\textbf{\kappa\text{-BI-LIPSCHITZ EQUIVALENCE OF REAL FUNCTION-GERMS}}

L. BIRBRAIR, J. C. F. COSTA, A. FERNANDES, AND M. A. S. RUAS

(Communicated by Mikhail Shubin)

\textbf{Abstract.} In this paper we prove that the set of equivalence classes of germs of real polynomials of degree less than or equal to \( k \), with respect to \( \kappa \)-bi-Lipschitz equivalence, is finite.

1. Introduction

Finiteness theorems of different kinds appear in the modern development of Singularity Theory. When one considers a classification problem, it is important to know if the problem is “tame” or not. In other words, how difficult the problem is and if there is any hope to develop a complete classification. For the problem of topological classification of polynomial function-germs, a finiteness result was conjectured by R. Thom [13] and proved by Fukuda [5]. He proved that the number of equivalence classes of polynomial function-germs of degree less than or equal to any fixed \( k \), with respect to topological equivalence, is finite. Note that R. Thom also discovered that this finiteness result does not hold for polynomial map-germs \( P: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) [13]. Finiteness theorems for polynomial map-germs, in the real and in the complex case, with respect to topological equivalence were the subject of investigation of various authors (see, for example, [12], [10], [3], [4]) and many interesting results were obtained in this direction. Mostowski [9] and Parusinski [11] proved that the set of equivalence classes of semialgebraic sets with a complexity bounded from below by any fixed \( k \) is finite. A finiteness result does not hold for polynomial function-germs with respect to bi-Lipschitz equivalence. Henry and Parusinski [6] showed that this problem is not tame, i.e., it has “moduli”.

Here we consider the problem of the \( \kappa \)-bi-Lipschitz classification of polynomial function-germs (\( \kappa \)-equivalence is the contact equivalence defined by Mather [7]). We show that this problem is still tame. The main idea of the proof is the following. First, we consider Lipschitz functions “of the same contact”. Namely, \( f \) and \( g \) are of the same contact if \( \frac{f}{g} \) is positive and bounded away from zero and infinity.

We show that two functions of the same contact are \( \kappa \)-bi-Lipschitz equivalent. The next step is related to the geometry of contact equivalence. Recall that two function-germs \( f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) are \( C^\infty \)-contact equivalent if there exists a
$C^\infty$-diffeomorphism in the product space $(\mathbb{R}^n \times \mathbb{R}, 0)$ which leaves $\mathbb{R}^n$ invariant and maps the graph $(f)$ to the graph $(g)$. This definition is due to Mather [7] for map-germs $f, g: \mathbb{R}^n \to \mathbb{R}^p$. Montaldi extended this notion by introducing a purely geometrical definition of contact: two pairs of submanifolds of $\mathbb{R}^n$ have the same contact type if there is a diffeomorphism of $\mathbb{R}^n$ taking one pair to the other, and relating this with the $\mathcal{K}$-equivalence of convenient map-germs [8]. This geometrical interpretation also exists for a topological version of $\mathcal{K}$-equivalence (cf. [11]).

In this paper, we give a definition of Montaldi’s construction for the bi-Lipschitz case and show that the existence of a bi-Lipschitz analogue to Montaldi’s construction and the Mostowski-Parusinski [11] theorem on Lipschitz stratifications.

2. Basic definitions and results

Definition 2.1. Two function-germs $f, g: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are called $\mathcal{K}$-bi-Lipschitz equivalent (or contact bi-Lipschitz equivalent) if there exist two germs of bi-Lipschitz homeomorphisms $h: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H: (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$
\begin{array}{ccc}
(\mathbb{R}^n, 0) & \overset{\text{id}}{\longrightarrow} & (\mathbb{R}^n \times \mathbb{R}, 0) \\
\downarrow h & & \downarrow H \\
(\mathbb{R}^n, 0) & \overset{\text{id}}{\longrightarrow} & (\mathbb{R}^n \times \mathbb{R}, 0)
\end{array}
\begin{array}{c}
\pi_n \\
\downarrow h \\
\pi_n
\end{array}
\begin{array}{c}
(\mathbb{R}^n, 0) \\
\downarrow h \\
(\mathbb{R}^n, 0)
\end{array}
$$

where $\text{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map and $\pi_n : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the canonical projection.

The function-germs $f$ and $g$ are called $C$-bi-Lipschitz equivalent if $h = \text{id}$.

In other words, two function-germs $f$ and $g$ are $\mathcal{K}$-bi-Lipschitz equivalent if there exists a germ of a bi-Lipschitz map $H: (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(x, y)$ can be written in the form $H(x, y) = (h(x), \hat{H}(x, y))$, where $h$ is a bi-Lipschitz map-germ, $\hat{H}(x, 0) = 0$ and $H$ maps the germ of the graph $(f)$ onto the graph $(g)$.

Recall that graph $(f)$ is the set defined as follows:

$$\text{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = f(x)\}.$$

Definition 2.2. Two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are called of the same contact at a point $x_0 \in \mathbb{R}^n$ if there exist a neighborhood $U_{x_0}$ of $x_0$ in $\mathbb{R}^n$ and two positive numbers $c_1$ and $c_2$ such that, for all $x \in U_{x_0}$, we have

$$c_1 f(x) \leq g(x) \leq c_2 f(x).$$

We use the notation: $f \approx g$.

Remark 2.3. It is clear that if two function-germs $f$ and $g$ are of the same contact, then the germs of their zero-sets are equal.

The main results of the paper are the following.
Theorem 2.4. Let \( f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be two germs of Lipschitz functions. Then \( f \) and \( g \) are \( C \)-bi-Lipschitz equivalent if and only if one of the following two conditions is true:

i) \( f \approx g \),

ii) \( f \approx -g \).

Theorem 2.5. Let \( \mathcal{P}_k(\mathbb{R}^n) \) be the set of all polynomials of \( n \) variables with degree less than or equal to \( k \). Then the set of equivalence classes of the germs at \( 0 \) of the polynomials in \( \mathcal{P}_k(\mathbb{R}^n) \), with respect to \( K \)-bi-Lipschitz equivalence, is finite.

3. Functions of the same contact

Proof of Theorem 2.4. Suppose that the germs of the Lipschitz functions \( f \) and \( g \) are \( C \)-bi-Lipschitz equivalent. Let \( H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0) \) be the germ of a bi-Lipschitz homeomorphism satisfying the conditions of Definition 2.1. Let \( V_+ \) be the subset of \( \mathbb{R}^n \times \mathbb{R} \) of points \((x, y)\) where \( y > 0 \), and let \( V_- \) be the subset of \( \mathbb{R}^n \times \mathbb{R} \) where \( y < 0 \). Clearly, we have one of the following possibilities:

1) \( H(V_+) = V_+ \) and \( H(V_-) = V_- \), or
2) \( H(V_+) = V_- \) and \( H(V_-) = V_+ \).

Let us consider the first possibility. In this case, the functions \( f \) and \( g \) have the same sign on each connected component of the set \( f(x) \neq 0 \). Moreover,

\[
|g(x)| = \| (x, 0) - (x, g(x)) \| = \| H(x, 0) - H(x, f(x)) \| \\
\leq c_2 \| (x, 0) - (x, f(x)) \| = c_2 |f(x)|,
\]

where \( c_2 \) is a positive real number. Using the same argument we can show

\[
c_1 |f(x)| \leq |g(x)|, \quad c_1 > 0.
\]

Hence, \( f \approx g \).

Let us consider the second possibility. Let \( \xi : (\mathbb{R}^n \times \mathbb{R}, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}, 0) \) be a map-germ defined as follows:

\[
\xi(x, y) = (x, -y).
\]

Applying the same arguments to a map \( \xi \circ H \), we will conclude that \( f \approx -g \).

Reciprocally, suppose that \( f \approx g \). Let us construct a map-germ

\[
H : (\mathbb{R}^n \times \mathbb{R}, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}, 0).
\]

Then the following may occur:

\begin{equation}
H(x, y) = \begin{cases} 
(x, 0) & \text{if } y = 0, \\
(x, g(y)/f(x)y) & \text{if } 0 \leq |y| \leq |f(x)|, \\
(x, y - f(x) + g(x)) & \text{if } 0 \leq |f(x)| \leq |y|, \\
(x, y) & \text{otherwise}.
\end{cases}
\end{equation}

The map \( H(x, y) = (x, \tilde{H}(x, y)) \) defined above is bi-Lipschitz. In fact, \( H \) is injective because, for any fixed \( x^* \), we can show that \( \tilde{H}(x^*, y) \) is a continuous and monotone function. Moreover, \( H \) is Lipschitz if \( 0 \leq |f(x)| \leq |y| \). Let us show that \( H \) is Lipschitz if \( 0 \leq |y| \leq |f(x)| \). By Rademacher’s theorem, in almost every \( x \) near \( 0 \in \mathbb{R}^n \), all the partial derivatives \( \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \) exist; hence the derivatives \( \frac{\partial \tilde{H}}{\partial x_i} \).
exist in almost every $x$ near $0 \in \mathbb{R}^n$. By the Mean Value Theorem and continuity of $\tilde{H}$, it is enough to show that the derivatives $\frac{\partial \tilde{H}}{\partial x_i}$ are bounded on the domain $0 \leq |y| \leq |f(x)|$, for all $i = 1, \ldots, n$. We have

$$\frac{\partial \tilde{H}}{\partial x_i} = \frac{(\frac{\partial g}{\partial x_i} f(x) - \frac{\partial f}{\partial x_i} g(x) ) y}{(f(x))^2} = \frac{\partial g}{\partial x_i} f(x) - \frac{\partial f}{\partial x_i} g(x) \frac{y}{f(x)}.$$ 

Since $|y| \leq |f(x)|$, then $\frac{y}{f(x)}$ is bounded. The expression $\frac{g(x)}{f(x)}$ is bounded since $f \approx g$. Moreover, $\frac{\partial g}{\partial x_i}$ and $\frac{\partial f}{\partial x_i}$ are bounded because $f$ and $g$ are Lipschitz functions.

Since $H^{-1}$ can be constructed in the same form as $\mathbf{1}$, we conclude that $H^{-1}$ is also Lipschitz and, thus, $H$ is a bi-Lipschitz map.  

\section{Montaldi's Construction}

\textbf{Definition 4.1.} Two germs of Lipschitz functions are called $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent (or contact equivalent in the sense of Montaldi) if there exists a germ of a bi-Lipschitz map $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and $M(\text{graph}(f)) = \text{graph}(g)$. The map $M$ is called a Montaldi map.

\textbf{Theorem 4.2.} Two germs of Lipschitz functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent if and only if they are $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent.

\textbf{Proof.} It is clear that the $\mathcal{K}$-bi-Lipschitz equivalence implies the $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence. To prove the converse, let $f$ and $g$ be $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent. Then

$$M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\} \quad \text{and} \quad M(\text{graph}(f)) = \text{graph}(g).$$

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be defined by $h(x) = \pi_n(M(x, f(x)))$.

\textbf{Claim 1.} $h$ is a bi-Lipschitz map-germ.

\textbf{Proof of Claim 1.} Since $g$ is a Lipschitz function, the projection $\pi_{n, \text{graph}(g)}$ is a bi-Lipschitz map. By the same argument, the map $x \mapsto (x, f(x))$ is bi-Lipschitz. The map $M$ is bi-Lipschitz by Definition $\mathbf{1.1}$.

\textbf{Claim 2.} One of the following assertions is true:

i) $f \approx g \circ h$,

ii) $f \approx -(g \circ h)$.

\textbf{Proof of Claim 2.} Since $M$ is a bi-Lipschitz map, it follows that there exist two positive numbers $c_1$ and $c_2$ such that

$$c_1 |f(x)| \leq ||M(x, f(x)) - M(x, 0)|| \leq c_2 |f(x)|.$$ 

By the above construction,

$$||M(x, f(x)) - M(x, 0)|| = ||(h(x), g(h(x))) - M(x, 0)|| \geq |g(h(x))|.$$ 

Therefore, $|g(h(x))| \leq c_2 |f(x)|$. 


Using the same procedure for the map $M^{-1}$, we obtain that
\[ c_1 |f(x)| \leq |g(h(x))|. \]

Since $M$ is a homeomorphism and $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$, the same argument as in Theorem 2.4 implies that, for all $x \in \mathbb{R}^n$,
\[ \sign f(x) = \sign g(h(x)), \]
or, for all $x \in \mathbb{R}^n$,
\[ \sign f(x) = -\sign g(h(x)). \]

Hence, Claim 2 is proved. \hfill \Box

End of the proof of Theorem 4.2 Using Claim 2 we obtain, by Theorem 2.4, that $f$ and $g \circ h$ are $C$-bi-Lipschitz equivalent. By Claim 1, $f$ and $g$ are $K$-bi-Lipschitz equivalent. \hfill \Box

5. Finiteness theorem

This section is devoted to a proof of the finiteness theorem (Theorem 2.5). Actually, this result follows from Theorem 4.2 and the equisingularity statement of the Mostowski-Parusinski theorem, but we are going to present an independent proof, using just the finiteness statement of the same theorem.

Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of $n$ variables with degree less than or equal to $k$. Let $\mathcal{S}_k(\mathbb{R}^n)$ be the set of all continuous semialgebraic functions $f: \mathbb{R}^n \to \mathbb{R}$ such that there exists a polynomial $P \in \mathcal{P}_k(\mathbb{R}^n)$; $|f(x)| = |P(x)|$, for all $x \in \mathbb{R}^n$. Clearly, there exists a positive integer number $K$ such that, for all $f \in \mathcal{S}_k(\mathbb{R}^n)$, the algebraic complexity of $f$ (see [2] for a definition) is less than or equal to $K$. For any function $f \in \mathcal{S}_k(\mathbb{R}^n)$, we associate the germ at $0 \in \mathbb{R}^n$ of the semialgebraic set $X_f \subset \mathbb{R}^n \times \mathbb{R}$, defined by $X_f = \mathbb{R}^n \times \{0\} \cup \{f(x) \neq 0\}$. We say that $X_f$ and $X_g$ are strongly bi-Lipschitz equivalent if there exists a germ of a bi-Lipschitz homeomorphism $h: (\mathbb{R}^n \times \{0\}) \to (\mathbb{R}^n \times \{0\})$ such that $h(X_f) = X_g$. Note that if $f$ and $g$ are $K$-bi-Lipschitz equivalent, the corresponding sets $X_f$ and $X_g$ are strongly bi-Lipschitz equivalent, but not vice versa. By the Mostowski-Parusinski Theorem ([9], [11]) the set of equivalence classes of the sets $X_f$ for $f \in \mathcal{S}_k(\mathbb{R}^n)$, with respect to strong bi-Lipschitz equivalence, is finite. We are going to show that the set of equivalence classes with respect to $K$-bi-Lipschitz equivalence is also finite.

Let $f \in \mathcal{S}_k(\mathbb{R}^n)$. Let $Y_1, \ldots, Y_p$ be the set of connected components of the set $f(x) \neq 0$. Let us define a transformation $F_i: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$. Let $(x, y)$ be a coordinate system in $\mathbb{R}^n \times \mathbb{R}$ such that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let

\[ H(x, y) = \begin{cases} (x, y) & \text{if } x \notin Y_i, \\ (x, y - f(x)) & \text{if } x \in Y_i. \end{cases} \]

Since $f(x)$ is a Lipschitz function, $F_i$ is a bi-Lipschitz map. The transformation $F_i$ maps the set $X_f$ to the set $X_{f_i}$, where $f_i \in \mathcal{S}_k(\mathbb{R}^n)$. In fact,

\[ f_i(x) = \begin{cases} f(x) & \text{if } x \notin Y_i, \\ -f(x) & \text{if } x \in Y_i. \end{cases} \]

We say that $f_i$ is obtained from $f$ by an elementary transformation. Let $\{f_{\alpha_1}, \ldots, f_{\alpha_p}\}$ be the set of all the functions which can be obtained from $f$ by a sequence of elementary transformations. Clearly, this set is finite.
By the Mostowski-Parusinski Theorem there exists a finite set of functions
\[ f^1, \ldots, f^m \in S_k(\mathbb{R}^n) \]
such that, for any \( g \in S_k(\mathbb{R}^n) \), there exists a number \( i \) such that \( X_g \) is strongly bi-Lipschitz equivalent to \( X_{f^i} \). Let us prove that there exists a function \( f^i_{\alpha j} \) obtained from \( f^i \) by a sequence of elementary transformations such that \( g \) is \( \mathcal{K}-\mathcal{M} \)-bi-Lipschitz equivalent to \( f^i_{\alpha j} \), i.e., there exists a Montaldi map between \( X_g \) and \( X_{f^i_{\alpha j}} \).

Suppose that there exists a germ of a bi-Lipschitz map \( h : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0) \) such that \( h(X_{f^i}) = X_g \). Let \( Y_1, \ldots, Y_p \) be the connected component of \( f^i(x) \neq 0 \). If \( h \) is not a Montaldi map, then there exists a family of the components \( Y_1, \ldots, Y_r, r \leq p \) such that the image of the set \( Y_j \times \{0\} \) belongs to a part of graph \( (g) \), and the image of the part of graph \( (f^i) \) above \( Y_j \) belongs to \( \mathbb{R}^n \times \{0\} \). Let \( f^i_j \) be the function obtained from \( f^i \) by an elementary transformation \( F^i_j \). Then the map \( h \circ F^i_j \) is a Montaldi map on the set \( Y_j \times \mathbb{R} \) and is equal to \( h \) on the complement of \( Y_j \times \mathbb{R} \). When we apply this construction for all the sets \( Y_1, \ldots, Y_r \) we obtain a Montaldi map \( M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0) \) such that \( M(X_{f^i_{\alpha j}}) = X_g \), for some \( f^i_{\alpha j} \). Since \( P_k(\mathbb{R}^n) \subset S_k(\mathbb{R}^n) \), the set of equivalence classes in \( P_k(\mathbb{R}^n) \), with respect to \( \mathcal{K}-\mathcal{M} \)-bi-Lipschitz equivalence, is finite.

By Theorem 4.2 the set of equivalence classes with respect to \( \mathcal{K} \)-equivalence is also finite. The theorem is proved. \[ \square \]

The authors are grateful to the referee for valuable suggestions.

References


Departamento de Matemática, Universidade Federal do Ceará, Av. Mister Hull s/u Campus do PICI, Bloco 914, CEP 60455-760 Fortaleza-CE, Brazil

Departamento de Matemática (IBILCE), Universidade Estadual Paulista, São José de Rio Preto, SP 15054-000 Brazil

Departamento de Matemática, Universidade Federal do Ceará, Av. Mister Hull s/u Campus do PICI, Bloco 914, CEP 60455-760 Fortaleza-CE, Brazil

Institute of Sciences and Mathematics, University of São Paulo, São Carlos SP, Brazil