suppose \(u\) is a strong solution of \(ACP(A; C_f, C_x)\). Then \(u(0) = Cx\) and \(u' = Au + C_f\), so that \(u - Cx = 1 * u' = 1 * (Au + C_f)\). Therefore we have
\[
T * u - C(1 * T)(\cdot)x = T * (u - Cx) = (1 * T)A * u + 1 * (T * C_f)
\]
\[
= (T - C) * u + C(1 * T) * f
\]
\[
= T * u - C(1 * u) + C1 * (T * f).
\]
Since \(C\) is injective, we have \(1 * u = (1 * T)(\cdot)x + 1 * (T * f)\). By differentiation, we have \(u = T(\cdot)x + T * f\). This also proves that the strong solution of \(ACP(A; C_f, C_x)\) is unique.

(ii) Suppose \(u := T(\cdot)x + T * f \in C([0, \tau), [D(A)])\). Then \(u(0) = Cx\), \(Au \in C([0, \tau), X)\) and \(1 * u = (1 * T)(\cdot)x + (1 * T) * f\), so that, by the closedness of \(A\), we have
\[
1 * (Au) = A(1 * u) = T(\cdot)x - Cx + (T - C) * f = u - Cx - C(1 * f).
\]
By differentiation, we have \(u' = Au + C_f\).

Derivation of (ii) from Proposition 4. By Proposition 4, \(ACP(A; C_x + C(1 * f), 0)\) has a unique solution \(v\) given by \(v(t) = \int_0^t u(s)ds\) for \(t \in [0, \tau)\), where \(u = T(\cdot)x + T * f\). If \(u \in C([0, \tau), [D(A)])\), then by the closedness of \(A\), we have \(v(t) \in D(A)\) and \(Av(t) = \int_0^t Au(s)ds\) is continuously differentiable in \(X\) on \([0, \tau)\). Hence \(u = v' = Au + C_x + C(1 * f)\) is continuously differentiable and satisfies \(u'(t) = Au(t) + C_f(t), 0 < t < \tau\), and \(u(0) = T(0)x = Cx\), i.e., \(u\) is a strong solution of \(ACP(A; C_f, C_x)\).

(iii) Consider the following two cases.
If \(f \in C^1([0, \tau), X)\), then \(T * f \in C^1([0, \tau), X)\) so that
\[
A[1 * (T * f)] = A(1 * T) * f = (T - C) * f
\]
\[
= T * f - C(1 * f) \in C^1([0, \tau), X).
\]
Since \(A\) is closed, it follows from differentiation that \((T * f)(t) \in D(A)\) for all \(t \in [0, \tau)\) and \(A(T * f) = (T * f)' - C_f \in C([0, \tau), X)\), i.e., \(T * f \in C([0, \tau), [D(A)])\).
If \( f \in C([0, \tau], [D(A)]) \), then \( A(T \ast f) = T \ast Af \in C([0, \tau], X) \), and hence we also have \( T \ast f \in C([0, \tau], [D(A)]) \).

Since \( T(\cdot)x \in C([0, \tau], [D(A)]) \) for \( x \in D(A) \), in both the above cases, we have \( u := T(\cdot)x + T \ast f \in C([0, \tau], [D(A)]) \) for every \( x \in D(A) \). Therefore \( u \) is a strong solution of \( ACP(A; Cf, Cx) \) for every \( x \in D(A) \), by (ii).

\[ \square \]

3. Proof of Theorem 6

We first prove the following proposition.

**Proposition 7.** Let \( T(\cdot) \) and \( S(\cdot) \) be two commuting local \( C \)-semigroups with generators \( A \) and \( B \), respectively. Then the following hold.

(i) \( A + B \) is closable and satisfies:
\[
(A + B) \subset C^{-1}(A + B)C \quad \text{and} \quad \overline{A + B} \subset C^{-1}A + BC.
\]

(ii) If, in addition, either one of \( T(\cdot) \) and \( S(\cdot) \) is even a \((C_0)\)-semigroup, then
\[
C^{-1}A + BC = \overline{A + B}.
\]

**Proof:** (i) First, we show that \( A + B \) is closable. Let \( \{x_n\} \) be a null sequence in \( D(A + B) \) such that \( (A + B)x_n \) converges to a vector \( y \in X \). We need to show \( y = 0 \). Observe that \( S(t)T(s) = T(s)S(t) \) implies that \( S(t)Ax = AS(t)x \) for \( x \in D(A) \). Hence we have
\[
(1 \ast T)(t)(1 \ast S)(s)y = \lim_{n \to \infty} (1 \ast T)(t)(1 \ast S)(s)(A + B)x_n
= \lim_{n \to \infty} \left[ |T(t) - C|(1 \ast S)(s)x_n + (1 \ast T)(t)|S(s) - C|x_n \right]
= \left[ |T(t) - C|(1 \ast S)(s)0 + (1 \ast T)(t)|S(s) - C|0 \right] = 0
\]
and then \( T(s)y = 0 \) for all \( s, t \in (0, \tau) \), by differentiation. Then the strong continuity of \( T(\cdot) \) and \( S(\cdot) \) at 0 implies \( C^2y = 0 \) and hence \( y = 0 \). Therefore \( A + B \) is closable.

Let \( x \in D(A + B) = D(A) \cap D(B) \). Since \( A \) and \( B \) are generators, by Proposition 3(i), we have \( C^{-1}ACx = Ax \) and \( C^{-1}BCx = Bx \), so that \( Cx \in D(A) \cap D(B) = D(A + B) \) and \( ACx = CAx \) and \( BCx = CBx \). Hence \( (A + B)Cx = ACx + BCx = C(A + B)x \) and so \( x \in D(C^{-1}(A + B)C) \) and \( (A + B)x = C^{-1}(A + B)Cx \). Hence \( (A + B) \subset C^{-1}(A + B)C \). Next, we show \( \overline{A + B} \subset C^{-1}A + BC \). If \( x \in D(\overline{A + B}) \), then there is a sequence \( \{x_n\} \) in \( D(A + B) \) such that \( (x_n, (A + B)x_n) \to (x, \overline{A + B}x) \). As above, we have \( (A + B)Cx_n = C(A + B)x_n \to CA + BCx \). This with the fact that \( Cx_n \to Cx \) implies that \( Cx \in D(\overline{A + B}) \) and \( \overline{A + BC} = CA + BCx \) or \( \overline{A + B} = C^{-1}A + BC \).

(ii) Assume \( S(\cdot) \) is a \((C_0)\)-semigroup. It remains to show the inclusion:
\[
C^{-1}A + BC \subset \overline{A + B}.
\]

Let \( x \in D(C^{-1}\overline{A + BC}) \) and \( y := C^{-1}\overline{A + BC}x \). Then \( Cy = \overline{A + BC}x \). So, there is a sequence \( \{z_n\} \) in \( D(A + B) \) such that \( (z_n, (A + B)z_n) \to (Cx, Cy) \) strongly as \( n \to \infty \). Therefore we have for every \( s, t \in [0, \tau) \)
\[
(1 \ast T)(t)(1 \ast S)(s)Cy
= \lim_{n \to \infty} (1 \ast T)(t)(1 \ast S)(s)(A + B)z_n
= \lim_{n \to \infty} \left[ (1 \ast S)(t)[T(s) - C]z_n + (1 \ast T)(s)(S(t) - I)z_n \right]
= (1 \ast S)(t)[T(s) - C]Cx + (1 \ast T)(s)(S(t) - I)Cx.
\]
Since $T(\cdot), S(\cdot),$ and $C$ commute, it follows from the injectivity of $C$ that
\[
(1 * T)(s)[(1 * S)(t)y - (S(t) - I)x] = [T(s) - C][(1 * S)(t)x]
\]
for every $s, t \in [0, \tau)$. By the definition of generator, this implies that $(1 * S)(t)x \in D(A)$ and
\[
A(1 * S)(t)x = (1 * S)(t)y - (S(t) - I)x = (1 * S)(t)y - B(1 * S)(t)x.
\]
Hence we have for every $t \in [0, \tau)$
\[
(1 * S)(t)y = (A + B)(1 * S)(t)x = \frac{A + B}{A + B}(1 * S)(t)x.
\]
By differentiation, we have $S(t)x \in D(A + B)$ and $S(t)y = \frac{A + B}{A + B}S(t)x$. Since $S(0) = I$, this implies that $x \in D(A + B)$ and $y \in A + B$. Therefore $C^{-1}(A + BC)$$\subset A + B$.

**Proof of Theorem 6.** (i) Since $S(\cdot)$ commutes with $T(\cdot)$, clearly $V(\cdot)$ is a local $C$-semigroup commuting with $S(\cdot)$ and $T(\cdot)$. $V(\cdot)$ satisfies (1.3):
\[
\int_{0}^{t} S(u)(1 * V)(u)du = \int_{0}^{t} S(u) \int_{0}^{u} S(-s)T(s)dsdu = \int_{0}^{t} \int_{0}^{t} S(u - s)T(s)duds
\]
\[
= \int_{0}^{t} \int_{0}^{t - s} S(u)T(s)dsdu = \int_{0}^{t} S(u) \int_{0}^{t - u} T(s)dsdu
\]
\[
= \int_{0}^{t} S(u)(1 * T)(t - u)du.
\]
Suppose $V_{1}(\cdot)$ and $V_{2}(\cdot)$ are two functions satisfying (1.3). Then the function $V(\cdot) := V_{1}(\cdot) - V_{2}(\cdot)$ satisfies $\int_{0}^{t} S(u)(1 * V)(u)du = 0$ for all $t \in [0, \tau)$. Hence $S(t)(1 * V)(t) = 0$ for all $t \in [0, \tau)$. Since $S(t)$ is injective, we must have $V(\cdot) \equiv 0$.

(ii) Since $S(-t)$ is a $(C_{0})$-semigroup with generator $B$, an application of Proposition 7 (with $S(\cdot)$ therein replaced by $S(-t)$) yields that $A + B$ is closable and $C^{-1}(A + BC) = \frac{A + B}{A + B}$.

Since $(1 * T)(t)A \subset A(1 * T)(t) = T(t) - C$ for $t \in [0, \tau)$, and since $A$ is closed and $S(t)Ay = AS(t)y$ for $y \in D(A)$ we have $R(\int_{0}^{t} S(u)(1 * T)(t - u)du) \subset D(A)$ and
\[
\int_{0}^{t} S(u)(1 * T)(t - u)du A \subset A \int_{0}^{t} S(u)(1 * T)(t - u)du
\]
\[
= \int_{0}^{t} S(u)A(1 * T)(t - u)du
\]
\[
= \int_{0}^{t} S(u)[T(t) - C]du
\]
\[
= \frac{d}{dt} \int_{0}^{t} S(u)(1 * T)(t - u)du - \int_{0}^{t} S(u)Cdu
\]
\[
= \frac{d}{dt} \int_{0}^{t} S(u)(1 * V)(u)du - \int_{0}^{t} S(u)Cdu
\]
\[
= S(t)(1 * V)(t) - \int_{0}^{t} S(u)Cdu
\]
for all $t \in [0, \tau)$. This and (1.3) imply that

$$
(3.1) \quad \int_0^t S(u)(1 * V)(u)Adu \subset A \int_0^t S(u)(1 * V)(u)du
$$

$$
= S(t)(1 * V)(t) - \int_0^t S(u)Cdu.
$$

Differentiating (1.3), we obtain

$$
(3.2) \quad S(t)(1 * V)(t) = \int_0^t S(u)T(t - u)du \text{ for all } t \in [0, \tau).
$$

Since $1 * V$ commutes with $S(\cdot)$, it commutes with the generator $-B$, i.e., $(1 * V)(u)x \in D(B)$ and $B(1 * V)(u)x = (1 * V)(u)Bx$ for $x \in D(B)$, so that $S'(u)(1 * V)(u)x = -BS(u)(1 * V)(u)x = -S(u)B(1 * V)(u)x = -S(u)(1 * V)(u)Bx$ for all $u \in [0, \tau)$. Then, using integration by parts, the closedness of $B$, and (1.3), we obtain for $x \in D(B)$ and $t \in [0, \tau)$,

$$
S(t)(1 * V)(t)x = -\int_0^t S(u)(1 * V)(u)Bxdu + \int_0^t S(u)V(u)xdu
$$

$$
= -\int_0^t BS(u)(1 * V)(u)xdu + \int_0^t S(u)V(u)xdu
$$

$$
= -B \int_0^t S(u)(1 * T)(t - u)xdu + \int_0^t S(u)V(u)xdu.
$$

Combining this and (3.2), and by the closedness of $B$ again, we obtain

$$
(3.3) \quad \int_0^t S(u)(1 * T)(t - u)duB \subset B \int_0^t S(u)(1 * T)(t - u)du
$$

$$
= -\int_0^t S(u)T(t - u)du + \int_0^t S(u)V(u)du
$$

for every $t \in [0, \tau)$.

Now we obtain from (1.3), (3.2) and (3.3) that

$$
(3.4) \quad \int_0^t S(u)(1 * V)(u)Bdu \subset B \int_0^t S(u)(1 * V)(u)du
$$

$$
= \int_0^t S(u)V(u)du - S(t)(1 * V)(t).
$$

Hence, summing (3.1) and (3.4) and then taking the closure of $A + B$ we have for every $t \in [0, \tau)$

$$
(3.5) \quad \int_0^t S(u)(1 * V)(u)(A + B)du \subset (A + B) \int_0^t S(u)(1 * V)(u)du
$$

$$
= \int_0^t S(u)[V(u) - C]du.
$$

Since $A + B$ is closed, differentiation with respect to $t$ yields

$$
R(S(t)(1 * V)(t)) \subset D(A + B)
$$

and

$$
(3.6) \quad S(t)(1 * V)(t)A + B \subset \overline{A + BS(t)(1 * V)(t)} = S(t)[V(t) - C]$$
for all \( t \in [0, \tau) \). Since \( S(t) \) is injective, \((1 \ast V)(t)A + B \subset V(t) - C\). On the other hand, since \( S(\cdot) \) commutes with \( V(\cdot) \), we have \( R((1 \ast V)(t))S(t) \subset D(A + B) \) and \( A + B(1 \ast V)(t)S(t) = [V(t) - C]S(t) \). Then, by the surjectivity of \( S(t) \), we obtain that \( R((1 \ast V)(t)) \subset D(A + B) \) and
\[
A + B(1 \ast V)(t) = V(t) - C.
\]
Thus we have shown
\[
(1 \ast V)(t)A + B \subset A + B(1 \ast V)(t) = V(t) - C.
\]
The conclusion of the theorem now follows from Proposition 3.

4. AN ILLUSTRATIVE EXAMPLE

Consider the following initial value problems in \( c_0 \):

\[
\begin{align*}
\begin{cases}
u_n(t) &= (1 + in)u_n(t) + e^{-n}q_n + \int_0^t e^{-n}g_n(s)ds, \ 0 < t < 1, \\
u_n(0) &= 0,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
u_n(t) &= (1 + in)u_n(t) + e^{-n}f_n(t), \ 0 < t < 1, \\
u_n(0) &= e^{-n}q_n.
\end{cases}
\end{align*}
\]

The family \( \{T(t)\}; 0 \leq t \leq 1 \), defined by \( T(t)x := (e^{-nt}x_n), \ x = (x_n) \in c_0 \), \( 0 \leq t \leq 1 \), is a local \( C \)-semigroup with \( C := \bigoplus_{n=1}^{\infty} e^{-n} \in B(c_0) \) and with generator \( A := \bigoplus_{n=1}^{\infty} n \). The diagonal matrix \( -B := -IA^2 = \bigoplus_{n=1}^{\infty} (-in^2) \) generates the \( C_0 \)-group \( S(\cdot) \), defined by \( S(t)x := (e^{-in^2t}x_n), \ x = (x_n) \in c_0, \ t \geq 0 \), which commutes with \( T(\cdot) \). By Theorem 6, \( A + B \) generates the local \( C \)-semigroup \( \{V(t); 0 \leq t < 1\} \), defined by \( V(t)x := (e^{-nt}e^{(1+in)t}x_n), \ x = (x_n) \in c_0 \).

If, for instance, \( g(t) = (g_n(t)) \in c_0 \) for all \( t \in [0, 1] \) and the functions \( \{g_n\} \) are uniformly continuous on \([0, 1]\), then \( g \in C([0, 1]; c_0) \). Now it follows from Proposition 4 that, for any \( q \in c_0 \), (4.1) has a unique solution \( v \in C([0, 1]; c_0) \), which is given by
\[
v(t) = \int_0^t V(s)qds + \int_0^t \int_0^s V(s-r)g(r)drds
= \left( e^{-n} \left[ \frac{1}{n(1 + in)} e^{(1+in)t} - 1 \right] q_n + \int_0^t \int_0^s e^{(1+in)(s-r)} g_n(r)drds \right),
\]
\( 0 \leq t < 1 \).

If \( (n^2f_n(t)) \in c_0 \) for all \( t \in [0, 1] \) and the functions \( \{n^2f_n\} \) are uniformly continuous on \([0, 1]\), then \( f \in C([0, 1]; [D(A + B)]) \). It follows from Theorem 5 that, for any \( q \in c_0 \) with \( \lim_{n \to \infty} n^2q_n = 0 \), (4.2) has a unique solution \( u \in C([0, 1]; c_0) \), which is given by
\[
u(t) = V(t)q + \int_0^t V(t-s)g(s)ds
= \left( e^{-n} \left[ e^{(1+in)t} q_n + \int_0^t e^{(1+in)(t-s)} f_n(s)ds \right] \right), \ 0 \leq t < 1.
\]
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