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RANDOMIZED UMD BANACH SPACES AND DECOUPLING INEQUALITIES FOR STOCHASTIC INTEGRALS

MARK C. VERAAR

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Abstract. In this paper we prove the equivalence of decoupling inequalities for stochastic integrals and one-sided randomized versions of the UMD property of a Banach space as introduced by Garling.

1. Introduction

In recent years, decoupling inequalities have been used to construct theories of stochastic integration in UMD Banach spaces [4, 13, 15]. The basic idea underlying this approach is to use abstract decoupling inequalities to estimate stochastic integrals

$$\int_0^T \phi(t)\, dW(t),$$

where ϕ is a process with values in a UMD space E and W is a standard Brownian motion, with its decoupled analogue

$$\int_0^T \phi(t)\, d\tilde{W}(t),$$

where $\tilde{W}$ is a standard Brownian motion independent of ϕ and W. This decoupled integral is easier to handle, as it is defined in a pathwise sense. Indeed, using a general two-sided decoupling inequality for E-valued tangent sequences, McConnell [13] was able to show that a strongly measurable E-valued process is stochastically integrable with respect to W if and only if its trajectories, viewed as E-valued functions, are stochastically integrable with respect to $\tilde{W}$. His techniques depend heavily on the equivalence of the UMD property and geometric notions related to ζ-convexity. Decoupling inequalities for tangent sequences may be found in [7, 9, 13, 14, 17].

Earlier, Garling [4] had derived a two-sided decoupling inequality for stochastic integrals of elementary E-valued processes directly from the definition of the UMD property. More precisely, he proved that a Banach space E is a UMD space if and

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only if for some (respectively, for all) $1 < p < \infty$ there exist constants $0 < c \leq C < \infty$ such that for all elementary E-valued processes ϕ we have

$$c\,\mathbb{E}\Big\|\int_0^T \phi(t)\,d\tilde{W}(t)\Big\|^p \leq \mathbb{E}\Big\|\int_0^T \phi(t)\,dW(t)\Big\|^p \leq C\,\mathbb{E}\Big\|\int_0^T \phi(t)\,d\tilde{W}(t)\Big\|^p. \tag{1.1}$$

These inequalities, combined with the operator-theoretic approach to stochastic integration of Banach space-valued functions developed in [16], was used in [15] to construct a systematic theory of stochastic integration for E-valued processes. In particular, necessary and sufficient conditions for L^p-stochastic integrability were obtained, analogues of the Itô isometry and the Burkholder-Davis-Gundy inequalities were proved, and McConnell's result was recovered as a corollary via standard stopping time arguments.

Various applications of the decoupling inequalities in (1.1) require only one of the two a priori estimates. An analysis of the proof of (1.1) in [4] shows moreover that one-sided decoupling inequalities can be derived from one-sided versions of the UMD property which were introduced subsequently by Garling in [5]. These properties are called UMD^- and UMD^+ below. These properties can be used as in [15] to obtain generalized theories of stochastic integration in which the necessary and sufficient conditions and two-sided estimates for stochastic integrals are replaced by necessary conditions or sufficient conditions, respectively, with one-sided estimates.

The stochastic integration theory in [15] has many consequences and applications. For instance, many results in the theory of stochastic evolution equations in Hilbert spaces (cf. [3] and the references therein), have analogues in UMD^- Banach spaces. Therefore, we believe it is important to know the largest class of spaces for which one can construct a stochastic integration theory as in [15]. The aim of the present paper is to show that this is the class of UMD^- Banach spaces. It is shown that the validity of the second one-sided a priori estimate in (1.1) for all elementary processes implies the $\mathrm{UMD}^-_{\mathrm{PW}}$ property. With the same ideas one can prove that E has property $\mathrm{UMD}^+_{\mathrm{PW}}$ if for some $1 < p < \infty$ the left estimate in (1.1) holds for all elementary E-valued processes, so we include this too. The proofs are based on Skorohod embedding techniques from [4], the Maurey-Pisier characterization of finite cotype and estimates for randomized sums in spaces of finite cotype.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 1}, \mathbb{P})$ be a filtered probability space, and let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space. Both probability spaces are assumed to be rich enough for constructions as below. We shall consider random variables and processes on $(\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, \mathbb{P} \times \tilde{\mathbb{P}})$. On this probability space we use the filtration $(\mathcal{F}_n \otimes \tilde{\mathcal{F}})_{n\geq 1}$. In most cases our random variables and processes are extensions to $\Omega \times \tilde{\Omega}$ of variables and processes on Ω or $\tilde{\Omega}$. Integration over Ω and $\tilde{\Omega}$ will be denoted by $\mathbb{E}$ and $\tilde{\mathbb{E}}$.

Let $(r_n)_{n\geq 1}$ be a Rademacher sequence on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_n = \sigma(r_k, k = 1, \ldots, n)$. Recall that a martingale difference sequence $(d_n)_{n=1}^N$ is a *Paley-Walsh martingale difference sequence* if it is a martingale difference sequence with respect to the filtration $(\mathcal{G}_n)_{n=0}^N$.

Recall that a Banach space E is a *UMD(p) space* for $p \in (1, \infty)$ if there exists a constant $C_p > 0$ such that for every $N \geq 1$, every martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$ and every $\{-1, 1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$, we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p\,\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}}.$$

Similarly, we say E is a $\mathrm{UMD}_{\mathrm{PW}}(p)$-space if one only considers Paley-Walsh martingales in the definition of $\mathrm{UMD}(p)$. In [11], Maurey has shown that $\mathrm{UMD}_{\mathrm{PW}}(p)$ already implies $\mathrm{UMD}(p)$. It was shown by Burkholder in [1] that if E is $\mathrm{UMD}(p)$ space for some $p \in (1,\infty)$, then E is a $\mathrm{UMD}(p)$ space for all $p \in (1,\infty)$. Spaces with this property will be referred to as *UMD spaces*. For the theory of UMD spaces we refer the reader to [1, 2] and references given therein.

Let $(\tilde{r}_n)_{n\geq 1}$ be a Rademacher sequence on $\tilde{\Omega}$.

Definition 1.1. Let E be a Banach space and let $p \in (1,\infty)$.

(1) The space E is *a* $UMD^-_{\mathrm{PW}}(p)$ *space* if there exists a constant $C_p^- > 0$ such that for every $N \geq 1$ and every Paley-Walsh martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$, we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p^- \Big(\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p\Big)^{\frac{1}{p}}. \tag{1.2}$$

(2) The space E is *a* $UMD^+_{\mathrm{PW}}(p)$ *space* if there exists a constant $C_p^+ > 0$ such that for every $N \geq 1$ and every Paley-Walsh martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$, we have

$$\Big(\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p^+ \Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}}. \tag{1.3}$$

The corresponding notion of UMD^- and UMD^+ spaces, where arbitrary martingale difference sequences are allowed, has been studied by Garling in [5]. It was shown there that if E is a $\mathrm{UMD}^{\pm}(p)$ space for some $p \in (1,\infty)$, then E is a $\mathrm{UMD}^{\pm}(p)$ space for all $p \in (1,\infty)$. Thus, both definitions are independent of $p \in (1,\infty)$ and spaces with this property will be referred to as *UMD*$^-$ and *UMD*$^+$ *spaces*. In [5] these properties are called LERMT (Lower Estimates for Random Martingale Transforms) and UERMT (Upper Estimates for Random Martingale Transforms) respectively. We preferred the notation UMD^- and UMD^+, since it emphasizes the relation with UMD. Here the superscript $^-$ stands for Lower and the superscript $^+$ stands for Upper. Similarly, one can show that $\mathrm{UMD}^{\pm}_{\mathrm{PW}}(p)$ are p-independent and these will denoted by $UMD^{\pm}_{\mathrm{PW}}$. It seems to be an open problem if $\mathrm{UMD}^-_{\mathrm{PW}}$ implies UMD^- and if $\mathrm{UMD}^+_{\mathrm{PW}}$ implies UMD^+.

We list some results on UMD^- and UMD^+ spaces, the proofs of which can be found in [5]:

- If E is a UMD^+ space, then its dual E^* is a UMD^- space. If E^* is a UMD^- space, then its predual E is a UMD^+ space.
- Every UMD^- space has finite cotype. Every UMD^+ space is super-reflexive.
- E is a UMD space if and only if it is both UMD^- and UMD^+.

Similar results hold for $\mathrm{UMD}^-_{\mathrm{PW}}$ and $\mathrm{UMD}^+_{\mathrm{PW}}$ spaces.

It was shown in [5] that l^1 is a UMD^- space. It can be shown that if E is a UMD^- space and if (S, Σ, μ) is a σ-finite measure space, then $L^p(\Omega; E)$ is a UMD^- space for all $p \in [1,\infty)$. A similar result holds for UMD^+ for $p \in (1,\infty)$.

Apart from trivial cases, the space $L^1(S,\mu)$ is an example of a UMD^- space that is not UMD. It appears to be unknown if there exist UMD^+ or $\mathrm{UMD}^+_{\mathrm{PW}}$ spaces that are not UMD (cf. [6, Problem 4.2]).

2. Main result

Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration induced by W. Similarly, let $\tilde{W}$ be a Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ be the augmented filtration induced by $\tilde{W}$.

Let E be a real Banach space. A process $\phi : [0,\infty) \times \Omega \to E$ will be called an *elementary process* if it is of the form

$$\phi(t,\omega) = \mathbf{1}_{[0]}(t)\xi_0(\omega) + \sum_{n=1}^{N} \mathbf{1}_{(t_{n-1},t_n]}(t)\xi_n(\omega),$$

where $0 \leq t_0 < \cdots < t_N < \infty$, ξ_n is an elementary $\mathcal{F}_{t_{n-1}}$-measurable random variable, $n = 1, \ldots, N$ and ξ_0 is $\mathcal{F}_0$-measurable. The stochastic integral $\int_0^\infty \phi(t)\, dW(t)$ is defined in the usual way and is an element of $L^p(\Omega; E)$ for all $p \in [1,\infty)$.

Theorem 2.1 (Garling)**.** *For a UMD space E and $p \in (1,\infty)$ the following statements hold:*

(1) *There exists a constant $c_p > 0$ such that for all elementary processes ϕ,*

$$\mathbb{E}\left\|\int_0^\infty \phi(t)\, dW(t)\right\|^p \leq c_p^p \mathbb{E}\,\tilde{\mathbb{E}}\left\|\int_0^\infty \phi(t)\, d\tilde{W}(t)\right\|^p. \tag{2.1}$$

(2) *There exists a constant $c_p > 0$ such that for all elementary processes ϕ,*

$$\mathbb{E}\,\tilde{\mathbb{E}}\left\|\int_0^\infty \phi(t)\, d\tilde{W}(t)\right\|^p \leq c_p^p \mathbb{E}\left\|\int_0^\infty \phi(t)\, dW(t)\right\|^p. \tag{2.2}$$

Conversely, if (2.1) *and* (2.2) *hold for all elementary processes ϕ, then E is a UMD space.*

Inspection of the proof in [4, Theorem 2] shows that (2.1) only requires UMD^- and (2.2) only requires UMD^+. The main result of this paper reads as follows.

Theorem 2.2. *Let E be a Banach space E and let $p \in (1,\infty)$. The following statements hold:*

(1) *If there exists a constant $c_p > 0$ such that* (2.1) *holds for all elementary processes, then E is a $\mathrm{UMD}^-_{\mathrm{PW}}$ space.*
(2) *If there exists a constant $c_p > 0$ such that* (2.2) *holds for all elementary processes, then E is a $\mathrm{UMD}^+_{\mathrm{PW}}$ space.*

Although these results are in some sense not surprising, they appear to be new and nontrivial to prove.

For the proof we need some lemmas. The first lemma is well known and follows from the strong Markov property.

Lemma 2.3. *Let $\tau_0 = 0$ and define inductively*

$$\tau_n = \inf\{t \geq \tau_{n-1} : |W_t - W_{\tau_{n-1}}| = 1\}, \quad 1 \leq n \leq N.$$

Then $(\tau_n)_{n=1}^N$ is an increasing sequence of stopping times and $(\Delta\tau_n, \Delta W_n)_{n=1}^N$ is an i.i.d. sequence of random vectors, where

$$\Delta\tau_n = \tau_n - \tau_{n-1}, \quad \Delta W_n = W_{\tau_n} - W_{\tau_{n-1}}, \quad 1 \leq n \leq N.$$

Moreover $(\Delta W_n)_{n=1}^N$ is a Rademacher sequence adapted to $(\mathcal{F}_{\tau_n})_{n=1}^N$.

The next lemma gives some important properties of the independent Brownian motion $\tilde{W}$ at random times. Such stopped Brownian motions $\tilde{W}$ are not Gaussian random variables in general, but in this case they inherit some important properties.

Lemma 2.4. *For $1 \leq n \leq N$, let $\Delta\tilde{W}_n = \tilde{W}_{\tau_n} - \tilde{W}_{\tau_{n-1}}$. Then $(\Delta\tilde{W}_n)_{n=1}^N$ is an i.i.d. sequence of symmetric random variables, which is independent of $(\Delta W_n)_{n=1}^N$. Furthermore, each ΔW_n has finite moments of all orders.*

Proof. For all $1 \leq n \leq N$, $\Delta\tilde{W}_n$ is symmetric, because $\Delta\tilde{W}_n(\omega,\cdot)$ is symmetric for each $\omega \in \Omega$. It follows from the strong Markov property of $(W,\tilde{W})$ that $(\Delta W_n, \Delta\tilde{W}_n)_{n=1}^N$ is an i.i.d. sequence. So in order to prove the independence of $(\Delta\tilde{W}_n)_{n=1}^N$ and $(\Delta W_n)_{n=1}^N$, it is enough to show that $\Delta W_1 = W_{\tau_1}$ and $\Delta\tilde{W}_1 = \tilde{W}_{\tau_1}$ are independent. The following argument is shown to us by Tuomas Hytönen. For every Brownian motion B on Ω we introduce the following two stopping times:

$$\tau_\pm^B = \inf\{t \geq 0 : B_t = \pm 1\}.$$

Note that $\tau_1 = \tau_-^W \wedge \tau_+^W$ and for the Brownian motion $-W$, we have $\tau_+^{-W} = \tau_-^W$ and $\tau_-^{-W} = \tau_+^W$. Let $B \in \mathbb{R}$ be some Borel measurable set. Since $(W,\tilde{W})$ is identically distributed with $(-W,\tilde{W})$ it follows that

$$\begin{aligned}\mathbb{P}\,(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) &= \mathbb{P}\,(\tau_+^W < \tau_-^W, \tilde{W}_{\tau_1} \in B)\\ &= \mathbb{P}\,(\tau_-^{-W} < \tau_+^{-W}, \tilde{W}_{\tau_1} \in B) = \mathbb{P}\,(W_{\tau_1} = -1, \tilde{W}_{\tau_1} \in B).\end{aligned}$$

Clearly,

$$\mathbb{P}\,(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) + \mathbb{P}\,(W_{\tau_1} = -1, \tilde{W}_{\tau_1} \in B) = \mathbb{P}\,(\tilde{W}_{\tau_1} \in B).$$

Hence

$$\mathbb{P}\,(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) = \frac{1}{2}\mathbb{P}\,(\tilde{W}_{\tau_1} \in B) = \mathbb{P}\,(W_{\tau_1} = 1)\mathbb{P}\,(\tilde{W}_{\tau_1} \in B).$$

The same holds for -1. This proves the independence.

For $0 < p < \infty$, we have

$$\mathbb{E}\,\tilde{\mathbb{E}}\,|\Delta\tilde{W}_n|^p = \mathbb{E}\,\tilde{\mathbb{E}}\,|\tilde{W}_{\tau_1}|^p = g_p^p \mathbb{E}\,\tau_1^{p/2},$$

where g_p^p is the p-th moment of a standard Gaussian random variable and the statement follows from the elementary fact that τ_1 has finite moments of all orders. □

Below we will consider adapted and measurable processes $\phi : [0,\infty)\times\Omega \to E$ that take values in a finite-dimensional subspace of E. Since n-dimensional subspaces of E are isomorphic to $\mathbb{R}^n$, one may construct the stochastic integral for such processes ϕ that satisfy $t \mapsto \phi(t,\omega) \in L^2(0,\infty,E)$ for almost all $\omega \in \Omega$. By the Burkholder-Davis-Gundy inequalities we have for all $p \in (1,\infty)$ and for ϕ as above, $\int_0^\infty \phi(t)\,dW(t) \in L^p(\Omega;E)$ if $\phi \in L^p(\Omega;L^2(0,\infty;E))$. In this case the decoupled stochastic integral $\int_0^\infty \phi(t)\,d\tilde{W}(t)$ is defined pathwise as an element of $L^p(\Omega;L^p(\tilde{\Omega};E))$. Moreover, if (2.1) or (2.2) holds for all elementary processes one may extend this to all processes as above. In fact, Garling proved (2.1) and (2.2) for this class of processes.

The next lemma is a variation of an example in [5]. We include a proof for convenience.

Lemma 2.5. *Let $E = c_0$ and $p \in [1,\infty)$. There does not exist a constant $c_p > 0$ such that for all elementary processes ϕ, (2.1) holds.*

Proof. Assume there exists a constant $c_p > 0$ such that for all elementary processes ϕ, (2.1) holds. Then we may extend (2.1) to all measurable and adapted processes $\phi \in L^p(\Omega; L^2(0,\infty;E))$ that take values in a finite-dimensional subspace of E. For each $N \geq 1$, we will construct a process ϕ as above and such that

$$\Big(\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p\Big)^{1/p} = N \text{ and } \Big(\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|^p\Big)^{1/p} \leq K_p\sqrt{N}.$$

Here $K_p > 0$ is some universal constant. This gives a contradiction.

We modify an example in [5] in such a way that the martingale differences arise as stochastic integrals. We use the notation of Lemmas 2.3 and 2.4. Fix an integer $N \geq 1$. Let $D = \{-1,1\}^N$, and for each $e = (e_n)_{n=1}^N \in D$ define the process $\phi_e : [0,\infty)\times\Omega \to \mathbb{R}$ by

$$\phi_e(t) = \begin{cases} e_n \mathbf{1}_{A_{e,n}} & \text{for } \ t \in (\tau_{n-1},\tau_n],\ n = 1,\dots,N,\\ 0 & \text{for } \ t = 0 \text{ or } t > \tau_N, \end{cases}$$

where $A_{e,1} = \Omega$ and for $2 \leq n \leq N$,

$$A_{e,n} = \{\Delta W_1 = e_1, \dots, \Delta W_{n-1} = e_{n-1}\}.$$

Then each ϕ_e is stochastically integrable with

$$\int_0^\infty \phi_e(t)\,dW(t) = \sum_{n=1}^N \Delta W_n e_n \mathbf{1}_{A_{e,n}}.$$

Define $\phi : [0,\infty)\times\Omega \to l^\infty(D)$ by $\phi = (\phi_e)_{e\in D}$. Then ϕ is stochastically integrable and for almost all $\omega \in \Omega$ and $e \in D$ we have $\Big|\Big(\int_0^\infty \phi(t)\,dW(t)\Big)(\omega)(e)\Big| \leq N$. For almost all $\omega \in \Omega$ and $e = (\Delta W_n(\omega))_{n=1}^N$ we have $\Big|\Big(\int_0^\infty \phi(t)\,dW(t)\Big)(\omega)(e)\Big| = N$. This shows that

$$\Big(\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p_{l^\infty(D)}\Big)^{1/p} = N, \quad \text{for all } p \in [1,\infty).$$

On the other hand, we have

$$\int_0^\infty \phi(t)\,d\tilde{W}(t) = \sum_{n=1}^N \Delta\tilde{W}_n v_n,$$

where for $1 \leq n \leq N$, $v_n = (e_n \mathbf{1}_{A_{e,n}})_{e\in D}$.

For $\omega \in \Omega$ and $e \in D$ let $k(\omega,e)$ be 0 if $\Delta W_1(\omega) \neq e_1$ and let $k(\omega,e)$ be the maximum of all integers $n \leq N$ such that $\Delta W_i(\omega) = e_i$ for all $i \leq n$ if $\Delta W_1(\omega) = e_1$. For almost all $\omega \in \Omega$ and for all $e \in D$, $\Big(\sum_{n=1}^N \Delta\tilde{W}_n v_n\Big)(\omega)(e)$ is equal to

$$\begin{array}{ll} -\Delta\tilde{W}_{k(\omega,e)+1}(\omega,\cdot)\Delta W_{k(\omega,e)+1}(\omega) + \sum_{n=1}^{k(\omega,e)} \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega), & \text{if } k(\omega,e) < N,\\ \sum_{n=1}^N \Delta\tilde{W}_n(\omega,\cdot)\Delta W_n(\omega), & \text{if } k(\omega,e) = N. \end{array}$$

Of course we have for all $k \leq N$,

$$-\tilde{W}_k\Delta W_k + \sum_{n=1}^{k-1}\Delta\tilde{W}_n\Delta W_n = 2\sum_{n=1}^{k-1}\Delta\tilde{W}_n\Delta W_n - \sum_{n=1}^{k}\Delta\tilde{W}_n\Delta W_n.$$

We obtain that for almost all $\omega \in \Omega$,

$$\Big\| \int_0^\infty \phi(t,\omega)\, d\tilde{W}(t) \Big\|_{l^\infty(D)} \leq 3 \sup_{k \leq N} \Big| \sum_{n=1}^k \Delta \tilde{W}_n(\omega,\cdot) \Delta W_n(\omega) \Big|.$$

Since for almost all $\omega \in \Omega$, $(\Delta \tilde{W}_n(\omega,\cdot))_{n=1}^N$ is a sequence of independent centered Gaussian random variables on $\tilde{\Omega}$, we have by the Lévy-Octaviani inequalities for independent symmetric random variables (see [9, Section 1.1]) for almost all $\omega \in \Omega$,

$$\begin{aligned} \tilde{\mathbb{E}} \sup_{k \leq N} \Big| \sum_{n=1}^k \Delta \tilde{W}_n(\omega,\cdot) \Delta W_n(\omega) \Big|^p &\leq 2^p \tilde{\mathbb{E}} \Big| \sum_{n=1}^N \Delta \tilde{W}_n(\omega,\cdot) \Delta W_n(\omega) \Big|^p \\ &= 2^p \tilde{\mathbb{E}} \Big| \sum_{n=1}^N \Delta \tilde{W}_n(\omega,\cdot) \Big|^p = 2^p \tilde{\mathbb{E}} \, |\tilde{W}_{\tau_N}(\omega,\cdot)|^p = 2^p g_p^p \tau_N(\omega)^{p/2}. \end{aligned}$$

Here g_p^p is the p-th moment of a standard Gaussian random variable. We may conclude that

$$\Big(\mathbb{E}\, \tilde{\mathbb{E}} \Big\| \int_0^\infty \phi(t)\, d\tilde{W}(t) \Big\|_{l^\infty(D)}^p \Big)^{1/p} \leq 6 g_p (\mathbb{E}\, \tau_N^{p/2})^{1/p}.$$

Recall that the sequence $(\tau_n - \tau_{n-1})_{n=1}^N$ is identically distributed. For $p = 2$ we obtain

$$\begin{aligned} (\mathbb{E}\, \tau_N^{p/2})^{1/p} = (\mathbb{E}\, \tau_N)^{1/2} &= \Big(\mathbb{E} \sum_{n=1}^N \tau_n - \tau_{n-1} \Big)^{1/2} \\ &= \Big(\sum_{n=1}^N \mathbb{E}\, (\tau_n - \tau_{n-1}) \Big)^{1/2} = \Big(\sum_{n=1}^N \mathbb{E}\, \tau_1 \Big)^{1/2} = \sqrt{N} \sqrt{\mathbb{E}\, \tau_1}. \end{aligned}$$

For $1 \leq p < 2$ we have by Hölder's inequality,

$$(\mathbb{E}\, \tau_N^{p/2})^{1/p} \leq (\mathbb{E}\, \tau_N)^{1/2} = \sqrt{N} \sqrt{\mathbb{E}\, \tau_1}.$$

Finally for $p > 2$, by the triangle inequality in $L^{p/2}(\Omega)$,

$$\begin{aligned} (\mathbb{E}\, \tau_N^{p/2})^{1/p} &= \Big(\mathbb{E} \Big(\sum_{n=1}^N \tau_n - \tau_{n-1} \Big)^{p/2} \Big)^{1/p} \leq \Big(\sum_{n=1}^N (\mathbb{E}\, (\tau_n - \tau_{n-1})^{p/2})^{2/p} \Big)^{1/2} \\ &= \Big(\sum_{n=1}^N (\mathbb{E}\, \tau_1^{p/2})^{2/p} \Big)^{1/2} = \sqrt{N} (\mathbb{E}\, \tau_1^{p/2})^{1/p}. \end{aligned}$$

By Lemma 2.4 this proves that, for all $p \in [1, \infty)$ and some universal constant K_p,

$$\Big(\mathbb{E}\, \tilde{\mathbb{E}} \Big\| \int_0^\infty \phi(t)\, d\tilde{W}(t) \Big\|^p \Big)^{1/p} \leq K_p \sqrt{N}.$$

Since $l^\infty(D)$ can be identified isometrically with a finite-dimensional subspace of c_0, this completes the proof. □

Corollary 2.6. *Let E be a Banach space. If there exists a constant $c_p > 0$ such that for all elementary processes* (2.1) *holds, then E has finite cotype.*

Proof. It follows from the above example that c_0 is not finitely representable in E. Hence the Maurey-Pisier Theorem (see [12]) implies that E has finite cotype. □

Proof of Theorem 2.2. We may assume that the martingale starts at zero (see [1, Remark 1.1]). Let $(r_n)_{n=1}^N$ be a Rademacher sequence on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(d_n)_{n=1}^N$ be an E-valued martingale difference sequence with respect to the filtration $(\sigma(r_1, r_2, \dots, r_n))_{n=0}^N$. We may write $d_n = r_n f_n(r_1, \dots, r_{n-1})$ for $n = 1, \dots, N$, for some $f_n : \{-1, 1\}^{n-1} \to E$. Let $(\tilde{r}_n)_{n=1}^N$ be a Rademacher sequence on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

(1): We will show that there exists a constant $C_p^- > 0$ only depending on E such that

$$\mathbb{E} \Big\| \sum_{n=1}^N d_n \Big\|^p \le (C_p^-)^p \mathbb{E}\, \tilde{\mathbb{E}} \Big\| \sum_{n=1}^N \tilde{r}_n d_n \Big\|^p. \tag{2.3}$$

We use the notation of Lemmas 2.3 and 2.4. Define a process $\phi : [0, \infty) \times \Omega \to E$ by

$$\phi(t) = \begin{cases} f_n(\Delta W_1, \dots, \Delta W_{n-1}) & \text{for } t \in (\tau_{n-1}, \tau_n],\ n = 1, \dots, N, \\ 0 & \text{for } t = 0 \text{ or } t > \tau_N. \end{cases}$$

The process ϕ is stochastically integrable and we have

$$\begin{aligned} \mathbb{E} \Big\| \int_0^\infty \phi(t)\, dW(t) \Big\|^p &= \mathbb{E} \Big\| \sum_{n=1}^N \Delta W_n f_n(\Delta W_1, \dots, \Delta W_{n-1}) \Big\|^p \\ &= \mathbb{E} \Big\| \sum_{n=1}^N r_n f_n(r_1, \dots, r_{n-1}) \Big\|^p = \mathbb{E} \Big\| \sum_{n=1}^N d_n \Big\|^p. \end{aligned}$$

Also, we have

$$\tilde{\mathbb{E}}\, \mathbb{E} \Big\| \int_0^\infty \phi(t)\, d\tilde{W}(t) \Big\|^p = \mathbb{E}\, \tilde{\mathbb{E}} \Big\| \sum_{n=1}^N \Delta \tilde{W}_n f_n(\Delta W_1, \dots, \Delta W_{n-1}) \Big\|^p.$$

By Lemma 2.4, Corollary 2.6 and [10, Proposition 9.14], we have

$$\mathbb{E}\, \tilde{\mathbb{E}} \Big\| \sum_{n=1}^N \Delta \tilde{W}_n x_n \Big\|^p \le K_p \mathbb{E}\, \tilde{\mathbb{E}} \Big\| \sum_{n=1}^N \tilde{r}_n x_n \Big\|^p,$$

where $(x_n)_{n=1}^N$ is a sequence in E and $K_p > 0$ is some constant depending only on E and p. By conditioning (cf. [8, Lemma 3.11]) this result extends to

$$\mathbb{E}\, \tilde{\mathbb{E}} \Big\| \sum_{n=1}^N \Delta \tilde{W}_n X_n \Big\|^p \le K_p \mathbb{E}\, \tilde{\mathbb{E}} \Big\| \sum_{n=1}^N \tilde{r}_n X_n \Big\|^p, \tag{2.4}$$

where $(X_n)_{n=1}^N$ is a sequence of E-valued random variables independent of $(\Delta \tilde{W}_n)_{n=1}^N$ and independent of $(\tilde{r}_n)_{n=1}^N$. By Lemmas 2.3 and 2.4, we may apply (2.4) to the random variables $X_n = f_n(\Delta W_1, \dots, \Delta W_{n-1})$ for $1 \le n \le N$ to

obtain:

$$\begin{aligned}\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde{W}_n f_n(\Delta W_1,\dots,\Delta W_{n-1})\Big\|^p &\le K_p^p\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n f_n(\Delta W_1,\dots,\Delta W_{n-1})\Big\|^p\\ &= K_p^p\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n f_n(r_1,\dots,r_{n-1})\Big\|^p \overset{\text{(i)}}{=} K_p^p\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n r_n f_n(r_1,\dots,r_{n-1})\Big\|^p\\ &= K_p^p\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p.\end{aligned}$$

In (i), we used that $(r_1,\dots,r_N,\tilde{r}_1,\dots,\tilde{r}_N)$ and $(r_1,\dots,r_N,r_1\tilde{r}_1,\dots,r_N\tilde{r}_N)$ are identically distributed. By assumption we have

$$\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p \le c_p^p\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|^p.$$

We may conclude that (2.3) holds with constant c_pK_p.

(2): We will show that there exists a constant $C_p^+ > 0$ only depending on E such that

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p \le (C_p^+)^p\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p. \tag{2.5}$$

Let ϕ be as before. By Lemmas 2.3, 2.4 and [10, Lemma 4.5] and the same arguments as before we have

$$\begin{aligned}\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n d_n\Big\|^p &= \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n f_n(r_1,\dots,r_{n-1})\Big\|^p\\ &= \mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \tilde{r}_n f_n(\Delta W_1,\dots,\Delta W_{n-1})\Big\|^p\\ &\le \frac{1}{(\mathbb{E}\,\tilde{\mathbb{E}}\,|\tilde{W}_1|)^p}\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N \Delta\tilde{W}_n f_n(\Delta W_1,\dots,\Delta W_{n-1})\Big\|^p.\end{aligned}$$

By assumption we have

$$\mathbb{E}\,\tilde{\mathbb{E}}\Big\|\int_0^\infty \phi(t)\,d\tilde{W}(t)\Big\|^p \le c_p^p\mathbb{E}\Big\|\int_0^\infty \phi(t)\,dW(t)\Big\|^p.$$

We may conclude that (2.5) holds with constant $\dfrac{c_p}{\mathbb{E}\,\tilde{\mathbb{E}}\,|\tilde{W}_1|}$. □

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Delft Institute of Applied Mathematics, Technical University of Delft, P.O. Box 5031, 2600 GA Delft, The Netherlands

E-mail address: M.C.Veraar@math.tudelft.nl