TRACE CLASS CRITERIA FOR BILINEAR HANKEL FORMS OF HIGHER WEIGHTS

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Abstract. In this paper we give a complete characterization of higher weight Hankel forms, on the unit ball of $\mathbb{C}^d$, of Schatten-von Neumann class $S_p$, $1 \leq p \leq \infty$. For this purpose we give an atomic decomposition for certain Besov-type spaces. The main result is then obtained by combining the decomposition and our earlier results.

1. Introduction

Hankel operators on the unit disc have been studied extensively; see [Pe1] for a systematic treatment. One of the main topics is to study Schatten-von Neumann properties of Hankel operators; see [Pe1] and [Pe2]. In [JP] Janson and Peetre introduced Hankel forms of higher weights on the unit disc. Their Schatten-von Neumann properties were studied in [Ro] and [Z].

In [P1] Peetre introduced Hankel forms of higher weights on the unit ball in $\mathbb{C}^d$. Their Schatten-von Neumann, $S_p$, properties were studied in [Su] for $2 \leq p \leq \infty$. See also [FR] for a different approach.

The results for $2 \leq p \leq \infty$ in [Su] were proved by using interpolation between $S_2$ and $S_\infty$ (bounded operators) and boundedness of certain matrix-valued Bergman projections, but the case of $1 \leq p < 2$ was left open there.

In this paper we extend the results in [Su] to $1 \leq p \leq \infty$. For this purpose we study the atomic decomposition for some Besov spaces of vector-valued holomorphic functions, see Section 4, which then gives $S_1$ properties. Our results follow by interpolation, and we get a full characterization for $1 \leq p \leq \infty$. Some of the proofs in this paper are based on techniques used in [Su] and will therefore be given briefly. The reader is referred to that article for more details.

The paper is organized as follows. In section 2 we recall briefly some notation and we prove Theorem 2.1 generalizing the result for $p = 2$ in [Su]. Section 3 is devoted to duality relations for the spaces of symbols. In Section 4 we give an atomic decomposition for a certain space of symbols, which will be used in Section 5 to prove the $S_1$ criterion.

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2. Preliminaries

2.1. The Banach space $\mathcal{H}^p_{\nu,s}$ for $1 \leq p \leq \infty$. Let $dm$ denote the Lebesgue measure on the unit ball $\mathbb{B} \subset \mathbb{C}^d$ and let $d\nu(z)$ be the measure $(1 - |z|^2)^{d-1}dm(z)$. For $d < \nu < \infty$ let $d\nu(z)$ be the measure $c_\nu(1 - |z|^2)^\nu d\nu(z)$, where $c_\nu$ is chosen such that

$$\int_{\mathbb{B}} d\nu(z) = 1.$$  

The closed subspace of all holomorphic functions in $L^2(d\nu)$ is denoted by $L^2_\nu(d\nu)$ and is called a weighted Bergman space. Note that the space $L^2_\nu(d\nu)$ has a reproducing kernel $K_z(w) = (1 - \langle w, z \rangle)^{-\nu}$, that is,

$$f(z) = (f, K_z)_\nu = \int_{\mathbb{B}} f(w)\overline{K_z(w)} d\nu(w), \quad f \in L^2_\nu(d\nu), \quad z \in \mathbb{B}. \quad (2.1)$$

Denote by $B(z, w)$ the Bergman operator on $V = \mathbb{C}^d$ as in [L], namely

$$B(z, w) = (1 - \langle z, w \rangle)(I - z \otimes w^*), \quad (2.2)$$

where $z \otimes w^*$ stands for the rank one operator given by $(z \otimes w^*)(v) = \langle v, w \rangle z$.

The Bergman metric at $z \in \mathbb{B}$, when we identify the tangent space with $V$, is $\langle B(z, z)^{-1}u, v \rangle$ for $u, v \in V$. We note that

$$B(z, w)^{-1} = (1 - \langle z, w \rangle)^{-2}((1 - \langle z, w \rangle)I + z \otimes w^*). \quad (2.3)$$

Let $B^t(z, w)$ denote the dual of $B(z, w)$ acting on the dual space $V'$ of $V$. When acting on a vector $v' \in V'$ it is

$$B^t(z, w)v' = (1 - \langle z, w \rangle)v'(I - zw^*). \quad (2.4)$$

For a nonnegative integer $s$, let $\otimes^s V'$ be the tensor product of $s$ copies of $V'$ and let $\otimes^0 V' = \mathbb{C}$. The space $\otimes^s V'$ is equipped with a natural Hermitian inner product induced by that of $V'$. Denote by $\otimes^s V'$ the subspace of symmetric tensors of length $s$ and denote by $\otimes^s B^t(z, z)$ the operator on $\otimes^s V'$ induced by the action of $B^t(z, z)$ on $V'$, where $\otimes^0 B^t(z, z) = I$. Recall, generally, that if $A$ acts on $V'$, $\otimes^s A$ acts on $\otimes^s V'$ by

$$(\otimes^s A)(u_1 \otimes u_2 \otimes \cdots \otimes u_s) = (Au_1) \otimes (Au_2) \otimes \cdots \otimes (Au_s).$$

For example, in the case $s = 2$ the operator $\otimes^2 B^t(z, z)$ becomes

$$(1 - |z|^2)^2 (I \otimes I - I \otimes A_z - A_z \otimes I + A_z \otimes A_z),$$

where $A_z = \bar{z} \otimes z^*$. Let $L^p_{\nu,s} = L^p_{\nu} (\mathbb{B}, \otimes^s V')$ be the space of functions $G : \mathbb{B} \to \otimes^s V'$ such that

$$\|G\|_{\nu,s,p} = \left( \int_{\mathbb{B}} \langle (1 - |z|^2)^{2
u} \otimes^s B^t(z, z)G(z), G(z) \rangle^{p/2} d\nu(z) \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$, and let $L^\infty_{\nu,s}$ be the space of functions $G : \mathbb{B} \to \otimes^s V'$ such that

$$\|G\|_{\nu,s,\infty} = \sup_{z \in \mathbb{B}} \langle (1 - |z|^2)^{2\nu} \otimes^s B^t(z, z)G(z), G(z) \rangle^{1/2} < \infty.$$  

Let $\mathcal{H}^p_{\nu,s}$ be the closed subspace of all holomorphic functions in $L^p_{\nu,s}$, $1 \leq p \leq \infty$. 


Also, we need the group $G$ of biholomorphic mappings of $\mathbb{B}$. Let $P_z$ be the orthogonal projection of $\mathbb{C}^d$ onto $\mathbb{C}_z$ and let $Q_z = I - P_z$. Put $s_z = (1 - |z|^2)^{1/2}$ and define a linear fractional mapping $\varphi_z$ on $\mathbb{B}$ by (see [Rul])

$$
\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z \rangle}.
$$

If $g \in G$ and $g(z) = 0$, then there is a unique unitary operator $U : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$
g = U \varphi_z.
$$

Define the complex Jacobian $J_g$ by $J_g(w) = \det(g'(w))$. Now, let $z_0 \in \mathbb{B}$. Then by arguments in Remark 3.1 in [Su] it follows that there is a constant $c$ with $|c| = 1$ such that

$$
J_{\varphi_{z_0}}(w)^{2\nu/(d+1)} = c \cdot \frac{(1 - |z_0|^2)^\nu}{(1 - \langle w, z_0 \rangle)^{2\nu}}.
$$

The next theorem gives the reproducing properties for $H^{p}_{\nu,s}$.

**Theorem 2.1.** Let $1 \leq p \leq \infty$. There is a nonzero constant $c$ such that, for any $G \in H^p_{\nu,s}$ and any $v \in \cap^s V'$,

$$
\langle G(z), v \rangle = c \int_{\mathbb{B}} \langle \otimes^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle (1 - |w|^2)^{2\nu} \, dw(w),
$$

where

$$
K_{\nu,s}(w, z) = (1 - \langle w, z \rangle)^{-2\nu} \otimes^s B^t(w, z)^{-1}.
$$

The proof of this theorem is given at the end of this subsection.

**Remark 2.2.** Consider $H^2_{\nu,s} \subset L^2_{\nu,s}$. According to Lemma 3.5 in [Su] the orthogonal projection operator $P_{\nu,s}$ of $L^2_{\nu,s}$ onto $H^2_{\nu,s}$, is given by

$$
P_{\nu,s}G(z) = c \int_{\mathbb{B}} (1 - |w|^2)^{2\nu} K_{\nu,s}(z, w) \otimes^s B^t(w, w) G(w) \, dw(w).
$$

Namely, for any $G \in L^2_{\nu,s}$ and any $v \in \cap^s V'$ it follows that

$$
\langle P_{\nu,s}G(z), v \rangle = c \int_{\mathbb{B}} \langle \otimes^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle (1 - |w|^2)^{2\nu} \, dw(w).
$$

The orthogonal projection operator has the following boundedness property.

**Proposition 2.3.** If $1 \leq p < \infty$, then $P_{\nu,s} : L^p_{\nu,s} \to H^p_{\nu,s}$ is bounded.

**Proof.** The case $1 < p < \infty$ is just Corollary 7.4 in [Su]. Now, consider the case $p = 1$. Let $F \in L^1_{\nu,s}$. Then it follows from Theorem 2.1 above and Lemma 7.1 in [Su] that

$$
\left\| \otimes^s B^t(z, z)^{1/2} P_{\nu,s}F(z) \right\| \leq C_s \int_{\mathbb{B}} T(z, w) \left\| \otimes^s B^t(w, w)^{1/2} F(w) \right\| (1 - |w|^2)^{2\nu} \, dw(w),
$$

where

$$
T(z, w) = \frac{(1 - |z|^2)^{s/2}(1 - |w|^2)^{s/2}}{|1 - \langle z, w \rangle|^{2\nu+s}}.
$$
Thus, by Fubini-Tonelli’s theorem and Proposition 1.4.10 in \cite{Ru} it follows that
\[ \|P_{\nu,s}F\|_{\nu,s,1} \leq C_s \int_{\mathbb{B}} \left\| \otimes^a B^t(w,w)^{1/2} F(w) \right\| \left(1 - |w|^2\right)^{2\nu} \cdot \left( \int_{\mathbb{B}} T(z,w)(1 - |z|^2)^{\nu} \, dt(z) \right) \, dt(w) \leq C'_s \int_{\mathbb{B}} \left\| \otimes^a B^t(w,w)^{1/2} F(w) \right\| \left(1 - |w|^2\right)^{\nu} \, dt(w) = C'_s \|F\|_{\nu,s,1}. \]
\[ \square \]

Note that it is proved in \cite{Su}, using the complex interpolation method of Banach spaces, that \( \mathcal{H}^p_{\nu,s} = (\mathcal{H}^2_{\nu,s}, \mathcal{H}^{\infty}_{\nu,s})_{[1-2/p]} \) if \( 2 < p < \infty \); see Theorem 8.2 in \cite{Su}. However, Proposition 2.5 allows us to use the same proof as in \cite{Su} to get the following result.

**Corollary 2.4.** If \( 1 < p < \infty \), then
\[ \mathcal{H}^p_{\nu,s} = (\mathcal{H}^2_{\nu,s}, \mathcal{H}^{\infty}_{\nu,s})_{[1-1/p]} \cdot \]

Now we go back to Theorem 2.1. First we need a proposition.

**Proposition 2.5.** Let \( s \) be a nonnegative integer and let \( \nu > d, 2\nu > \alpha > d \). Then there is a constant \( C_s > 0 \) such that
\[ (1 - |z|^2)^{2\nu - \alpha} \left\| K_{\nu,s}(\cdot,z) \otimes^a B^t(z,z)^{1/2} v \right\|_{\alpha,s,1} \leq C_s \|v\| \]
for all \( z \in \mathbb{B} \) and all \( v \in \otimes^a V' \).

**Proof.** Let \( v \in \otimes^a V' \). It follows from Lemma 7.1 in \cite{Su} and Proposition 1.4.10 in \cite{Ru} that
\[ \left\| K_{\nu,s}(\cdot,z) \otimes^a B^t(z,z)^{1/2} v \right\|_{\alpha,s,1} = \int_{\mathbb{B}} \left\| \otimes^a B^t(w,w)^{1/2} \otimes^a B^t(w,z)^{-1} \otimes^a B^t(z,z)^{1/2} v \right\| \frac{(1 - |w|^2)^\alpha}{|1 - \langle w,z\rangle|^{2\nu}} \, dt(w) \leq C_s \|v\| \int_{\mathbb{B}} \frac{(1 - |z|^2)^{s/2}(1 - |w|^2)^{2s/2}}{|1 - \langle w,z\rangle|^{2\nu + s}} \, dt(w) \leq C'_s (1 - |z|^2)^{2\nu - 2\alpha} \|v\|. \]
\[ \square \]

**Lemma 2.6.** Let \( z \in \mathbb{B} \). Then there is a constant \( C_s > 0 \) such that, for any \( v \in \otimes^a V' \) and any \( 1 \leq p \leq \infty \), it follows that
\[ \left\| (1 - |z|^2)^{\nu} K_{\nu,s}(\cdot,z) \otimes^a B^t(z,z)^{1/2} v \right\|_{\nu,s,p} \leq C_s \|v\|. \]

**Proof.** Let \( T_z = (1 - |z|^2)^{\nu} K_{\nu,s}(\cdot,z) \otimes^a B^t(z,z)^{1/2} \). By Proposition 2.5 and by Lemma 7.1 in \cite{Su} it follows that \( \|T_z v\|_{\nu,s,1} \leq C_s \|v\| \) and \( \|T_z v\|_{\nu,s,\infty} \leq C'_s \|v\| \) respectively, for all \( v \in \otimes^a V' \). Thus the result follows from Riesz-Thorin’s interpolation theorem.
\[ \square \]

Now we can prove Theorem 2.1

**Proof of Theorem 2.1.** Let \( G \in \mathcal{H}^p_{\nu,s}, 1 \leq p \leq \infty \). Then it follows from Lemma 2.6 that, for all \( v \in \otimes^a V' \),
\[ \int_{\mathbb{B}} \left| \langle \otimes^a B^t(w,w) G(w), K_{\nu,s}(w,z) v \rangle \right| \, dw^2(s) \leq \|G\|_{\nu,s,p} \|K_{\nu,s}(\cdot,z) v\|_{\nu,s,q} \leq \infty. \]
In particular, if \( z = 0 \), then
\[
\int_B \left| \langle \otimes B^t(w, w) G(w), v \rangle \right| (1 - |w|^2)^{2\nu} \, dw(w) < \infty.
\]
By the mean-value property for holomorphic functions and rotation invariance for integration,
\[
\int_B \left( (1 - |w|^2)^{2\nu} \otimes^s B^t(w, w) G(w), v \right) \, dw = c' \langle G(0), v \rangle,
\]
where \( c' \neq 0 \) only depends on \( d, \nu \) and \( s \). Hence, there exists a nonzero constant \( c \) such that, for all \( G \in \mathcal{H}_p^{\nu, s} \) and all \( v \in \otimes^s V' \),
\begin{equation}
\langle G(0), v \rangle = c \langle G, v \rangle_{\nu, s, 2},
\end{equation}
where \( \langle \cdot, \cdot \rangle_{\nu, s, 2} \) is the \( \mathcal{H}_p^{\nu, s} \)-pairing. Now, define an isometry \( \pi_{\nu, s} \) on \( \mathcal{H}_p^{\nu, s} \) by
\[
\pi_{\nu, s} : g \in G, S(z) \rightarrow \left( \otimes^s (dg^{-1}(z))^t \right) S(g^{-1}z) \left( J_{g^{-1}(z)} \right)^{2\nu/(d+1)},
\]
as in [Su]. Let \( z_0 \in \mathbb{B} \). For notational convenience we prove the reproducing property only for \( s = 1 \); the case for general \( s \) is identically the same. On the one hand,
\begin{equation}
\langle \pi_{\nu, 1}(\varphi_{z_0})G(0), v \rangle = \langle G(z_0), J_{\varphi_{z_0}}(0)^{2\nu/(d+1)}(\varphi_{z_0}^{\prime}(0))^t v \rangle.
\end{equation}
By equation (2.8), \((1 - |z_0|^2)^{(d+1)/2} < |J_{\varphi_{z_0}}(w)| < (1 - |z_0|^2)^{-d-1} \) on \( \mathbb{B} \), so \( \pi_{\nu, 1}(\varphi_{z_0})G \in \mathcal{H}_p^{\nu, 1} \). However, using equation (2.8) above for \( \pi_{\nu, 1}(\varphi_{z_0})G \) and the transformation properties
\[
B(\varphi_{z_0}(w), \varphi_{z_0}(z)) = \varphi_{z_0}^{\prime}(w) B(w, z) \left( \varphi_{z_0}^{\prime}(z) \right)^t
\]
(see equation (9) in [Su]) and
\[
K_{\nu, 1}(\varphi_{z_0}(w), \varphi_{z_0}(z)) = J_{\varphi_{z_0}}(w)^{-2\nu/(d+1)} \cdot J_{\varphi_{z_0}}(z)^{-2\nu/(d+1)} \cdot \left( \varphi_{z_0}^{\prime}(w)^t \right)^{-1} K_{\nu, 1}(w, z) \left( \varphi_{z_0}^{\prime}(z) \right)^{-1}
\]
(see equation (9) in [Su] and Theorem 2.2.5 in [Ru]), the left-hand side in equation (2.9) above is
\[
\langle G(z_0), u \rangle = c(G, K_{\nu, s}(\cdot, z_0)u)_{\nu, s, 2},
\]
where \( u = J_{\varphi_{z_0}}(0)^{2\nu/(d+1)}(\varphi_{z_0}^{\prime}(0))^t v \). Since \( v \) is arbitrary, then so is \( u \in \otimes^s V' \), which proves the theorem. \( \square \)

2.2. Hankel forms of higher weights. Let \( H_1 \) and \( H_2 \) be Hilbert spaces and let \( T : H_1 \rightarrow H_2 \) be a linear operator. Define the singular numbers \( s_n(T) = \inf\{\|T - K\| : \text{rank}(K) \leq n\}, n \geq 0 \). If \( T \) is compact, these singular numbers are equal to the eigenvalues of \( |T| = (T^*T)^{1/2} \). We denote by \( S_p \) the ideal of operators for which \( \{s_n(T)\}_{n \geq 0} \in l^p, 0 < p \leq \infty \); see [S].

The transvectant \( T_s \) on \( L_2^s(d\nu) \otimes L_2^s(d\nu) \) (introduced in [PI]; see also [P2], [PZ] and [Su]) is defined by
\begin{equation}
T_s(f, g)(z) = \sum_{k=0}^s \binom{s}{k} (-1)^k \partial^{s-k} f(z) \otimes \partial^k g(z)
\end{equation}
Proof of Lemma 2.7. There is a constant $C_\nu > 0$ such that

$$||T_\nu(f, g)||_{\nu,1} \leq C_\nu ||f||_\nu ||g||_\nu$$

for all $f, g \in L^2_\nu(dt_\nu)$.

First we need a lemma, which actually is a consequence of Theorem 4.1 in [Su], but we give an independent and easier proof.

**Lemma 2.8.** There is a constant $C_{\nu,s} > 0$ such that

$$\int_B \left(\otimes^s B^t(z, z) \partial^s f(z), \partial^s f(z) \right) (1 - |z|^2)^\nu \, dt(z) \leq C_{\nu,s} ||f||_\nu$$

for all $f \in L^2_\nu(dt_\nu)$.

**Proof.** First,

$$\partial^s f(z) = c_\nu(\nu)_s \int_B \frac{f(w) \otimes^s \bar{w}}{(1 - \langle z, w \rangle)^{\nu+s}} \cdot (1 - |w|^2)^\nu \, dt(w),$$

so that

$$\left|\otimes^s B^t(z, z)^{1/2} \partial^s f(z) \right| \leq C_{\nu,s} \int_B |f(w)| \cdot \left|B^t(z, z)^{1/2} \bar{w} \right|^2 \cdot (1 - |w|^2)^\nu \, dt(w).$$

We can estimate

$$\left|B^t(z, z)^{1/2} \bar{w} \right| = s_z \left(\|s_z P_z \bar{w}\|^2 + \|Q_z \bar{w}\|^2 \right)^{1/2} \leq \sqrt{2} \cdot s_z |1 - \langle z, w \rangle|^{1/2}.$$ 

Hence,

$$\left|\otimes^s B^t(z, z)^{1/2} \partial^s f(z) \right| \leq C_{\nu,s} \int_B T(z, w)|f(w)|(1 - |w|^2)^\nu \, dt(w),$$

where

$$T(z, w) = \frac{(1 - |z|^2)^{s/2}}{|1 - \langle z, w \rangle|^{\nu+s/2}}.$$ 

Now, the result follows by exactly the same arguments as in the proof of Theorem 7.2 in [Su] (where we let $t = -(\nu - d)/4$). \hfill \Box

**Proof of Lemma 2.7.** The transvectant is a linear combination of terms $\partial^k f(z) \otimes \partial^{s-k} g(z)$ so we need only to estimate $\|\partial^k f(z) \otimes \partial^{s-k} g(z)\|_{\nu,s,1}$ for $0 \leq k \leq s$. First we observe that

$$\left|\otimes^s B^t(z, z)^{1/2} \partial^k f(z) \otimes \partial^{s-k} g(z) \right|$$

$$= \left|\otimes^s B^t(z, z)^{1/2} \partial^k f(z) \right| \cdot \left|\otimes^{s-k} B^t(z, z)^{1/2} \partial^{s-k} g(z) \right|.$$
Thus by Hölder’s inequality and Lemma 2.8 it follows that
\[
\int_\mathbb{B} \left\| \otimes^s B^t(z, z)^{1/2} \partial^k f(z) \otimes \partial^{s-k} g(z) \right\| (1 - |z|^2)\nu \, dt(z)
\leq C\|f\|_{\nu,k}\|g\|_{\nu,s-k} \leq C_\nu\|f\|_{\nu}\|g\|_{\nu}.
\]
\[
\square
\]

The Hankel bilinear form $H^p_k$ on $L^2_\nu(dt_\nu) \otimes L^2_\nu(dt_\nu)$ is defined by
\[
(2.11) \quad H^p_k(f, g) = \int_\mathbb{B} \langle \otimes^s B^t(z, z) T_\nu(f, g)(z), F(z) \rangle \, dt_\nu(z)
\]
where $F : \mathbb{B} \to \otimes^s \mathcal{V}$ is holomorphic. We call $F$ the symbol of the corresponding Hankel form. We remark that
\[
H^p_k(f, g) = \int_\mathbb{B} f(z)g(z)\overline{F(z)} \, dt_\nu(z).
\]
This is the classical Hankel form studied in [JPR].

With the form $H^p_k$ one can associate the operator $A^p_k$ defined by
\[
H^p_k(f, g) = \langle f, A^p_k g \rangle_\nu
\]
as in [JPR]. Notice that $A^p_k$ is an anti-linear operator on $L^2_\nu(dt_\nu)$. To get a linear operator one combines $A^p_k$ with a conjugation, i.e., one instead considers the operator \( \overline{A^p_k} : g \to \overline{A^p_k}g \). We say that $H^p_k$ is of Schatten-von Neumann class $\mathcal{S}_p$, for $0 < p < \infty$, if and only if $\overline{A^p_k} : L^2_\nu(dt_\nu) \to L^2_\nu(dt_\nu)$ is of class $\mathcal{S}_p$.

3. Duality of $\mathcal{H}^p_{\nu,s}$

In this section we determine the dual space $(\mathcal{H}^p_{\nu,s})^*$ of $\mathcal{H}^p_{\nu,s}$, $1 \leq p < \infty$.

**Lemma 3.1.** Let $1 \leq p < \infty$. If $\Phi \in (L^p_{\nu,s})^*$, then there is a function $G \in L^q_{\nu,s}$ such that
\[
\Phi(F) = \int_\mathbb{B} \langle \otimes^s B^t(z, z) F(z), G(z) \rangle (1 - |z|^2)^{2\nu} \, dt(z)
\]
and $\|\Phi\| = \|G\|_{\nu,s,q}$ where $1/q + 1/p = 1$.

**Proof.** Define $A(z) = (1 - |z|^2)^\nu \otimes^s B^t(z, z)^{1/2}$ and $(MA F)(z) = A(z) F(z)$. Then $M_A$ is an isometry from $L^p_{\nu,s}$ onto $L^p$, where $L^p = \{ F : \mathbb{B} \to V : \|F\|_p < \infty \}$ and
\[
\|F\|_p = \left( \int_\mathbb{B} \|F(z)\|^p \, dt(z) \right)^{1/p}.
\]
Consider $\Theta = \Phi M_A^{-1}$. Then $\Theta$ is a bounded linear functional on $L^p$ and $\Theta(AF) = \Phi(F)$. Then we can find a function $H \in L^q$ such that
\[
\Phi(F) = \int_\mathbb{B} \langle (AF)(z), H(z) \rangle \, dt(z)
\]
with $\|\Theta\| = \|H\|_q$. Let $G = M_A^{-1}H$. Then $G \in L^q_{\nu,s}$ and
\[
\Phi(F) = \int_\mathbb{B} \langle \otimes^s B^t(z, z) F(z), G(z) \rangle (1 - |z|^2)^{2\nu} \, dt(z).
\]
Also $\|\Phi\| = \|G\|_{\nu,s,q}$. \(\square\)
Theorem 3.2. For $1 \leq p < \infty$ we have $(\mathcal{H}_{p,s}^v)^* = \mathcal{H}_{p,s}^q$, under the integral pairing

$$(F,G)_{\nu,s,2} = \int_B \langle \otimes^s B^t(z,z)F(z),G(z) \rangle (1-|z|^2)^{2\nu}d\nu(z), \quad F \in \mathcal{H}_{p,s}^v, \ G \in \mathcal{H}_{p,s}^v,$$

where $1/p + 1/q = 1$. Namely, for any bounded linear functional $\Phi : \mathcal{H}_{p,s}^v \to \mathbb{C}$ there is a function $G \in \mathcal{H}_{p,s}^q$ such that $\Phi(F) = \langle F,G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}_{p,s}^v$ with

$$C\|G\|_{\nu,s,q} \leq \|\Phi\| \leq \|G\|_{\nu,s,q}.$$ 

Proof. By Hölder’s inequality, every function $G \in \mathcal{H}_{p,s}^q$ defines a bounded linear functional $\tilde{\Phi}$ on $\mathcal{H}_{p,s}^v$ under the above integral pairing with $\|\Phi\| \leq \|G\|_{\nu,s,q}$.

Conversely, let $\Phi \in (\mathcal{H}_{p,s}^v)^*$. By the Hahn-Banach theorem we can extend $\Phi$ to a bounded linear functional $\hat{\Phi}$ on $L_{p,s}^v$ such that $\Phi(F) = \hat{\Phi}(F)$ for all $F \in \mathcal{H}_{p,s}^v$ with $\|\Phi\| = \|\hat{\Phi}\|$. By Lemma 3.1 there is a function $H \in L_{p,s}^v$ such that

$$\Phi(F) = \int_B \langle \otimes^s B^t(z,z)F(z),H(z) \rangle (1-|z|^2)^{2\nu}d\nu(z)$$

for all $F \in L_{p,s}^v$, with $\|\hat{\Phi}\| = \|H\|_{\nu,s,q}$. However, Theorem 2.1 implies that, for any $F \in \mathcal{H}_{p,s}^v,$

$$F(z) = (P_{\nu,s}F)(z) = c \int_B (1-|w|^2)^{2\nu}K_{\nu,s}(w,z)^* \otimes^s B^t(w,w)F(w)d\nu(w).$$

Substituting this into formula (3.1) and using Fubini-Tonelli’s theorem we get that

$$\Phi(F) = \hat{\Phi}(F) = \int_B \langle \otimes^s B^t(w,w)F(w),\nu,H(w) \rangle (1-|w|^2)^{2\nu}d\nu(w).$$

Let $G = P_{\nu,s}H$. By Proposition 2.3, $\|P_{\nu,s}H\|_{\nu,s,q} \leq C\|H\|_{\nu,s,q}$. Then $G \in \mathcal{H}_{p,s}^q$, $\Phi(F) = \langle F,G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}_{p,s}^v$, and $C\|G\|_{\nu,s,q} \leq \|\Phi\|$. □

4. Atomic decomposition of $\mathcal{H}_{p,s}^v$

Following [JPR], we denote by $l^1(B, \otimes^s V')$ the space of all functions $a : B \to \otimes^s V'$, with support in $\{z_j\}_{j=1}^\infty \subset B$, such that

$$\|a\|_{l^1} = \sum_{j=1}^\infty \|a(z_j)\| < \infty.$$ 

Also, denote by $l^\infty(B, \otimes^s V')$ the space of all functions $a : B \to \otimes^s V'$ such that

$$\|a\|_{l^\infty} = \sup_{z \in B} \|a(z)\| < \infty.$$ 

Then it is elementary that

$$l^\infty(B, \otimes^s V') = (l^1(B, \otimes^s V'))^*,$$

under the pairing

$$\langle a, b \rangle' = \sum_{j=1}^\infty \langle a(z_j), b(z_j) \rangle$$

where $a \in l^1(B, \otimes^s V')$ with support $\{z_j\}_{j=1}^\infty \subset B$ and $b \in l^\infty(B, \otimes^s V')$. Namely, for any bounded linear functional $\Phi : l^1(B, \otimes^s V') \to \mathbb{C}$ there is a function $b$ in $l^\infty(B, \otimes^s V')$ such that $\Phi(a) = \langle a, b \rangle'$ for all $a \in l^1(B, \otimes^s V')$ with $\|\Phi\| = \|b\|_{l^\infty}$. 

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Theorem 4.1. It follows that \( F \in \mathcal{H}_{\nu,s}^1 \) if and only if there is a sequence \( \{z_j\}_{j=1}^{\infty} \subset \mathbb{B} \) and a sequence \( \{a_j\}_{j=1}^{\infty} \in l^1(\mathbb{B}, \odot^s V') \) such that

\[
F(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \odot B^t(z_j, z_j)^{1/2} a_j.
\]

Proof. By Proposition 2.4 for any \( v \in \odot^s V' \) and any \( z \in \mathbb{B} \),

\[
\left\| K_{\nu,s}(\cdot, z) \odot B^t(z, z)^{1/2} v \right\|_{\nu,s,1} \leq C_s(1 - |z|^2)^{-\nu} \|v\|.
\]

Thus, the operator \( T : l^1(\mathbb{B}, \odot^s V') \to \mathcal{H}_{\nu,s}^1 \) defined by

\[
(Ta)(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \odot B^t(z_j, z_j)^{1/2} a_j
\]
is bounded, where \( a_j = a(z_j) \) and the support of \( a \) is \( \{z_j\}_{j=1}^{\infty} \). We need to prove that \( T \) is onto. Consider \( T^* : (\mathcal{H}_{\nu,s}^1)^* \to (l^1(\mathbb{B}, \odot^s V'))^* \), \( T^*(\Phi)(a) = \Phi(Ta) \), which is bounded, where \( \Phi \in (\mathcal{H}_{\nu,s}^1)^* \) and \( a \in l^1(\mathbb{B}, \odot^s V') \). By Theorem 3.2 for any \( \Phi \in (\mathcal{H}_{\nu,s}^1)^* \) there is a \( G \in \mathcal{H}_{\nu,s}^\infty \) such that \( \Phi(F) = (F, G)_{\nu,s,2} \) for all \( F \in \mathcal{H}_{\nu,s}^1 \), with \( C\|G\|_{\nu,s,\infty} \leq \|\Phi\| \leq \|G\|_{\nu,s,\infty} \). Now, let \( a \in l^1(\mathbb{B}, \odot^s V') \) with support \( \{z_j\}_{j=1}^{\infty} \subset \mathbb{B} \).

By the reproducing property in Theorem 2.1 it follows that

\[
T^*(\Phi)(a) = \Phi(Ta) = (Ta, G)_{\nu,s,2} = c \sum_{j=1}^{\infty} \langle a_j, (1 - |z_j|^2)^\nu \odot B^t(z_j, z_j) \rangle G(z_j).
\]

Hence, by (4.1) and Theorem 3.2 it follows that

\[
\frac{1}{c} \cdot \|T^*\Phi\|_{(l^1)^*} = \sup_{z \in \mathbb{B}} \left\| (1 - |z|^2)^\nu \odot B^t(z, z) \right\| = \|G\|_{\nu,s,\infty} \geq \|\Phi\|.
\]

On the one hand, (4.2) yields that \( \ker T^* = \{0\} \) and consequently the range of \( T \) is dense in \( \mathcal{H}_{\nu,s}^1 \). On the other hand, (4.2) yields that the range of \( T^* \) is closed and so is the range of \( T \) by the Closed Range Theorem. \( \square \)

5. Trace class \( S_1 \)

We consider now the trace class property of \( H_p^\nu \) in (2.11).

Theorem 5.1. The Hankel form \( H_p^\nu \) is of trace class \( S_1 \) if and only if \( F \in \mathcal{H}_{\nu,s}^1 \).

Combining the results in [Su] we have now a complete characterization of the Schatten-von Neumann class Hankel forms.

Theorem 5.2. The Hankel form \( H_p^\nu \) is of Schatten-von Neumann class \( S_p \) if and only if \( F \in \mathcal{H}_{\nu,s}^1 \) for \( 1 \leq p \leq \infty \).

Proof of Theorem 5.2. It follows from Lemma 5.5 below and Theorem 1.1(a) in [Su] that the operator \( \Gamma : F \to H_p^\nu \) is bounded from \( \mathcal{H}_{\nu,s}^1 \) into \( \mathcal{S}_1 \) and from \( \mathcal{H}_{\nu,s}^\infty \) into \( \mathcal{S}_\infty \), respectively. Since \( \mathcal{S}_p = (\mathcal{S}_1, \mathcal{S}_\infty)_{1-\frac{1}{p}} \) if \( 1 < p < \infty \), then it follows by Riesz-Thorin’s interpolation theorem and Corollary 2.4 that \( \Gamma \) is bounded from \( \mathcal{H}_{\nu,s}^1 \) into \( \mathcal{S}_p \) if \( 1 < p < \infty \).

On the other hand, it follows from Lemma 5.6 below and Theorem 1.1(a) in [Su] that \( \tilde{T}_s \), defined in (5.5), is bounded from \( \mathcal{S}_1 \) into \( \mathcal{H}_{\nu,s}^1 \) and from \( \mathcal{S}_\infty \) into \( \mathcal{H}_{\nu,s}^\infty \), respectively. Again, by interpolation \( \tilde{T}_s \) is bounded from \( \mathcal{S}_p \) into \( \mathcal{H}_{\nu,s}^p \) if \( 1 < p < \infty \).
Also, if \(H^p_F \in S_p\) for \(1 \leq p < \infty\), then \(\hat{T}_s(H^p_F) = F\), which follows by the same arguments as in the proof of Lemma 8.6 in [Su].

The proof of Theorem 5.4 will be divided into a few lemmas. We will first show in Lemma 5.3 that every \(H^p_F\) is of trace class \(S_1\) if \(F\) is in \(H^1_{\nu,s}\) and then in Lemma 5.5 that \(H^1_{\nu,s}\) can be continuously embedded into \(H^\infty_{\nu,s}\). Using these results we prove, in Lemma 5.6 that \(F \rightarrow H^p_F\) is bounded from \(H^1_{\nu,s}\) into \(S_1\). Finally, in Lemma 5.6 we find a bounded mapping \(\hat{T}_s\) from the trace class \(S_1\) into \(H^1_{\nu,s}\) such that \(\hat{T}_s(H^p_F) = F\).

**Lemma 5.3.** If \(F \in H^1_{\nu,s}\), then \(H^p_F \in S_1\).

**Proof.** Let \(F \in H^1_{\nu,s}\). By Theorem 1.1 \(F = \sum_{j=1}^{\infty} F_j\) where

\[F_j(w) = (1 - |z_j|^2)^{\nu} K_{\nu,s}(w, z_j) \otimes s B^t(z_j, z_j)^{1/2} a_j\]

for some \(\{z_j\}_{j=1}^{\infty} \subset B\) and some \(\{a_j\}_{j=1}^{\infty} \in l^1(\mathbb{B}, \otimes s V')\). We claim that

\[
\text{rank } H^p_F \leq M_s \quad \text{for all } j = 1, 2, 3, \ldots,
\]

where \(M_s\) depends only on \(s\) and \(d\). Accepting temporarily the claim and using Theorem 1.1(a) in [Su] we get that

\[
\|H^p_F\|_{S_1} \leq \sum_{j=1}^{\infty} \|H^s_{F_j}\|_{S_1} \leq M_s \sum_{j=1}^{\infty} \|H^s_{F_j}\|_{S_\infty} \leq M'_s \sum_{j=1}^{\infty} \|F_j\|_{\nu,s,\infty} \leq M''_s \sum_{j=1}^{\infty} ||a_j||.
\]

Now we go back to claim (5.1). By Lemma 5.4 \(T_s(f, g) \in H^1_{\nu,s}\) for all \(f, g \in L^2(\mu, \nu)\). Thus, by the reproducing property in Theorem 2.1

\[H^p_{F_j}(f, g) = c \left\langle T_s(f, g)(z_j), (1 - |z_j|^2)^{\nu} \otimes s B^t(z_j, z_j)^{1/2} a_j \right\rangle.
\]

Fix \(z_0 \in B\). Then \(T_s(f, g)(z_0)\) is a sum of finitely many rank one forms where the number \(M_s\) of summands depends only on \(s\) and \(d\). To see this, we consider \(f(z_0) = (f, K_{\nu,s}(z_0))\). Since

\[
\partial^{s-k} f(z_0) \otimes \partial^k g(z_0) = \left\langle f, \partial^{s-k} K_{\nu,s}(z_0) \right\rangle \otimes \left\langle g, \partial^k K_{\nu,s}(z_0) \right\rangle,
\]

then \((f, g) \rightarrow \partial^{s-k} f(z_0) \otimes \partial^k g(z_0)\) is a rank one form. Thus, the bilinear form \((f, g) \rightarrow T_s(f, g)(z_0)\) has rank at most \(M_s\) and so has \(H^p_{F_j}\).

**Lemma 5.4.** The operator \(I : H^1_{\nu,s} \rightarrow H^\infty_{\nu,s}\), \(I(F) = F\), is bounded.

**Proof.** First, let \(F \in H^1_{\nu,s}\). Then \(H^p_F \in S_1\) by Lemma 5.3 Hence \(H^p_F \in S_\infty\), so by Theorem 1.1(a) in [Su] it follows that \(F \in H^\infty_{\nu,s}\). Thus \(I\) is well-defined.

Now, assume that \(F_n \rightarrow F\) in \(H^1_{\nu,s}\) and that \(I(F_n) \rightarrow G\) in \(H^\infty_{\nu,s}\). We shall prove that \(I(F) = G\). On the one hand, since \(F_n \rightarrow F\) in \(H^1_{\nu,s}\), then there is a subsequence \(I(F_{n_k})\) converging pointwise to \(I(F)\). On the other hand, since \(I(F_n) \rightarrow G\) in \(H^\infty_{\nu,s}\), then \(I(F_{n_k}) \rightarrow G\) pointwise. Thus \(I(F) = G\) and the operator \(I\) is bounded by the Closed Graph Theorem.

**Lemma 5.5.** The operator \(\Gamma : H^1_{\nu,s} \rightarrow S_1, \Gamma(F) = H^p_F\), is bounded.

**Proof.** The operator \(\Gamma\) is well defined by Lemma 5.3. We use the Closed Graph Theorem. Assume that \(F_n \rightarrow F\) in \(H^1_{\nu,s}\) and that \(\Gamma(F_n) \rightarrow B\) in \(S_1\). We shall prove that \(H^p_F = B\). On the one hand, by Theorem 1.1(a) in [Su] and Lemma 5.3 it follows that

\[
\|H^p_{F_n} - F\|_{S_\infty} \leq C\|F_n - F\|_{\nu,s,\infty} \leq C'\|F_n - F\|_{\nu,s,1}
\]
so that $H^2_{P_n} \to H^2_F$ in $S_\infty$. On the other hand,
\[ \| \Gamma(F_n) - B \|_{S_\infty} \leq \| \Gamma(F_n) - B \|_{S_1} \]
so that $H^2_{P_n} \to B$ in $S_\infty$. Thus $H^2_F = B$ so that $\Gamma$ has the closed graph property. Hence, $\Gamma$ is bounded. \qed

We recall the transvectant $\tilde{T}_s : S_\infty \rightarrow \mathcal{H}^1_{\nu,s}$ defined in \cite{Su} (see also \cite{FR} and \cite{PZ}), where $\mathcal{A}_s(\mathbb{B} \times \mathbb{B})$ consists of all holomorphic functions $G : \mathbb{B} \times \mathbb{B} \to \mathcal{O} V'$. We recall further that the transvectant $\tilde{T}_s$ in (2.10) can be defined for any holomorphic function $G(z,w)$ on $\mathbb{B} \times \mathbb{B}$, namely
\[
(T_sG)(z,w) = \sum_{k=0}^s \binom{s}{k} (-1)^k \frac{\partial_k^k \partial_{\nu,s}^{-k} G(z,w)}{(\nu)_k (\nu)_{s-k}}.
\]
For bounded bilinear forms $A$ on $L^2_\nu(dt_\nu)$, we define
\[
(\tilde{T}_s(A))(z) = (T_sG)(z,z),
\]
where $G(z,w) = A(K_z,K_w)$.

**Lemma 5.6.** The operator $\tilde{T}_s : S_1 \rightarrow \mathcal{H}^1_{\nu,s}$ defined in (5.2) is bounded. Also, $\tilde{T}_s(H^2_F) = F$ if $H^2_F \in S_1$.

**Proof.** First, let $B \in S_1$ be of rank one. Then there exists $\phi, \varphi \in L^2_\nu(dt_\nu)$ such that
\[
B(f,g) = \langle f, \phi \rangle \nu \langle g, \varphi \rangle \nu
\]
for all $f, g \in L^2_\nu(dt_\nu)$. Then $\|B\|_{S_1} = \|\phi\|_{\nu} \|\varphi\|_{\nu}$ and $\tilde{T}_s(B)(z) = T_s(\phi, \varphi)(z)$, so by Lemma 2.7 it follows that
\[
\| \tilde{T}_s(B) \|_{\nu,s,1} \leq C_s \|\phi\|_{\nu} \|\varphi\|_{\nu} \leq C_s \|B\|_{S_1}.
\]
In general, if $B \in S_1$ we can write $B = \sum_{n=1}^N B_n$, rank $B_n = 1$ such that
\[
\|B^n\|_{S_1} = \sum_{n=1}^N \|B_n\|_{S_1} \to \|B\|_{S_1}, \text{ as } N \to \infty,
\]
application of the transvectant $\tilde{T}_s$ to some $G \in \mathcal{H}^1_{\nu,s}$, it follows by Lemma 8.4 in \cite{Su} that $\tilde{T}_s(B^n) \to \tilde{T}_s(B)$ in $\mathcal{H}^\infty_{\nu,s}$. Hence $\tilde{T}_s(B) = G$ so that (5.3) holds for any $B \in S_1$.

Also, if $H^2_F \in S_1$, then $\tilde{T}_s(H^2_F) = F$. (As in the proof of Theorem 5.2 we refer to the proof of Lemma 8.6 in \cite{Su}.) \qed

**References**


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