POWERS AND ROOTS OF TOEPLITZ OPERATORS

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Abstract. We study the commutativity of two Toeplitz operators whose symbols are quasihomogeneous functions. We give a relationship between this commutativity and the roots (or powers) of the Toeplitz operators. We use this to characterize Toeplitz operators with symbols in $L^\infty(D)$ which commute with Toeplitz operators whose symbols are of the form $e^{ip\theta}r^m$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $dA$ denote normalized Lebesgue area measure. The Bergman space, denoted by $L^2_a$, is the Hilbert space of analytic functions on $\mathbb{D}$ that are square integrable with respect to $dA$. It is well known that $L^2_a$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and $(\sqrt{n + 1} z^n)_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2_a$. Let $P$ be the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L^2_a$. For a function $\phi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator with symbol $\phi$ is the operator $T_\phi$ from $L^2_a$ to $L^2_a$ defined by $T_\phi(f) = P(\phi f)$.

If $k_z(w) = \frac{1}{(1-\bar{z}w)^2} = \sum_{j=0}^{\infty} (1 + j) w^j \bar{z}^j$ is the Bergman reproducing kernel, then

$$T_\phi(f)(z) = P(\phi f)(z) = \int_{\mathbb{D}} \phi(w)f(w)k_z(w) \, dA(w).$$

The question to be studied in this paper is: When do two Toeplitz operators $T_\phi$ and $T_\psi$ commute? In 1964, Brown and Halmos [4] solved this problem for the analogously defined Toeplitz operators on the Hardy space. They showed that $T_\phi T_\psi = T_\psi T_\phi$ for some $\phi$ and $\psi \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$, if and only if either

(a) $\phi$ and $\psi$ are both analytic, or
(b) $\bar{\phi}$ and $\bar{\psi}$ are both analytic, or
(c) one of the two symbols is a linear function of the other.

We recall that a function in $L^\infty(\mathbb{T})$ is said to be analytic if all of its Fourier coefficients with negative indices are equal to 0.

The same question concerning Toeplitz operators on the Bergman space has a much more complicated answer. There are however some results which resemble those of [4]. In fact, Axler and Čučković proved in [2] that the condition that one of (a), (b) or (c) be true is still necessary and sufficient when the two symbols $\phi$ and $\psi$ are bounded harmonic functions on $\mathbb{D}$. Moreover, with Rao [3], they proved...
that if $\phi$ is a bounded analytic function and if $\psi$ is a bounded symbol such that $T_\phi$ and $T_\psi$ commute, then $\psi$ must be analytic too. When we consider arbitrary symbols, things are different. In [5] Ćuĉković and Rao used the Mellin transform to study the commutativity of multiplication of two Toeplitz operators $T_\phi$ and $T_\psi$ on the Bergman space and describe those operators which commute with $T_{e^{ip\theta}r^m}$ for $(m, p) \in \mathbb{N} \times \mathbb{N}$. In this paper we use our results from [7] to interpret and extend the results of [5]. We give some solutions in the case where the Toeplitz operators have symbols which are “quasihomogeneous” functions and show that these solutions are related to “$p^{th}$ roots” and powers of the Toeplitz operators.

As in [7] we say that a bounded symbol $f$ is quasihomogeneous of degree $k$ if it is of the form $e^{ik\theta} \phi$ where $\phi$ is a radial function. In this case we say that the Toeplitz operator $T_f$ is quasihomogeneous of degree $k$.

2. Preliminaries

The Mellin transform of a function $\psi \in L^1([0, 1], rdr)$ is defined by

$$\hat{\psi}(z) = \int_0^1 \psi(r)r^{z-1} dr.$$ 

It is easy to see that $\hat{\psi}$ is a bounded holomorphic function on the half-plane $\Pi = \{z : \Re z > 2\}$.

We denote the Mellin convolution of two functions $\phi$ and $\psi$ by $\phi *_M \psi$ and we define it by the equation

$$(\phi *_M \psi)(r) = \int_r^1 \phi(\frac{r}{t}) \psi(t) \frac{dt}{t}.$$ 

It is clear that the Mellin transform converts Mellin convolution into a pointwise product, i.e., that

$$(\hat{\phi *_M \psi})(r) = \hat{\phi}(r)\hat{\psi}(r).$$

We shall often use the following classical theorem (see [8, p. 102]).

**Theorem 1.** Suppose that $f$ is a bounded, holomorphic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $d_1, d_2, \cdots$, where

i) $\inf\{|d_n|\} > 0$ and

ii) $\sum_{n \geq 1} \Re(\frac{1}{d_n}) = \infty$.

Then $f$ vanishes identically on $\{z : \Re z > 0\}$.

**Remark 2.** We shall often apply this theorem to show that if $\psi \in L^1([0, 1], rdr)$ and if there exist $n_0 \in \mathbb{Z}_+, p \in \mathbb{N}$ such that

$$\hat{\psi}(n_0 + pk) = 0$$

for all $k \in \mathbb{N}$, then $\hat{\psi}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $\psi = 0$.

3. Powers of Toeplitz operators

The following lemma determines the values of powers of a bounded quasihomogeneous Toeplitz operator evaluated at any element of the orthonormal basis of $L^2_a$. 

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Lemma 3. Let $n \in \mathbb{N}, s \in \mathbb{Z}_+$ and let $\psi$ be a bounded radial function on $\mathbb{D}$. Then, for all $k \in \mathbb{N}$ we have
\[
\left( T_{e^{i\theta}} \psi \right)^n (\xi^k)(z) = \left[ \prod_{j=0}^{n-1} 2(k + js + s + 1)\hat{\psi}(2k + 2js + s + 2) \right] z^{k+ns} = \frac{\prod_{j=0}^{n-1} \hat{\psi}(2k + 2js + s + 2)}{\prod_{j=0}^{n-1} \hat{\mathbb{B}}(2k + 2js + s + 2)} z^{k+ns},
\]
where $\mathbb{B}$ denotes the constant function with value one.

Proof. The lemma is a consequence of the following direct calculation. We write
\[
T_{e^{i\theta}} \psi \xi^k (z) = \int_0^1 \int_0^{2\pi} \psi(r) r^k \sum_{j=0}^{\infty} (j + 1) e^{i(k+s-j)\theta} r^j z^j \frac{1}{\pi} rdrd\theta
\]
and interchange the integral over $[0, 2\pi]$ and the sum to see that
\[
T_{e^{i\theta}} \psi \xi^k (z) = 2(k + s + 1) \hat{\psi}(2k + s + 2) z^{k+s} = \frac{\hat{\psi}(2k + s + 2)}{\hat{\mathbb{B}}(2k + 2s + 2)} z^{k+s}.
\]
The lemma is proved by applying $T_{e^{i\theta}} \psi$ to $\xi^k$ $n$ times. $\square$

We have the following decomposition of $L^2(\mathbb{D}, dA)$ as
\[
L^2(\mathbb{D}, dA) = \bigoplus_{k \in \mathbb{Z}} e^{ik\theta} \mathcal{R}_k,
\]
where $\mathcal{R}_k$ is the space of functions on $[0, 1]$ that are square integrable with respect to the measure $rdr$. Thus every function $f \in L^2(\mathbb{D}, dA)$ has the decomposition
\[
f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r), \quad f_k \in \mathcal{R}_k.
\]
Moreover, if $f \in L^\infty(\mathbb{D}, dA) \subset L^2(\mathbb{D}, dA)$, then for each $r \in [0, 1)$,
\[
|f_k(r)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} d\theta \right| \leq \sup_{z \in \mathbb{D}} |f(z)|, \quad \forall k \in \mathbb{Z},
\]
and so the functions $f_k$ are bounded in the disk.

In [7] we proved the following results, which we will use in the proof of our main theorem.

Proposition 4. Let $\phi$ be a nonzero bounded radial function, let $p$ be a positive integer and let $f(re^{i\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$. Then
a) $T_f$ commutes with $T_{e^{ip\theta}} \phi$ if and only if $T_{e^{ik\phi}} f_k$ commutes with $T_{e^{ip\theta}} \phi$ for all $k \in \mathbb{Z}$.
b) If there exists $k \in \mathbb{Z}_-$ and a bounded radial function $f_k$ such that $T_{e^{ip\theta}} T_{e^{ik\phi}} f_k = T_{e^{ik\phi}} T_{e^{ip\theta}} \phi$, then $f_k$ must be equal to zero.
c) If there exists $k \in \mathbb{Z}_+$ and a bounded radial function $f_k$ such that $T_{e^{ip\theta}} T_{e^{ik\phi}} f_k = T_{e^{ik\phi}} T_{e^{ip\theta}} \phi$, then $f_k$ is unique up to a constant factor. In particular $f_0$ is a constant.
Let Lemma 6. Notation.

\[ S \equiv \pi \]

Then if \( F \) commutes with \( T \), each \( f_k \) is uniquely determined up to multiplication by a constant and is equal to 0 for \( k < 0 \).

Next we present two technical but easy results which permit us to prove Propositions 7 and 9, the principal results of this section.

Remark 5. Let \((a_l)_{l \in \mathbb{N}}\) and \((b_l)_{l \in \mathbb{N}}\) be two nonvanishing sequences and \( p \) and \( s \) two positive integers such that

\begin{equation}
\sum z^n = b_0 + b_1z + \cdots + b_s z^s \quad \text{for all } z \in \mathbb{N}.
\end{equation}

Then if

\[ A_k = \prod_{j=0}^{s-1} a_{k+jp} \quad \text{and} \quad B_k = \prod_{j=0}^{p-1} b_{k+j}, \]

we have

\[ A_k B_{k+p} = A_{k+p} B_k \quad \text{for all } k \in \mathbb{N}. \]

(Just multiply the \( p \) equations obtained by taking \( l = k, k+s, \ldots, k+(p-1)s \) in \((2)\) together to see that, if \((2)\) is true, then

\[ \frac{B_{k+p}}{B_k} = \frac{a_{k+ps}}{a_k} = \frac{A_{k+p}}{A_k} \quad \text{for all } k \in \mathbb{N}. \]

Notation. Let \( S \) and \( T \) be two functions (resp. two operators). We will say that \( S \equiv T \) if there exists a constant \( c \neq 0 \) such that \( S = cT \).

Lemma 6. Let \( F \) and \( G \) be two nonzero bounded holomorphic functions on the half plane \( \Pi = \{ z : \Re z > 2 \} \). If there exists \( p \in \mathbb{N} \) such that

\begin{equation}
F(z)G(z+p) = F(z+p)G(z) \quad \text{for all } z \in \Pi,
\end{equation}

then \( F \equiv G \).

Proof. Suppose that \((3)\) is true. Then, if (as above) we multiply the \( k \) equations obtained by taking \( z_n = z + np \) for \( n = 0, \ldots, k-1 \), we have

\begin{equation}
F(z)G(z + kp) = F(z + kp)G(z) \quad \text{for all } k \in \mathbb{N}.
\end{equation}

Now, let \( z_0 \in \Pi \) such that \( G(z_0) \neq 0 \) and let \( E = \{ k \in \mathbb{N} : G(z_0 + kp) = 0 \} \). If \( \sum_{k \in E} \Re \left( \frac{1}{z_0 + kp} \right) = \infty \), then Theorem 1 implies that \( G = 0 \). This contradicts the hypothesis of the lemma. Thus \( \sum_{k \in E^c} \Re \left( \frac{1}{z_0 + kp} \right) = \infty \), where \( E^c \) is the complement in \( \mathbb{N} \) of the set \( E \).

Now, equation \((4)\) implies that

\[ \frac{F(z_0 + kp)}{G(z_0 + kp)} = \frac{F(z_0)}{G(z_0)} \quad \text{for all } k \in E^c. \]

So, applying Theorem 1 to the function \( F - cG \), where \( c = \frac{F(z_0)}{G(z_0)} \), completes the proof.

Let \( p \) and \( s \) be two positive integers and \( \psi \) a bounded radial function.

If \((T_{e^{i\theta} \psi})^p\) is a Toeplitz operator, then it is the unique quasihomogeneous Toeplitz operator of degree \( ps \) (see Proposition 3 and Proposition 4 of [7]) which commutes with \( T_{e^{i\theta} \psi \psi} \). It is natural to ask whether all nonzero Toeplitz operators which are of quasihomogeneous of degree a multiple of \( s \) and which commute with \( T_{e^{i\theta} \psi} \) are of this form.
Proposition 7. Let $p$ and $s$ be two positive integers and $\phi$ and $\psi$ be two nonzero bounded radial functions such that

\begin{equation}
T_{e^{ip\theta} \phi} T_{e^{is\theta} \psi} = T_{e^{ip\theta} \phi} T_{e^{is\theta} \psi}.
\end{equation}

Then

\begin{equation}
(T_{e^{ip\theta} \phi})^s = (T_{e^{is\theta} \psi})^p.
\end{equation}

Proof. For all $k \in \mathbb{N}$, let

\begin{align*}
a_k &= \frac{\hat{\phi}(2k + p + 2)}{\| (2k + 2p + 2) } \quad \text{and} \quad b_k = \frac{\hat{\psi}(2k + s + 2)}{\| (2k + 2s + 2) },
\end{align*}

so that

\begin{align*}
T_{e^{ip\theta} \phi}(\xi^k)(z) &= a_k z^{k+p} \quad \text{and} \quad T_{e^{is\theta} \psi}(\xi^k)(z) = b_k z^{k+s}.
\end{align*}

Then equation (6) shows that $a_{k+p} b_k = b_{k+p} a_k$ for all $k \in \mathbb{Z}_+$, and so Remark 8 implies that

\begin{equation}
\prod_{j=0}^{s-1} a_{k+jp} \prod_{j=0}^{p-1} b_{k+jp} = \prod_{j=0}^{s-1} a_{k+jp} \prod_{j=0}^{p-1} b_{k+jp}.
\end{equation}

Let $F$ and $G$ be the two bounded holomorphic functions defined for all $z \in \Omega$ by

\begin{align*}
F(z) &= \prod_{j=0}^{p-1} \hat{\phi}(z + 2jp + 2s) \prod_{j=0}^{s-1} \hat{\phi}(z + 2jp + p),
\end{align*}

and

\begin{align*}
G(z) &= \prod_{j=0}^{p-1} \hat{\psi}(z + 2jp + 2p) \prod_{j=0}^{s-1} \hat{\psi}(z + 2js + s).
\end{align*}

Then equation (7) is equivalent to

\begin{align*}
F(2k + 2p) G(2k + 2p + 2) &= F(2k + 2p + 2) G(2k + 2) \quad \text{for all } k \in \mathbb{Z}_+.
\end{align*}

Now, applying Theorem 11 in the form of Remark 2 implies that

\begin{align*}
F(z) G(z + 2) &= F(z + 2p) G(z) \quad \text{for all } z \in \Omega.
\end{align*}

Finally, using Lemma 5 we obtain that

\begin{align*}
\prod_{j=0}^{s-1} \hat{\phi}(z + 2jp + p) \prod_{j=0}^{p-1} \hat{\psi}(z + 2js + s) &\quad \text{for all } z \in \Omega,
\end{align*}

and Lemma 3 completes the proof. \hfill \Box

Remark 8. i) We will assume that $(T_{e^{i\theta} \phi})^0 = I$, where $I$ is the identity operator of $L^2_\alpha$ onto $L^2_\alpha$.

ii) If $p$ and $s$ are both negative integers and if $T_{e^{ip\theta} \phi} T_{e^{is\theta} \psi} = T_{e^{ip\theta} \phi} T_{e^{is\theta} \psi}$, then by considering the adjoint operators we obtain

\begin{align*}
T_{e^{-i\theta \phi}} T_{e^{-i\theta \phi}} &= T_{e^{-i\theta \phi}} T_{e^{-i\theta \phi}},
\end{align*}

and so Proposition 7 implies that $(T_{e^{ip\theta} \phi})^{-s} = (T_{e^{is\theta} \psi})^{-p}$.

Now, by considering once again the adjoint operators we see that

\begin{align*}
(T_{e^{ip\theta} \phi})^{-s} &\equiv (T_{e^{is\theta} \psi})^{-p}.
\end{align*}
Proposition 9. Let \( \phi \) and \( \psi \) be two nonzero bounded radial functions and \( n, p \) and \( s \) be positive integers. Then
\[
(T^{e^{ip\theta}}\phi)^n = (T^{e^{ip\theta}}\psi)^{np} \implies T^{e^{ip\theta}}\phi = (T^{e^{ip\theta}}\psi)^{p}.
\]

Proof. For all \( k \in \mathbb{Z}_+ \), let
\[
a_k = 2(k + ps + 1)\hat{\phi}(2k + ps + 2) \quad \text{and} \quad b_k = 2(k + s + 1)\hat{\psi}(2k + s + 2),
\]
so that
\[
(T^{e^{ip\theta}}\phi)^n = (T^{e^{ip\theta}}\psi)^{np} \iff \prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{np-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+
\]
and
\[
T^{e^{ip\theta}}\phi = (T^{e^{ip\theta}}\psi)^{p} \iff a_k = \prod_{j=0}^{p-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+.
\]

Suppose that
\[
\prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{np-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+.
\]

We will prove that
\[
\prod_{j=0}^{p-1} a_{knps} = \prod_{j=0}^{np-s} b_{knps+js} \quad \text{for all } k \in \mathbb{Z}_+.
\]

We prove (9) by induction on \( k \). If we take \( k = 0 \) in equation (8), then we obtain
\[
\prod_{j=0}^{n-1} a_{jps} = \prod_{j=0}^{np-1} b_{jps} = \prod_{j=0}^{p-1} b_{jps} \prod_{j=0}^{np-1} b_{jps} = \prod_{j=0}^{p-1} b_{jps} \prod_{j=0}^{np-1} b_{jps+js}.
\]

Otherwise
\[
\prod_{j=0}^{n-1} a_{jps} = a_0 \prod_{j=1}^{n-1} a_{jps} = a_0 \prod_{j=0}^{n-1} a_{ps+jps},
\]

But equation (8) implies that
\[
\prod_{j=0}^{n-1} a_{ps+jps} = \prod_{j=0}^{np-1} b_{ps+jps},
\]

Thus
\[
a_0 = \prod_{j=0}^{p-1} b_{jps}.
\]
Now, assume (9) is true for knps. We show it is true for \((k + 1)\text{np}\). We set \(k\) equal to \(\text{np}\) in the left-hand side of (8) and obtain
\[
\prod_{j=0}^{n-1} a_{\text{np} + j\text{ps}} = a_{\text{np}} \prod_{j=0}^{n-2} a_{\text{np} + j\text{ps}}.
\]
Then
\[
a_{(k+1)\text{np}} \prod_{j=0}^{n-1} a_{\text{np} + j\text{ps}} = a_{\text{np}} \prod_{j=0}^{n-1} a_{\text{np} + j\text{ps}}.
\]
But
\[
\prod_{j=0}^{n-1} a_{\text{np} + j\text{ps}} = \prod_{j=0}^{np-1} b_{\text{np} + j\text{ps}}
\]
and
\[
\prod_{j=0}^{np-1} b_{\text{np} + j\text{ps}} = \prod_{j=0}^{p-1} b_{\text{np} + j\text{ps}} \prod_{j=0}^{np-1} b_{(k+1)\text{np} + j\text{ps}}.
\]
Thus (9) is proved for \((\text{np})_{k \in \mathbb{Z}_+}\). Hence, for all \(k \in \mathbb{Z}_+\), we have
\[
\hat{\psi}(2\text{np} + ps + 2) \prod_{j=0}^{p-1} \hat{\psi}(2\text{np} + 2js + 2s + 2) = \hat{\psi}(2\text{np} + p + 2) \prod_{j=0}^{p-1} \hat{\psi}(2\text{np} + 1 + 2)
\]
and, using equation (11) and Remark 2, we complete the proof. \(\square\)

Remark 10. In [7] (Proposition 6) we prove that if \(p > 0\) and \(\phi\) is a nonzero bounded radial function and if there exists a bounded radial function \(\psi\) such that \(T_\psi\) commutes with \(T_{e^{ip\theta}}\), then \(\psi\) must be a constant. Here is another proof of this proposition. In fact, using Proposition 7, we have \((T_\psi)^p \equiv I\), so Proposition 9 implies that \(T_\psi \equiv I\), and so, that \(\psi \equiv \hat{1}\) since \(I\) is the Toeplitz operator of symbol 1.

4. Main result

Let \(p\) be a positive integer. We start this section with the definition of the \(T\)-th root of a quasihomogeneous Toeplitz operator of degree \(p\) or \(-p\). This new notion plays an important role in the remainder of the paper.

Definition 11. Let \(\phi\) be a nonzero bounded radial function and \(p\) be a positive integer. We say that the Toeplitz operator \(T_{e^{ip\theta}}\) has a \(T\)-th root \(T_{e^{ip\theta}}\) if and only if there exists a nonzero bounded radial function \(\psi\) such that
\[
T_{e^{ip\theta}} = (T_{e^{ip\theta}})^p.
\]

Remark 12. i) The \(T\)-th root of a quasihomogeneous Toeplitz operator is unique. In fact, suppose that \(T_{e^{ip\theta}}\) has two \(T\)-th roots \(T_{e^{ip\theta}}\) and \(T_{e^{ip\theta}}\). Then \((T_{e^{ip\theta}})^p = (T_{e^{ip\theta}})^p\). Then, by Proposition 9 we have that \(T_{e^{ip\theta}} = T_{e^{ip\theta}}\), which implies that \(\psi = \hat{\psi}\).

ii) If the quasihomogeneous degree is negative we have an analogous definition of the \(T\)-th root. Let \(p\) be a positive integer and \(\phi\) be a bounded radial function. Then, we say that \(T_{e^{-ip\theta}}\) has a \(T\)-th root if there exists a bounded radial function \(\psi\) such that \(T_{e^{-ip\theta}} = (T_{e^{-ip\theta}})^p\). It is easy to see,
Thus i) is a direct consequence of assertion b) of Proposition 4.

Examples. i) \( T_{e^{i\theta}}(\frac{\phi}{1 + e^{i\theta}}) \) is the T-2\(^{th}\) root of \( T_{e^{2\theta}} \).

ii) \( T_{e^{i\theta}}(\frac{\phi}{1 + e^{2i\theta}}) \) is the T-2\(^{th}\) root of \( T_{e^{2\theta}} + 10 \).

Now, if \( T_{e^{i\theta}} \) is the T-p\(^{th}\) root of \( T_{e^{ip\theta}} \) and if \( (T_{e^{i\theta}})^k \) (for \( k \in \mathbb{N} \)) is a Toeplitz operator, then \( (T_{e^{i\theta}})^k \) is the unique nonzero quasihomogeneous Toeplitz operator of degree \( k \) which can commute with \( T_{e^{ip\theta}} \). What we prove below is that if \( T_{e^{ip\theta}} \) has a T-p\(^{th}\) root \( T_{e^{ip\theta}} \), then the only nonzero quasihomogeneous Toeplitz operator of degree \( s \) which commutes with \( T_{e^{ip\theta}} \) is an \( s\)\(^{th}\) power of \( T_{e^{ip\theta}} \), extending the result (Propositions 7 and 9) of section 3 in this case.

**Theorem 13.** Let \( \phi \) be a nonzero bounded radial function and let \( p \) be a positive integer. Assume that \( T_{e^{ip\theta}} \) has a T-p\(^{th}\) root \( T_{e^{ip\theta}} \). Suppose that

\[
f(\rho e^{i\theta}) = \sum_{k=\infty}^{\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)
\]

is such that

\[
(10) \quad T_f T_{e^{i\theta}} = T_{e^{i\theta}} T_f.
\]

Then

i) \( f_k = 0 \) for \( k < 0 \).

ii) If \( k \geq 0 \) and \( (T_{e^{i\theta}})^k \) is a Toeplitz operator, then either \( T_{e^{ik\theta}} f_k = (T_{e^{i\theta}})^k \) or \( f_k = 0 \).

iii) If \( k \geq 0 \) and \( (T_{e^{i\theta}})^k \) is not a Toeplitz operator, then \( f_k = 0 \).

**Proof.** Assertion a) of Proposition 4 implies that if equation \( 10 \) is true, then

\[
T_{e^{ik\theta}} f_k T_{e^{ip\theta}} = T_{e^{ip\theta}} T_{e^{ik\theta}} f_k, \quad \text{for all } k \in \mathbb{Z}.
\]

Thus i) is a direct consequence of assertion b) of Proposition 4.

Now, to prove ii), let \( k \) be a positive integer such that \( (T_{e^{i\theta}})^k \) is a Toeplitz operator. Then \( (T_{e^{i\theta}})^k \) is a quasihomogeneous Toeplitz operator of degree \( k \) which commutes with \( T_{e^{ip\theta}} \). So, if \( f_k \) is not identically equal to zero, then \( f_k \) is a bounded nonzero radial function such that \( T_{e^{ik\theta}} f_k \) commutes with \( T_{e^{ip\theta}} \). Thus, assertion c) of Proposition 4 implies that \( T_{e^{ik\theta}} f_k = (T_{e^{i\theta}})^k \).

Finally, let \( k \) be a positive integer such that \( (T_{e^{i\theta}})^k \) is not a Toeplitz operator and suppose that there exists a nonzero bounded radial function \( f_k \) such that \( T_{e^{ik\theta}} f_k \) commutes with \( T_{e^{ip\theta}} \). Then Proposition 4 implies that

\[
(T_{e^{ik\theta}} f_k)^p = (T_{e^{i\theta}})^k.
\]

Thus \( (T_{e^{ik\theta}} f_k)^p = (T_{e^{i\theta}})^{kp} \) and Proposition 4 implies that \( T_{e^{ik\theta}} f_k = (T_{e^{i\theta}})^k \), which contradicts our hypothesis. This proves iii).

Before starting with corollaries, we state an interesting theorem which follows from [5] and give an idea of its proof. In fact we will apply this theorem to see that if \( p \) is any positive integer and \( m \) is any nonnegative integer, then the Toeplitz operator \( T_{e^{ip\theta} r^m} \) always has a T-p\(^{th}\) root.
Theorem 14. Let $p \geq 1$ and $m \geq 0$ be two integers. For all integers $s$, such that $1 \leq s < p$, there exists a unique bounded radial function $\psi$ such that

$$T_{e^{is\theta}} T_{e^{ip\theta}} = T_{e^{is\theta}} T_{e^{ip\theta}}.$$  \hfill (11)

Proof. (This is a slight variation of the proof found in [5].) If $m \geq 0, p \geq 1$ and $1 \leq s < p$, we define the radial functions $f$ and $g$ by

$$f(r) = 2pr^{2s}(1 - r^{2p})^{-\frac{k}{2}}$$ and $$g(r) = 2pr^{m+p}(1 - r^{2p})^{-\frac{k}{2}}.$$  

Let $\psi$ be the radial function defined by

$$r^s\psi = f * M g.$$  

Čučkovic and Rao prove, using a long rather technical calculation, that $\psi$ is bounded. Here, we will show that $\psi$ satisfies (11). To do this, we need only verify that for $k \in \mathbb{Z}$,

$$\frac{2k + 2p + 2}{2k + m + p + 2} r^s\psi(2k + 2 + 2) = \frac{2k + 2s + 2}{2k + m + p + 2s + 2} r^s\psi(2k + 2).$$

By (11), we have $r^s\psi(2k + 2) = \hat{f}(2k + 2)(g(2k + 2)$. A simple substitution $t = r^{2p}$ shows that

$$\hat{f}(2k + 2) = B\left(\frac{2k + 2s + 2}{2p}, 1 - \frac{s}{2p}\right)$$ and $$\hat{g}(2k + 2) = B\left(\frac{2k + m + p + 2}{2p}, \frac{s}{2p}\right),$$

where $B$ denotes the beta function. Using the well-known identities $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}$ and $\Gamma(1 + z) = z\Gamma(z)$, where $\Gamma$ is the gamma function, it is easy to see that

$$\frac{(2k + 2s + 2)(2k + m + p + 2)}{(2k + 2p + 2)(2k + m + p + 2s + 2)} = \frac{\Gamma(z + 2p)}{\Gamma(z + 2p + 2p)} r^s\psi(2k + 2),$$

which finishes the proof.  

Remark 15. i) It is trivial that $T_{e^{is\theta}} T_{e^{ip\theta}}$ commutes with itself. So, if $p = s$, assertion c) of Proposition 4 implies that $\psi \equiv r^m$.

ii) We wish to highlight the following case. If $m = (2n + 1)p$ for $n \in \mathbb{N}$, then the function $\psi$ exists for all $s \in \mathbb{N}$. In fact, if we substitute $m = (2n + 1)p$ in (12) and use Theorem 1, we obtain for all $s \in \Pi$,

$$\frac{r^s\psi(z + 2p)}{r^s\psi(z)} = F(z + 2p) = \frac{\Gamma(z + 2p)}{\Gamma(z + 2p + 2p)} F(z),$$

where $F(z) = \prod_{j=0}^{n-1} \frac{z + 2jp + 2}{z + 2jp + 2s}$.

Now, using the identity $\Gamma(1 + z) = z\Gamma(z)$ repeatedly, we have

$$F(z) = 2p \prod_{j=0}^{n-1} \frac{z + 2jp + 2p}{z + 2jp + 2s},$$

which is a proper fraction in $z$ and can be written as

$$F(z) = \sum_{j=0}^{n} \frac{a_j}{z + 2jp + 2s}.$$  \hfill (13)

Since $\frac{1}{z + 2jp + 2s} = r^{2jp + 2s}(z)$, it follows by Lemma 4 that

$$r^s\psi(z) \equiv \sum_{j=0}^{n} a_j r^{2jp + 2s}(z),$$
where the $a_j$ are defined by (13), and so Theorem 1 implies that
\[
\psi(r) = \sum_{j=0}^{n} a_j r^{2jp+s}.
\]

Next, we give some easy but interesting consequences of Theorem 14.

**Corollary 16.** For all integers $m \geq 0$, $p \geq 1$, and $s \geq 1$ there exists a bounded radial function $\psi$ such that
\[
(T_{e^{ip\theta}})^p \equiv T_{e^{ip\theta}r^m}.
\]

**Proof.** Let $m \geq 0$, $p \geq 1$, and $s \geq 1$ be integers. Theorem 14 implies that there exists a bounded radial function $\psi$ such that
\[
T_{e^{ip\theta}}T_{e^{ip\theta}r^m} = T_{e^{ip\theta}r^m}T_{e^{ip\theta}r^m}.
\]
Using Proposition 7 we have $(T_{e^{ip\theta}})^p \equiv (T_{e^{ip\theta}r^m})^s$ and so, an application of Proposition 9 finishes the proof. □

In [4], Brown and Halmos studied multiplicativity of Toeplitz operators on the Hardy space and showed that the product of two Toeplitz operators $T_f$ and $T_g$ is equal to a third Toeplitz operator $T_h$ for some $f, g$ and $h$ in $L^\infty(\mathbb{T})$ if and only if $f$ is conjugate analytic or $g$ is analytic, that is, hardly ever. The question of when the product of two Toeplitz operators on the Bergman space is equal to a third is much more complicated and still open. Most work on this question shows that it is not often true that the product of two Toeplitz operators is a Toeplitz operator (see [1] and [6]). But, below, we show that, for certain nontrivial Toeplitz operators $T_{e^{ik\theta}}$, not only is $(T_{e^{ik\theta}})^2$ equal to a Toeplitz operator, but there exists a positive integer $k$ such that $(T_{e^{ik\theta}})^i$ is a Toeplitz operator for all positive integers $i \leq k$.

**Corollary 17.** Let $m \geq 0$ and $p \geq 1$ be two integers. If $T_{e^{ip\theta}r^m}$ has a $T$-th root $T_{e^{ip\theta}}$ then, for all integers $k$ with $1 \leq k \leq p$, the product $(T_{e^{ik\theta}})^k$ is a Toeplitz operator.

**Proof.** Let $k$ be an integer such that $1 \leq k \leq p$. By Theorem 14 we know that there exists a bounded radial function $\phi$ such that $T_{e^{ik\theta}}$ commutes with $T_{e^{ip\theta}r^m}$. So, Proposition 6 implies that
\[
(T_{e^{ik\theta}})^p \equiv (T_{e^{ip\theta}r^m})^k.
\]
Thus $(T_{e^{ik\theta}})^p = (T_{e^{ip\theta}r^m})^{kp}$ since $T_{e^{ip\theta}r^m}$ is the $T$-th root of $T_{e^{ip\theta}r^m}$, and so Proposition 9 finishes the proof. □

It is easily seen that if $f$ is a bounded analytic function on $\mathbb{D}$, then $T_f$ is just a multiplication operator. Thus for any integer $k \geq 1$, it is clear that $(T_f)^k$ is a Toeplitz operator of symbol $f^k$. By taking adjoints, we can see that the powers of a Toeplitz operator with conjugate analytic symbol are also Toeplitz operators. These are the trivial cases. The next corollary says there are nontrivial symbols $f$ such that $(T_f)^k$ is always a Toeplitz operator for all integers $k \geq 1$.

**Corollary 18.** There exist bounded radial functions $\psi$ such that for all integers $k \geq 1$ the product $(T_{e^{ik\theta}})^k$ is still a Toeplitz operator.
Proof. Let \( n \geq 0 \) and \( p \geq 1 \) be two integers. By Theorem 14 we know that the Toeplitz operator \( T_{e^{ip\theta r}} (2n+1)_p \) has a \( T \)-\( p \)th root \( T_{e^{i\theta \psi}} \), where \( \psi \) is a bounded radial function. Moreover, the assertion ii) of Remark 15 tells us that, for all integers \( k \geq 1 \), there exists a bounded radial function \( \psi_k \) such that \( T_{e^{ik\theta \psi_k}} \) commutes with \( T_{e^{ip\theta r}} (2n+1)_p \). Thus Proposition 7 implies that \( (T_{e^{ik\theta \psi_k}})^p \equiv (T_{e^{i\theta \psi}})^{kp} \) since \( T_{e^{ip\theta r}} (2n+1)_p = (T_{e^{i\theta \psi}})^p \) and, again, Proposition 9 finishes the proof. \( \square \)

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References


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