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COMPACT PERTURBATIONS OF ISOMETRIES

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ABSTRACT. We give some characterizations of isometries (contractions) which are perturbations with compact operators of a given arbitrary isometry, in terms of certain natural factorizations. As a consequence we obtain general parametric representations of these perturbations.

1. INTRODUCTION

Let L, H be separable, infinite-dimensional Hilbert spaces and $B(L,H)$ the Banach space of the bounded linear operators from L to H. For a closed subspace H_0 of H, we denote by $\mathbf{1}_{H_0}$ the identity on H_0 and by ι_{H_0} the embedding of H_0 into H.

In recent years several works have dealt with the study of the perturbations $V_2 = V_1 + X$ of an isometry $V_1 \in B(L,H)$, where X ranges over particular subsets of the Banach space $K(L,H)$ of the compact operators in $B(L,H)$, such as rank one operators ([Nak86], [Nak93]) or finite rank operators ([BT97], [BT00]). A first question in this study is the description of all such perturbations V_2 that remain isometric or, at least, contractive. Such descriptions are given in [Nak86], [Nak93] for X of rank one; namely, $V_2 = V_1 + X$ is an isometry (respectively contraction) if and only if

$$X = (\alpha - 1)h \otimes V_1^* h \tag{1.1}$$

for some unitary vector $h \in H$ and some complex scalar α such that $|\alpha| = 1$ (respectively $|\alpha| \leq 1$).

By a direct computation, this is in fact equivalent to a factorization of the type

$$V_2 = YV_1 \tag{1.2}$$

for some unitary (respectively contractive) $Y \in B(H)$ such that $Y - \mathbf{1}_H$ is a rank one operator (namely $Y = \mathbf{1}_H + (\alpha - 1)h \otimes h$).

The aim of this paper is to extend this characterization in the factored form (1.2) to the case when X ranges over the finite rank operators and over the whole $K(L,H)$. As a consequence, the spectral description of the factor Y yields parametric characterizations of the type (1.1).

Let us see first how a factorization of type (1.2) can be obtained for a contractive finite rank perturbation of an isometry.

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Proposition 1.1. *Let V_1 be an isometry and V_2 a contraction (respectively an isometry) in $B(L,H)$. Then $V_2 - V_1$ is a finite rank operator if and only if there is a contraction (respectively a unitary) Y in $B(H)$ such that $V_2 = YV_1$ and $Y - \mathbf{1}_H$ is a finite rank operator.*

Proof. Suppose V_2 is a contraction, put $X = V_2 - V_1$ and let $\mathcal{X} = \text{Im } X$ (which has finite dimension). One can observe that $\mathcal{X} \cap \text{Ker } V_1^* = \{0\}$. Indeed, if $x = Xy$ is a vector in $\mathcal{X} \cap \text{Ker } V_1^*$, then

$$\|x\|^2 = \langle V_1^* Xy, y\rangle + \langle y, V_1^* Xy\rangle + \|Xy\|^2 = \|(V_1 + X)y\|^2 - \|V_1 y\|^2 \leq 0,$$

so necessarily $x = 0$.

Let P be the orthogonal projection onto $\mathcal{X}$. Since $X^*(\mathbf{1}_H - P) = (\mathbf{1}_H - P)X = 0$, the inequality

$$(V_1 + X)^*(V_1 + X)^* \leq \mathbf{1}_H = V_1^* V_1$$

is equivalent to

$$(V_1^* + X^*)P(V_1 + X) + V_1^*(\mathbf{1}_H - P)V_1 \leq V_1^* P V_1 + V_1^*(I - P)V_1$$

or

$$(V_1^* + X^*)P(V_1 + X) \leq V_1^* P V_1.$$

Thus the map $Y_0 : PV_1 x \mapsto P(V_1 + X)x$ is a contraction defined on $PV_1 L$. But $PV_1 L = \mathcal{X}$ since $\mathcal{X} \cap \text{Ker } V_1^* = \{0\}$. Then $Y \in B(H)$ defined by $Y = \mathbf{1}_H - P + Y_0 P$ is the required contraction.

If V_2 is supposed to be an isometry, then all the inequalities above become equalities, which implies that Y is a unitary, and so the direct implication is proved. The converse is trivial. $\square$

2. Factorizations and perturbations with compact operators

Let $V_1 \in B(L,H)$ be an isometry and $V_2 \in B(L,H)$ a contraction. Since $V_2^* V_2 \leq V_1^* V_1$ there are contractions $Y \in B(H)$ such that $V_2 = YV_1$. Two such contractions necessarily coincide on $\text{Im } V_1$, as they are all extensions of the contraction Y_0 defined on $\text{Im } V_1$ into the closure of $\text{Im } V_2$ by $Y_0 : V_1 l \mapsto V_2 l$, $l \in L$. Clearly V_2 has closed range if and only if Y_0 has closed range.

In particular, if V_2 is an isometry, then Y_0 is a unitary, so one can find extensions Y that are at least partial isometries. Obviously if $\text{Im } V_1$ and $\text{Im } V_2$ have the same codimension, then Y_0 can be extended for instance to unitaries of the form $Y = Y_0 \oplus Y_1$, with Y_1 an arbitrary unitary sending $H \ominus \text{Im } V_1$ onto $H \ominus \text{Im } V_2$.

The question now is what else can be said about the factors Y under the additional assumption that V_2 is a contractive perturbation of V_1 with a compact operator. Several remarks are immediate in this case:

First it follows that V_2 is right semi-Fredholm, so in particular Y_0 has closed range. Moreover, if V_2 is an isometry, then its range has the same codimension as V_1, so, as remarked, there are plenty of unitary extensions Y of Y_0 satisfying (1.2).

Secondly, it follows from (1.2) that the operator $\iota_{\text{Im } V_2} Y_0 - \iota_{\text{Im } V_1}$ is necessarily compact. The aim of this section is to show that $V_2 - V_1$ is compact if and only if there is one extension Y that inherits this last property of Y_0, namely $Y - \mathbf{1}_H$ is compact. This is obviously true for any extension Y of Y_0 if $\text{Im } V_1$ has finite codimension, so in the following we consider the nontrivial case when $\text{Im } V_1$ has infinite codimension.

We first consider the particular case when the perturbation V_2 is an isometry. For $i = 1, 2$ set $H_i = V_i L$ and denote by P_i the orthogonal projection onto H_i. If $V_1 - V_2$ is compact, then so is $P_1 - P_2$, since

$$P_1 - P_2 = \frac{1}{2}[(V_1 - V_2)(V_1^* + V_2^*) + (V_1 + V_2)(V_1^* - V_2^*)].$$

It is useful to investigate first the geometry of two arbitrary closed subspaces H_1 and H_2 of H whose projections have compact difference. The next lemma shows that such subspaces always contain finite-codimensional parts which can be mapped one onto the other by unitaries that differ from the identity by a compact operator.

Lemma 2.1. *Let H_1 and H_2 be infinite-dimensional subspaces of H. Let P_1 and P_2 be the orthogonal projections onto H_1 and H_2 respectively, and suppose that $P_1 - P_2$ is compact. Then*

1) *$H_1 \cap H_2^\perp$ and $H_2 \cap H_1^\perp$ have finite dimensions, say c_1 and c_2.*

2) *There is a unitary U_0 mapping $H_1' = H_1 \ominus (H_1 \cap H_2^\perp)$ onto $H_2' = H_2 \ominus (H_2 \cap H_1^\perp)$ such that $\iota_{H_2'} U_0 - \iota_{H_1'}$ is compact.*

3) *If, for example, $c_1 \leq c_2$, then U_0 can be extended to an isometry V_0 from H_1 into H_2 satisfing*

a) *$\iota_{H_2} V_0 - \iota_{H_1}$ is compact;*

b) *$\dim(H_2 \ominus V_0 H_1) = c_2 - c_1$.*

In particular if $c_1 = c_2$, then V_0 can be chosen to be unitary.

Proof. To prove 1), observe that $\mathbf{1}_{H_1 \cap H_2^\perp} = (P_1 - P_2)|_{H_1 \cap H_2^\perp}$, so $\mathbf{1}_{H_1 \cap H_2^\perp}$ is compact, and this happens if and only if $\dim(H_1 \cap H_2^\perp) < \infty$. In a similar way it follows that $\dim(H_2 \cap H_1^\perp) < \infty$.

To prove 2), let $X : H_1 \to H_2$ be the operator angle between H_1 and H_2, i.e., the restriction of P_2 to H_1. Obviously X^* is the reverse operator angle, i.e., the restriction of P_1 to H_2. Since $P_2 - P_1$ is compact, it follows immediately that $\iota_{H_2} X - \iota_{H_1}$ is compact.

We also have $\operatorname{Ker} X = H_1 \cap H_2^\perp$ and $\operatorname{Ker} X^* = H_2 \cap H_1^\perp$, so the restriction X_0 of X to the subspace $H_1' = H_1 \ominus \operatorname{Ker} X$ considered as an operator from this subspace into $H_2' = H_2 \ominus \operatorname{Ker} X^* = \overline{\operatorname{Im} X}$ is a quasi-invertible operator (i.e., injective with dense range) from H_1' to H_2' for which $\iota_{H_2'} X_0 - \iota_{H_1'}$ is compact.

Now, since for $h \in H_1'$,

$$X_0^* X_0 h - h = P_1 P_2 h - h = P_1 P_2 h - P_1 h = P_1(P_2 - P_1)h,$$

we have that $X_0^* X_0 - \mathbf{1}_{H_1'}$ is compact. This implies easily that $|X_0| = (X_0^* X_0)^{1/2}$ is invertible in H_1' and that $|X_0| - \mathbf{1}_{H_1'}$ is compact, and therefore $|X_0|^{-1} - \mathbf{1}_{H_1'} = (\mathbf{1}_{H_1'} - |X_0|)|X_0|^{-1}$ is also compact.

We set $U_0 = X_0 |X_0|^{-1}$, which is a unitary from H_1' onto H_2'. Then

$$\iota_{H_2'} U_0 - \iota_{H_1'} = (\iota_{H_2'} X_0 - \iota_{H_1'})|X_0|^{-1} + \iota_{H_1'}(|X_0|^{-1} - \mathbf{1}_{H_1'})$$

is compact, which proves 2). Statement 3) is an immediate consequence of 2). □

Now, under the stronger assumption that H_1 and H_2 are not only ranges of projections but ranges of isometries with compact difference, then one can find unitaries differing from the identity by a compact operator which act not only between some finite-codimensional parts of H_1 and H_2, but between H_1 and H_2 themselves.

Theorem 2.2. *Let H_1, H_2 be infinite-dimensional closed subspaces of H, and let P_i be the orthogonal projection onto H_i $(i = 1, 2)$. The following are equivalent:*

1) *there exist two isometries V_1, V_2 in $B(L, H)$ with ranges H_1 and H_2 respectively, and $V_1 - V_2$ compact;*

2) *there is a unitary $U : H_1 \to H_2$ such that $\iota_{H_2} U - \iota_{H_1}$ is compact;*

3) *$P_1 - P_2$ is compact and* $\dim(H_1 \cap H_2^{\perp}) = \dim(H_2 \cap H_1^{\perp})$.

Proof. The implication 1) $\Rightarrow$ 2) is contained in the remarks at the beginning of this section (with $U = Y_0$). For the converse pick any isometry $V_1 \in B(L, H)$ with range H_1 (this is possible since L has infinite dimension) and put $V_2 = UV_1$. The implication 3) $\Rightarrow$ 2) follows directly from Lemma 2.1 with $c_1 = c_2$.

It remains to show that 1) $\Rightarrow$ 3)). Obviously $P_1 - P_2$ is compact, so by Lemma 2.1 the dimensions $c_1 = \dim(H_1 \cap H_2^{\perp})$ and $c_2 = \dim(H_2 \cap H_1^{\perp})$ are finite. To see that they are equal, suppose for example that $c_1 \leq c_2$. Then by Lemma 2.1 there is an isometry $V_0 : H_1 \to H_2$ such that $\dim(H_2 \ominus V_0 H_1) = c_2 - c_1$ and $\iota_{H_2} V_0 - \iota_{H_1}$ is compact.

Due to the equivalence of 1) and 2), one can also find a unitary W from H_1 to H_2 such that $\iota_{H_2} W - \iota_{H_1}$ is compact. Therefore $W - V_0 : H_1 \to H_2$ is compact, and so $WW^* - V_0 W^* = \mathbf{1}_{H_2} - V_0 W^*$ is compact. Since the isometry $V_0 W^*$ has the range of codimension $c_2 - c_1 < \infty$ and since the difference of the two isometries $\mathbf{1}_{H_2}$ and $V_0 W^*$ is compact, we have

$$0 = (\dim H_2 \ominus \mathbf{1}_{H_2} H_2) = \dim(H_2 \ominus V_0 W^* H_2) = c_2 - c_1,$$

so $c_1 = c_2$ and the theorem is proved. □

The characterization 3) in Theorem 2.2 in terms of projections allows a nice interplay between (infinite-codimensional) subspaces and their orthogonal complements, which leads to the main result of the section.

Theorem 2.3. *Let V_1, V_2 be arbitrary isometries in $B(L, H)$. Then $V_1 - V_2$ is compact if and only if there is a unitary U in $B(H)$ such that $U - \mathbf{1}_H$ is compact and $V_2 = UV_1$.*

Proof. Let $H_i = V_i L$, $i = 1, 2$. As mentioned in the beginning of the section we may suppose that $K_1 = H \ominus H_1$ and $K_2 = H \ominus H_2$ have infinite dimension.

By Theorem 2.2 there is a unitary $U_0 : H_1 \to H_2$ such that $V_2 = U_0 V_1$ and $\iota_{H_2} U_0 - \iota_{H_1}$ is compact.

Moreover, since the subspaces H_1 and H_2 satisfy condition 3) in Theorem 2.2, $P_1 - P_2$ is compact and $\dim(H_1 \cap K_2) = \dim(K_1 \cap H_2)$. But, as $P_1 - P_2 = (\mathbf{1}_H - P_1) - (\mathbf{1}_H - P_2)$, the subspaces K_1 and K_2 themselves satisfy the same condition. Therefore from Theorem 2.2, K_1 and K_2 are also ranges of isometries with compact difference or, equivalently, there is a unitary $U_1 : K_1 \to K_2$ such that $\iota_{K_2} U_1 - \iota_{K_1}$ is compact. But then $U = U_0 \oplus U_1$ satisfies $V_2 U = V_1$ and $U - \mathbf{1}_H$ is compact, which proves the direct implication. The converse is trivial. □

A dilation argument can be used to prove the following analogous result for contractive perturbations.

Theorem 2.4. *Let V be an isometry and T a contraction in $B(L, H)$. Then $T - V$ is compact if and only if there is a contraction $Y \in B(H)$ such that $Y - \mathbf{1}_H$ is compact and $T = YV$.*

Proof. It is enough to prove the direct implication. The equality

$$\mathbf{1}_L - T^*T = \frac{1}{2}[(V^* - T^*)(V + T) + (V^* + T^*)(V - T)]$$

shows that $\mathbf{1}_L - T^*T$ is compact; therefore $D_T = (\mathbf{1}_L - T^*T)^{1/2}$ is compact.

We consider the isometries V_1 and V_2 from H to $H \oplus L$ defined by $V_1 := \begin{pmatrix} V \\ 0 \end{pmatrix}$ and $V_2 := \begin{pmatrix} T \\ D_T \end{pmatrix}$. Then $V_1 - V_2$ is compact, so by Theorem 2.3 there is a unitary U in $B(H \oplus L)$, $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ such that $V_2 = UV_1$, so $T = U_{11}V$.

Obviously U_{11} is a contraction in $B(H)$, and since $U - \mathbf{1}_{H\oplus H}$ is compact, it follows that $U_{11} - \mathbf{1}_H$ is compact. Therefore we can choose $Y = U_{11}$, which satisfies all the requirements needed, and the proof is complete. □

As a consequence we derive a generalization of the parametric form (1.1) for general compact perturbations.

Corollary 2.5. *Let V be an isometry and K a compact operator in $B(L, H)$. Then $V + K$ is an isometry if and only if there is an orthonormal sequence of vectors $(e_n)_n \subset H$ and $(\alpha_n)_n \subset \mathbb{T}$ such that $\lim_{n\to\infty} \alpha_n = 1$ and*

$$K = \sum_{n=0}^{\infty} (\alpha_n - 1)e_n \otimes V^* e_n. \tag{2.1}$$

Proof. By Theorem 2.3, $V' = V + K$ is an isometry if and only if there is a unitary U such that $V' = UV$ and $U - \mathbf{1}_H$ is compact. It follows that $U - \mathbf{1}_H$ is diagonalisable, so there is an orthonormal base $(e_n)_{n\geq 0}$ of H and a sequence $(\beta_n)_{n\geq 0}$ of scalars (the eigenvalues of $U - \mathbf{1}_H$) such that $U - \mathbf{1}_H = \sum_{n=0}^{\infty} \beta_n e_n \otimes e_n$.

It follows that $K = \sum_{n=0}^{\infty} (\alpha_n - 1)e_n \otimes V^* e_n$, with $\alpha_n = \beta_n + 1 \in \mathbb{T}$ as eigenvalues for the unitary U. □

It is obvious that, in the parametric description in Corollary 2.5 for compact perturbations, the finite rank case corresponds to a finite sum in (2.1). This and Corollary 2.5 give one consequence concerning compact perturbations of isometries as iterated rank-one perturbations.

Corollary 2.6. a) *Suppose that the isometry $V' \in B(L, H)$ is a compact perturbation of the isometry $V \in B(L, H)$. Then V' is the norm limit of a sequence $V_n \in B(L, H)$ of isometries such that $V_0 = V$ and V_n either coincides or is a rank-one perturbation of V_{n-1} for every $n \geq 1$.*

b) *In particular if V' is a finite rank perturbation of V there is a finite sequence $V_0 = V, V_1, \ldots, V_n = V'$ of isometries such that each V_k is a rank one perturbation of V_{k-1}.*

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References

[BT97] Chafiq Benhida and Dan Timotin, *Functional models and finite-dimensional perturbations of the shift*, Integral Equations Operator Theory **29** (1997), no. 2, 187–196. MR1472098 (98d:47032)

[BT00] ———, *Finite rank perturbations of contractions*, Integral Equations Operator Theory **36** (2000), no. 3, 253–268. MR1753419 (2001f:47011)

[Cla72] Douglas N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191. MR0301534 (46:692)

[CT04] Gilles Cassier and Dan Timotin, *Power boundedness and similarity to contractions for some perturbations of isometries*, Journal of Mathematical Analysis and Applications **293** (2004), 160–180. MR2052538 (2004m:47044)

[Nak86] Yoshihiro Nakamura, *One-dimensional perturbations of isometries*, Integral Equations Operator Theory **9** (1986), no. 2, 286–294. MR0837520 (87k:47031)

[Nak93] ———, *One-dimensional perturbations of the shift*, Integral Equations Operator Theory **17** (1993), no. 3, 373–403. MR1237960 (94h:47019)

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