WEIGHTED COMPOSITION OPERATOR ON THE FOCK SPACE

SEI-ICHIRO UEKI

(Communicated by Joseph A. Ball)

Abstract. We characterize the boundedness and compactness of a weighted composition operator on the Fock space. Our results use a certain integral transform. We also estimate the essential norm of a weighted composition operator. The result could be extended to the higher-dimensional case.

1. Introduction

Throughout this paper, let $dA$ denote the usual Lebesgue measure on $\mathbb{C}$. The Fock space $\mathcal{F}^2$ is the space of all entire functions $f$ on $\mathbb{C}$ for which

$$\|f\|^2_{\mathcal{F}^2} = \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty.$$ 

$\mathcal{F}^2$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{F}^2} = \frac{1}{2\pi} \int_{\mathbb{C}} f(z)g(z) e^{-|z|^2} dA(z).$$

The reproducing kernel function for $\mathcal{F}^2$ is given by $K(z, w) = \exp(\langle z, w \rangle)$ (see [5]). We also use the normalized kernel function $k_w(z) = \exp(\langle z, w \rangle) - |w|^2$.

Many authors have studied various concrete operators on the Fock space. For these studies, refer to [1, 2, 4, 5, 8, 10, 11]. In this paper, our object is the weighted composition operator on $\mathcal{F}^2$. Let $\varphi$ and $u$ be entire functions on $\mathbb{C}$. The weighted composition operator $uC_\varphi$ is defined by $uC_\varphi f = u \cdot (f \circ \varphi)$ for an entire function $f$. The purpose of this paper is to characterize the boundedness and the compactness of $uC_\varphi$. Our results will be expressed in terms of the integral transform

$$B_\varphi(|u|^2)(w) = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |e^{\langle z, w \rangle}|^2 e^{-|w|^2} e^{-|z|^2} dA(z).$$

The following two theorems are our main results.

Theorem 1. Let $\varphi$ be an entire function on $\mathbb{C}$ and $u \in \mathcal{F}^2$. Then $uC_\varphi$ is a bounded operator on $\mathcal{F}^2$ if and only if $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$. 

Received by the editors October 25, 2005 and, in revised form, December 5, 2005. 
2000 Mathematics Subject Classification. Primary 47B38; Secondary 30D15. 
Key words and phrases. Weighted composition operators, Fock space, entire function. 
©2006 American Mathematical Society 
Reverts to public domain 28 years from publication.
Theorem 2. Suppose that \( \varphi \) and \( u \) are entire functions on \( \mathbb{C} \) such that \( uC_\varphi \) is bounded on \( \mathcal{F}^2 \). Then there is a positive constant \( C \) such that
\[
\limsup_{|w| \to \infty} B_\varphi(|u|^2)(w) \leq \|uC_\varphi\|^2 \leq C \limsup_{|w| \to \infty} B_\varphi(|u|^2)(w).
\]
In particular, \( uC_\varphi \) is a compact operator on \( \mathcal{F}^2 \) if and only if
\[
\lim_{|w| \to \infty} B_\varphi(|u|^2)(w) = 0.
\]

2. PROOFS OF THE MAIN THEOREMS

In order to prove our results, we need several lemmas. In this section, let \( \varphi \) be an entire function and \( u \in \mathcal{F}^2 \).

The following lemma appears in [6, 7]. However, we give another proof. Our proof is simpler than the proof in [7] and remains valid in the higher-dimensional case.

Lemma 1. There is a positive constant \( C \) such that for each entire function \( f \) and \( z \in \mathbb{C} \),
\[
|f(z)|^2 e^{-\frac{|w|^2}{2}} \leq C \int_{D(z,1)} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w),
\]
where \( D(z,1) = \{ w \in \mathbb{C} : |w - z| < 1 \} \).

Proof. Since \( |f|^2 \) is subharmonic, we have
\[
|f(0)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta,
\]
for all \( r > 0 \). Multiplying both sides by \( 2\pi re^{-\frac{\theta^2}{2}} \) and integrating with respect to \( r \) from 0 to 1 give

\[
(1) \quad |f(0)|^2 \leq \frac{1}{2\pi(1-e^{-\frac{\theta^2}{2}})} \int_{D(0,1)} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w).
\]

For fixed \( c \in \mathbb{C} \), we set \( I_c f(z) = k_c(z)f(z-c) \). By [3, Proposition 2], we see that \( I_c \) is a unitary operator on \( \mathcal{F}^2 \) and \( I_c^{-1} = I_{-c} \). Since \( |f(z)|^2 e^{-\frac{|w|^2}{2}} = |I_{-z} f(0)|^2 \), it follows from (1) and a change of variable that
\[
|f(z)|^2 e^{-\frac{|w|^2}{2}} = |I_{-z} f(0)|^2 \leq \frac{1}{2\pi(1-e^{-\frac{\theta^2}{2}})} \int_{D(0,1)} |I_{-z} f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w)
\]
\[
= \frac{1}{2\pi(1-e^{-\frac{\theta^2}{2}})} \int_{D(0,1)} e^{-\frac{|w|^2}{2} - \text{Re}(w,z)} |f(w+z)|^2 e^{-\frac{|w|^2}{2}} dA(w)
\]
\[
= \frac{1}{2\pi(1-e^{-\frac{\theta^2}{2}})} \int_{D(0,1)} |f(w+z)|^2 e^{-\frac{|w+z|^2}{2}} dA(w)
\]
\[
= \frac{1}{2\pi(1-e^{-\frac{\theta^2}{2}})} \int_{D(z,1)} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w).
\]

The proof is complete. \( \square \)
Lemma 2. Define the positive measure \( \mu \) by
\[
\mu(E) = \frac{1}{2\pi} \int_{\varphi^{-1}(E)} |u(z)|^2 e^{-\frac{|z|^2}{2}} dA(z),
\]
where \( E \) is a Borel subset of \( \mathbb{C} \). Then
\[
\int_{D(w,1)} e^{\frac{|z|^2}{2}} d\mu(z) \leq e^{\frac{1}{2}} B_\varphi(|u|^2)(w),
\]
for all \( w \in \mathbb{C} \).

Proof. For each \( z \in D(w,1) \), we see that
\[
|k_w(z)|^2 = e^{Re(z,w) - \frac{|w|^2}{2}} \geq e^{-\frac{1}{2} |w|^2}.
\]
Hence we get
\[
e^{-\frac{1}{2}} \int_{D(w,1)} e^{\frac{|z|^2}{2}} d\mu(z) \leq \int_{D(w,1)} |k_w(z)|^2 d\mu(z) \leq \int_{\mathbb{C}} |k_w(z)|^2 d\mu(z).
\]
By the definitions of the measure \( \mu \) and the integral operator \( B_\varphi(|u|^2) \), we see that
\[
\int_{\mathbb{C}} |k_w(z)|^2 d\mu(z) = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |k_w(\varphi(z))|^2 e^{-\frac{|z|^2}{2}} dA(z) = B_\varphi(|u|^2)(w).
\]
So we obtain the desired inequality. \( \square \)

Proof of Theorem \[1\] First, we suppose that \( uC_\varphi \) is bounded on \( \mathcal{F}^2 \). Thus we have
\[
\|uC_\varphi k_w\|_{\mathcal{F}^2} \leq C\|k_w\|_{\mathcal{F}^2} = C,
\]
for some constant \( C > 0 \) and all \( w \in \mathbb{C} \). The formula for the normalized kernel function \( k_w \) shows that \( \|uC_\varphi k_w\|_{\mathcal{F}^2} = B_\varphi(|u|^2)(w) \), and so \( B_\varphi(|u|^2)(w) \leq C \) for all \( w \in \mathbb{C} \). This implies that \( B_\varphi(|u|^2) \in L^\infty(\mathbb{C}) \).

Next, we prove the other direction. By the definition of the measure \( \mu \), we obtain
\[
\|uC_\varphi f\|_{\mathcal{F}^2} = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |f(\varphi(z))|^2 e^{-\frac{|z|^2}{2}} dA(z) = \int_{\mathbb{C}} |f(z)|^2 d\mu(z).
\]
It follows from Lemma \[1\] that
\[
\|uC_\varphi f\|_{\mathcal{F}^2} = \int_{\mathbb{C}} |f(z)|^2 d\mu(z)
\quad \leq C \int_{\mathbb{C}} e^{\frac{|z|^2}{2}} d\mu(z) \int_{\mathbb{C}} \chi_{D(z,1)}(w) |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w),
\]
where \( \chi_{D(z,1)}(w) \) is the characteristic function of the set \( D(z,1) \). Since \( \chi_{D(z,1)}(w) = \chi_{D(w,1)}(z) \), Lemma \[2\] and Fubini’s theorem show that
\[
\|uC_\varphi f\|_{\mathcal{F}^2} \leq C e^{\frac{1}{2}} \int_{\mathbb{C}} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \int_{\mathbb{C}} \chi_{D(w,1)}(z) e^{\frac{|z|^2}{2}} d\mu(z)
\quad \leq C e^{\frac{1}{2}} \int_{\mathbb{C}} |f(w)|^2 e^{-\frac{|w|^2}{2}} B_\varphi(|u|^2)(w) dA(w)
\quad \leq 2\pi C e^{\frac{1}{2}} \sup_{w \in \mathbb{C}} B_\varphi(|u|^2)(w) \|f\|_{\mathcal{F}^2}^2.
\]
This implies that \( uC_\varphi \) is bounded on \( \mathcal{F}^2 \). \( \square \)
For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, define $R_n f(z) = \sum_{k=0}^{\infty} a_k z^k$, acting on $\mathcal{F}^2$. Note that $\|R_n\| = 1$ on $\mathcal{F}^2$. By [9, 5.1 Proposition], if $uC_{\varphi}$ is bounded on $\mathcal{F}^2$, then

$$\|u C_{\varphi}\|_{\mathcal{L}} = \lim_{n \to \infty} \|u C_{\varphi} R_n\|.$$ \hfill (2)

Moreover, the following lemma holds.

**Lemma 3.** For each entire function $f$ and $w \in \mathbb{C}$,

$$|R_n f(w)| \leq \|f\|_{\mathcal{F}^2} \left\{ \sum_{k=n}^{\infty} \frac{|w|^{2k}}{2^k k!} \right\}^{\frac{1}{2}}.$$ \hfill Proof.

The orthogonality of $w^k$ shows that $\|R_n K w\|_{\mathcal{F}^2} = \sum_{k=n}^{\infty} \frac{|w|^{2k}}{2^k k!}$. Since $R_n$ is self-adjoint, we have

$$|R_n f(w)| = |\langle R_n f, K w \rangle_{\mathcal{F}^2}| = |\langle f, R_n K w \rangle_{\mathcal{F}^2}| \leq \|f\|_{\mathcal{F}^2}\|R_n K w\|_{\mathcal{F}^2} = \|f\|_{\mathcal{F}^2} \left\{ \sum_{k=n}^{\infty} \frac{|w|^{2k}}{2^k k!} \right\}^{\frac{1}{2}}.$$ \hfill □

**Proof of Theorem** \cite{2}. First we prove the lower estimate. Take a compact operator $K$ on $\mathcal{F}^2$. Since $k_w \to 0$ weakly as $|w| \to \infty$, we see $\|K k_w\|_{\mathcal{F}^2} \to 0$ as $|w| \to \infty$. Therefore,

$$\|u C_{\varphi} - K\| \geq \limsup_{|w| \to \infty} \|u C_{\varphi} - K\|_{\mathcal{F}^2} \geq \limsup_{|w| \to \infty} (\|u C_{\varphi} k_w\|_{\mathcal{F}^2} - \|K k_w\|_{\mathcal{F}^2}) = \limsup_{|w| \to \infty} \|u C_{\varphi} k_w\|_{\mathcal{F}^2}.$$ 

Thus

$$\|u C_{\varphi}\|_{\mathcal{L}}^2 \geq \|u C_{\varphi} - K\|_{\mathcal{L}}^2 \geq \limsup_{|w| \to \infty} \|u C_{\varphi} k_w\|_{\mathcal{F}^2}^2 = \limsup_{|w| \to \infty} B_{\varphi}(|u|^2)(w).$$

Next, we prove the upper estimate. Take $f \in \mathcal{F}^2$ with $\|f\|_{\mathcal{F}^2} \leq 1$. Using the same argument as in the proof of Theorem \cite{1} we get

$$\|u C_{\varphi} R_n f\|_{\mathcal{F}^2}^2 = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |(R_n f)(\varphi(z))|^2 e^{-\frac{|z|^2}{2}} d\mu(z) = \int_{\mathbb{C}} |R_n f(z)|^2 d\mu(z) \leq C \int_{\mathbb{C}} e^{-\frac{|z|^2}{2}} d\mu(z) \int_{\mathbb{C}} \chi_{D(z,1)}(w)|R_n f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w)$$

$$= C \int_{\mathbb{C}} \chi_{D(z,1)}(w)|R_n f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \int_{\mathbb{C}} \chi_D(w,1)(z) e^{-\frac{|z|^2}{2}} d\mu(z) \leq C e^{\frac{1}{2}} \int_{\mathbb{C}} |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} B_{\varphi}(|u|^2)(w) dA(w)$$

$$= e^{\frac{1}{2}} C \left( \int_{\{ |w| \geq r \}} + \int_{\{ |w| < r \}} \right) |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} B_{\varphi}(|u|^2)(w) dA(w),$$ \hfill (3)
for fixed $r > 0$. Since $\|R_n f\|_{L^2} \leq 1$, we see that
\[
\int_{\{ |w| \geq r \}} |R_n f(w)|^2 e^{-\frac{\|w\|^2}{2}} B_\varphi(|w|^2)(w) dA(w) \leq 2\pi \sup_{|w| \geq r} B_\varphi(|w|^2)(w).
\]
By Lemma 3, we have
\[
\int_{\{ |w| < r \}} |R_n f(w)|^2 e^{-\frac{\|w\|^2}{2}} B_\varphi(|w|^2)(w) dA(w) \leq \sup_{w \in \mathbb{C}} B_\varphi(|w|^2)(w) \int_{\mathbb{C}} e^{-\frac{\|w\|^2}{2}} dA(w) \sum_{k=n}^{\infty} \frac{r^{2k}}{2^k k!}.
\]
Since $\sum_{k=n}^{\infty} \frac{r^{2k}}{2^k k!} \to 0$ as $n \to \infty$ and, by Theorem 1, $B_\varphi(|w|^2)$ is bounded, so
\[
\sup_{w \in \mathbb{C}} B_\varphi(|w|^2)(w) \int_{\mathbb{C}} e^{-\frac{\|w\|^2}{2}} dA(w) \sum_{k=n}^{\infty} \frac{r^{2k}}{2^k k!} \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence, taking the supremum over entire functions $f$ with $\|f\|_{L^2} \leq 1$, and letting $n \to \infty$ in (3), we obtain
\[
\lim_{n \to \infty} \sup_{\|f\|_{L^2} \leq 1} \|u C_\varphi R_n f\|_{L^2}^2 \leq 2\pi e^{\frac{r}{2}} C \sup_{|w| \geq r} B_\varphi(|w|^2)(w).
\]
Combining this with (2), we get $\|u C_\varphi\|_e^2 \leq 2\pi e^{\frac{r}{2}} C \sup_{|w| \geq r} B_\varphi(|w|^2)(w)$. Letting $r \to \infty$, we have
\[
\|u C_\varphi\|_e^2 \leq 2\pi e^{\frac{r}{2}} C \lim_{r \to \infty} \sup_{|w| \geq r} B_\varphi(|w|^2)(w) = 2\pi e^{\frac{r}{2}} C \lim_{r \to \infty} \sup_{|w| \geq r} B_\varphi(|w|^2)(w).
\]
This completes the proof. \qed

**Remark.** The $N$-dimensional Fock space $\mathcal{F}^2(\mathbb{C}^N)$ is defined by
\[
\mathcal{F}^2(\mathbb{C}^N) = \left\{ f \text{ is entire} : \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} |f(z)|^2 e^{-\frac{|z|^2}{2}} dm_{2N}(z) < \infty \right\},
\]
where $m_{2N}$ denotes the Lebesgue measure on $\mathbb{C}^N$. All arguments used in this paper can apply to the $N$-dimensional case. Thus, Theorem 1 and Theorem 2 are still valid in $\mathcal{F}^2(\mathbb{C}^N)$.

**Acknowledgement**

The author would like to thank the referee for pointing out mistakes.

**References**


Department of Mathematics, Nippon Institute of Technology, Miyashiro, Saitama 345-8501, Japan

E-mail address: sueki@camel.plala.or.jp

Current address: 2-18-4, Miyashiro-Machi Chuo, Minamisaitama, Saitama 345-0821 Japan