A MINIMUM FIXED POINT THEOREM
FOR SMOOTH FIBER PRESERVING MAPS

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(Communicated by Paul Goerss)

This paper is dedicated to my advisor, Robert F. Brown

ABSTRACT. Let \( p : E \to B \) be a smooth fiber bundle. Given a smooth fiber preserving map \( f : E \to E \), we will show that \( f \) can be deformed by a smooth, fiber preserving homotopy to a smooth map \( g : E \to E \) such that the number of fixed points of \( g \) is equal to the fiberwise Nielsen number of \( f \).

For a given map \( f : X \to X \), where \( X \) is a compact ANR, the Nielsen number of \( f \), denoted \( N(f) \), is a lower bound for the number of fixed points of maps homotopic to \( f \). Wecken proved that if \( X \) is a triangulated manifold of dimension greater than or equal to 3, there is a map \( g \) homotopic to \( f \) that has \( N(f) \) fixed points [6]. A space is said to be Wecken if every self map of it has this property. Brown later proved that topological manifolds of dimension at least 3 are Wecken [1]. The corresponding theorem in the smooth category was proved by Jiang in [4]. He showed that for a smooth manifold \( M \) of dimension \( \geq 3 \), if \( f : M \to M \) is a smooth map, then \( f \) can always be smoothly deformed to a map \( g \) with exactly \( N(f) \) fixed points.

The goal of this note is to apply Jiang’s smooth Wecken theorem to prove a smooth version of a Wecken-type theorem for fiber preserving maps of Heath, Keppelmann and Wong [3]. In the setting of this theorem, \( p : E \to B \) is a fibration of compact connected ANR’s. Then the pair \((f, \bar{f})\) is called a fiber preserving map of \( p \) if \( f : E \to E \), \( \bar{f} : B \to B \) and the condition \( \bar{f}p = pf \) is satisfied. The fiberwise Nielsen number \( N_{\bar{f}}(f, p) \) of \((f, \bar{f})\), also known as the naive addition formula, is then defined to be

\[
N_{\bar{f}}(f, p) = \sum_{x \in \xi} N(f_x),
\]

where \( \xi \) is a set consisting of one point from each essential fixed point class of \( \bar{f} \). If \( g : E \to E \) is homotopic to \( f \) by a fiber preserving homotopy, then \( g \) has at least \( N_{\bar{f}}(f, p) \) fixed points.

**Theorem 1** (Heath, Keppelmann, Wong). Let \((f, \bar{f})\) be a fiber preserving map from \( p \) to itself with the property that \( \bar{f} \) is homotopic to a map \( \bar{g} \) that has exactly \( N(\bar{f}) \) fixed points. Suppose further that every fiber over the unique set of essential representatives for \( \bar{g} \) is a Wecken space. Then there is a fiber preserving map \((g, \bar{g})\) that is fiber homotopic to \((f, \bar{f})\) with the property that \( g \) has exactly \( N_{\bar{f}}(f, p) \) fixed points.
For the fiber Wecken theorem in the smooth category, we must assume that we have a smooth fiber bundle. This consists of a smooth surjective map \( p : E \to B \), where \( E \) and \( B \) and the fiber, \( F \), are smooth compact manifolds with or without boundary. Furthermore, \( B \) can be covered by a system of local coordinate neighborhoods \( \{U_\alpha\} \) such that there are diffeomorphisms \( \phi_U : U_\alpha \times F \to p^{-1}(U_\alpha) \) satisfying \( p\phi_U(x, y) = x \).

We will prove the following.

**Theorem 2.** If \( p : E \to B \) is a smooth fiber bundle with \( \text{dim } B, F \geq 3 \) and \( (f, \bar{f}) \) is a smooth fiber preserving homotopy of \( p \), then there exists a smooth fiber preserving map \( (g, \bar{g}) \) smoothly fiber preserving homotopic to \( (f, \bar{f}) \) such that \( g \) has \( N_F(f, p) \) fixed points.

**Proof.** First, we can directly apply Jiang’s smooth Wecken theorem to \( \bar{f} \). This gives us a smooth map \( \bar{h} : B \to B \) that is smoothly homotopic to \( f \), by a smooth homotopy \( \alpha_t \), where \( \alpha_0 = \bar{f} \) and \( \alpha_1 = h \), such that \( h \) has \( N(f) \) fixed points. By the smooth covering homotopy theorem \([2]\), there exists a smooth lift \( \tilde{\alpha}_t : E \to E \) of \( \alpha_t \circ p \) since \( (f, \bar{f}) \) is a smooth fiber preserving map of \( p \). Let \( h = \tilde{\alpha}_1 \circ \bar{p} ; \) then \( (h, \bar{h}) \) is a fiber preserving map of \( p \) that is smoothly fiber homotopic to \( (f, \bar{f}) \).

Suppose \( b \) is a fixed point of \( \bar{h} \) and that \( b \) is contained in some local coordinate chart \( V \) that has the local trivialization property. Since \( \bar{h} \) has isolated fixed points, we can find an open neighborhood \( U \) of \( b \) where \( U \subseteq \bar{h}^{-1}(V) \cap V \) and \( U \) has no additional fixed points of \( \bar{h} \). We may assume that \( U \) is the interior of a geodesic ball with center \( b \) and radius 1 (we can always rescale). This implies that any two points in \( U \) can be joined by a unique arc length geodesic that is contained in \( U \).

Let \( h_b = \bar{h}|_{p^{-1}(b)} : p^{-1}(b) \to p^{-1}(b) \). Applying the smooth Wecken theorem to \( h_b \), there exists a smooth map \( g_b \) smoothly homotopic to \( h_b \), by a homotopy we will call \( h_t \), where \( h_0 = g_b \) and \( h_1 = h_b \), such that \( g_b \) has \( N(h_b) \) fixed points. Since \( h_b \) is homotopic to \( f_b \) by the homotopy \( h_t \) above, it follows that \( g_b \) has \( N(f_b) \) fixed points. The local triviality conditions on \( U \) and \( V \) give us a homotopy

\[
\phi_V^{-1} \circ h \circ \phi_U : \{ b \} \times F \to \{ b \} \times F
\]
such that

\[
\phi_V^{-1} \circ h \circ \phi_U(b, y) = (b, \bar{h}_t(b, y)),
\]

where \( \tilde{h}_t \) is a smooth homotopy on \( \{ b \} \times F \). Let \( \tilde{h}_0(b, y) = \bar{g}(b, y) \) and \( \tilde{h}_1(b, y) = \bar{h}(b, y) \).

Since \( (h, \bar{h}) \) is a fiber preserving map of \( p \), we have that \( \phi_V^{-1} \circ h \circ \phi_U : U \times F \to V \times F \) is of the form

\[
\phi_V^{-1} \circ h \circ \phi_U(x, y) = (\bar{h}(x), \bar{h}(x, y)),
\]

where \( \bar{h} \) is a smooth map on \( U \times F \). For each \( x \in U \), there exists a unique arc-length parameter geodesic \( \gamma_x : I \to U \), where \( \gamma_x(0) = b \), \( \gamma_x(1) = x \), \( \gamma_x \) depends smoothly on its endpoints and varies continuously with \( x \). Define \( k_t : U \times F \to F \) by

\[
k_t(x, y) = \bar{h}(\gamma_x(t), y).
\]

Then \( k_t \) is a continuous homotopy, where \( k_0(x, y) = \bar{h}(b, y) \) and \( k_1(x, y) = \bar{h}(x, y) \). We can now define a homotopy \( c_t : U \times F \to F \) as follows:

\[
c_t(x, y) = \begin{cases} 
\bar{h}_2t(b, y), & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\bar{h}_{2t-1}(x, y), & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
By standard smooth approximation techniques, \( c_t \) can be approximated by a smooth homotopy \( \tilde{c}_t \) with \( \tilde{c}_0(x, y) = \tilde{g}(b, y) \) and \( \tilde{c}_1(x, y) = \tilde{h}(x, y) \).

Consider a smooth monotone increasing function \( B : I \to I \), such that \( B \) equals 0 on the interval \([0, \frac{1}{2}]\) and \( B \) equals 1 on the interval \([\frac{1}{2}, 1]\). Define \( \tilde{l}_t : U \times F \to V \times F \) by
\[
\tilde{l}_t(x, y) = (\tilde{h}(x), \tilde{c}_{t+(-1-t)B(dist(b,x))}(x, y))
\]
where \( dist(b, x) \) is the length of the unique minimal geodesic \( \gamma_x \) joining \( b \) to \( x \). If \( \phi_U^{-1}(z) = (x, y) \), we use \( \tilde{l}_t \) to define a smooth homotopy \( l_t : p^{-1}(U) \to p^{-1}(V) \) by
\[
l_t(z) = \phi_V \circ \tilde{l}_t \circ \phi_U^{-1}(z) = \phi_V(\tilde{h}(x), \tilde{c}_{t+(-1-t)B(dist(b,x))}(x, y)).
\]
Consider
\[
l_0(z) = \phi_V(\tilde{h}(x), \tilde{c}_{B(dist(b,x))}(x, y)).
\]
When \( z \in p^{-1}(b) \), then \( l_0(z) = g_b(z) \). If \( dist(b, x) \geq \frac{1}{4} \), then \( l_0(z) = h(z) \). Note that \( l_t \) is a homotopy ending at \( h(z) \).

Extend \( l_t \) to a smooth fiber preserving homotopy, which we will call \( L_t \), defined on all of \( E \) by taking defining \( L_t \) to be \( h \) outside of the neighborhood of \( p^{-1}(b) \). Define \( g \) to be \( L_0 \). Now \( g \) is a smooth self map of \( E \) that is smoothly homotopic to \( h \), where \( g|_{p^{-1}(b)} = g_b \) and \( g = h \) outside of a neighborhood of \( p^{-1}(b) \). The map \( g \) has \( N(f_b) \) fixed points on the fiber \( p^{-1}(b) \) over the fixed point \( b \). Repeated application of this process produces a map \( g \) that has \( N(f, p) = \sum_{x \in \xi} N(f_x) \) fixed points, where \( \xi \) is any set of representatives for the essential fixed point classes of \( f \).

**References**


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