PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 5, May 2007, Pages 1465–1475
S 0002-9939(06)08626-6
Article electronically published on November 29, 2006

POWERS AND ROOTS OF TOEPLITZ OPERATORS

ISSAM LOUHICHI

(Communicated by Joseph A. Ball)

Abstract. We study the commutativity of two Toeplitz operators whose symbols are quasihomogeneous functions. We give a relationship between this commutativity and the roots (or powers) of the Toeplitz operators. We use this to characterize Toeplitz operators with symbols in $L^\infty(\mathbb{D})$ which commute with Toeplitz operators whose symbols are of the form $e^{ip\theta}r^m$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let dA denote normalized Lebesgue area measure. The Bergman space, denoted by L^2_a, is the Hilbert space of analytic functions on $\mathbb{D}$ that are square integrable with respect to dA. It is well known that L^2_a is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and $(\sqrt{n+1}z^n)_{n\in\mathbb{N}}$ is an orthonormal basis of L^2_a. Let P be the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto L^2_a. For a function $\phi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator with symbol ϕ is the operator T_ϕ from L^2_a to L^2_a defined by $T_\phi(f) = P(\phi f)$.

If $k_z(w) = \frac{1}{(1-\bar{z}w)^2} = \sum_{j=0}^\infty (1+j)w^j\bar{z}^j$ is the Bergman reproducing kernel, then

$$T_\phi(f)(z) = P(\phi f)(z) = \int_{\mathbb{D}} \phi(w)f(w)\overline{k_z(w)}\, dA(w).$$

The question to be studied in this paper is: When do two Toeplitz operators T_ϕ and T_ψ commute? In 1964, Brown and Halmos [4] solved this problem for the analogously defined Toeplitz operators on the Hardy space. They showed that $T_\phi T_\psi = T_\psi T_\phi$ for some ϕ and $\psi \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$, if and only if either

(a) ϕ and ψ are both analytic, or
(b) $\bar{\phi}$ and $\bar{\psi}$ are both analytic, or
(c) one of the two symbols is a linear function of the other.

We recall that a function in $L^\infty(\mathbb{T})$ is said to be analytic if all of its Fourier coefficients with negative indices are equal to 0.

The same question concerning Toeplitz operators on the Bergman space has a much more complicated answer. There are however some results which resemble those of [4]. In fact, Axler and Čučković proved in [2] that the condition that one of (a), (b) or (c) be true is still necessary and sufficient when the two symbols ϕ and ψ are bounded harmonic functions on $\mathbb{D}$. Moreover, with Rao [3], they proved

Received by the editors March 26, 2005 and, in revised form, December 20, 2005.
2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 47L80.
Key words and phrases. Toeplitz operators, Bergman space, Mellin transform.

that if ϕ is a bounded analytic function and if ψ is a bounded symbol such that T_ϕ and T_ψ commute, then ψ must be analytic too. When we consider arbitrary symbols, things are different. In [5] Čučković and Rao used the Mellin transform to study the commutativity of multiplication of two Toeplitz operators T_ϕ and T_ψ on the Bergman space and describe those operators which commute with $T_{e^{ip\theta}r^m}$ for $(m,p) \in \mathbb{N}\times\mathbb{N}$. In this paper we use our results from [7] to interpret and extend the results of [5]. We give some solutions in the case where the Toeplitz operators have symbols which are "quasihomogeneous" functions and show that these solutions are related to "p^{th} roots" and powers of the Toeplitz operators.

As in [7] we say that a bounded symbol f is quasihomogeneous of degree k if it is of the form $e^{ik\theta}\phi$ where ϕ is a radial function. In this case we say that the Toeplitz operator T_f is quasihomogeneous of degree k.

2. Preliminaries

The Mellin transform of a function $\psi \in L^1([0,1], rdr)$ is defined by

$$\widehat{\psi}(z) = \int_0^1 \psi(r) r^{z-1}\, dr.$$

It is easy to see that $\widehat{\psi}$ is a bounded holomorphic function on the half-plane $\Pi = \{z : \Re z > 2\}$.

We denote the Mellin convolution of two functions ϕ and ψ by $\phi *_M \psi$ and we define it by the equation

$$(\phi *_M \psi)(r) = \int_r^1 \phi(\frac{r}{t})\psi(t)\frac{dt}{t}.$$

It is clear that the Mellin transform converts Mellin convolution into a pointwise product, i.e., that

$$(\widehat{\phi *_M \psi})(r) = \widehat{\phi}(r)\widehat{\psi}(r). \tag{1}$$

We shall often use the following classical theorem (see [8, p. 102]).

Theorem 1. *Suppose that f is a bounded, holomorphic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $d_1, d_2, \cdots$, where*

i) $\inf\{|d_n|\} > 0$ *and*
ii) $\sum_{n\geq 1} \Re(\frac{1}{d_n}) = \infty$.

Then f vanishes identically on $\{z : \Re z > 0\}$.

Remark 2. We shall often apply this theorem to show that if $\psi \in L^1([0,1], rdr)$ and if there exist $n_0 \in \mathbb{Z}_+, p \in \mathbb{N}$ such that

$$\widehat{\psi}(n_0 + pk) = 0 \text{ for all } k \in \mathbb{N},$$

then $\widehat{\psi}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $\psi = 0$.

3. Powers of Toeplitz operators

The following lemma determines the values of powers of a bounded quasihomogeneous Toeplitz operator evaluated at any element of the orthonormal basis of L^2_a.

Lemma 3. *Let $n \in \mathbb{N}, s \in \mathbb{Z}_+$ and let ψ be a bounded radial function on $\mathbb{D}$. Then, for all $k \in \mathbb{N}$ we have*

$$\begin{aligned}\left(T_{e^{is\theta}\psi}\right)^n(\xi^k)(z) &= \Big[\prod_{j=0}^{n-1} 2(k+js+s+1)\widehat{\psi}\big(2k+2js+s+2\big)\Big] z^{k+ns} \\ &= \frac{\prod_{j=0}^{n-1}\widehat{\psi}\big(2k+2js+s+2\big)}{\prod_{j=0}^{n-1}\widehat{\mathbb{1}}(2k+2js+2s+2)} z^{k+ns},\end{aligned}$$

where $\mathbb{1}$ denotes the constant function with value one.

Proof. The lemma is a consequence of the following direct calculation. We write

$$T_{e^{is\theta}\psi}(\xi^k)(z) = \int_0^1\int_0^{2\pi} \psi(r)r^k \sum_{j=0}^{\infty}(j+1)e^{i(k+s-j)\theta}r^j z^j \frac{1}{\pi}\, rdrd\theta$$

and interchange the integral over $[0, 2\pi]$ and the sum to see that

$$\begin{aligned}T_{e^{is\theta}\psi}(\xi^k)(z) &= 2(k+s+1)\widehat{\psi}(2k+s+2)z^{k+s} \\ &= \frac{\widehat{\psi}(2k+s+2)}{\widehat{\mathbb{1}}(2k+2s+2)} z^{k+s}.\end{aligned}$$

The lemma is proved by applying $T_{e^{is\theta}\psi}$ to ξ^k n times. □

We have the following decomposition of $L^2(\mathbb{D}, dA)$ as

$$L^2(\mathbb{D}, dA) = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R},$$

where $\mathcal{R}$ is the space of functions on $[0, 1]$ that are square integrable with respect to the measure rdr. Thus every function $f \in L^2(\mathbb{D}, dA)$ has the decomposition

$$f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r), \qquad f_k \in \mathcal{R}.$$

Moreover, if $f \in L^\infty(\mathbb{D}, dA) \subset L^2(\mathbb{D}, dA)$, then for each $r \in [0, 1)$,

$$\mid f_k(r) \mid = \frac{1}{2\pi} \mid \int_0^{2\pi} f(re^{i\theta})e^{-ik\theta}\, d\theta \mid \leq \sup_{z\in\mathbb{D}} \mid f(z) \mid, \quad \forall k \in \mathbb{Z},$$

and so the functions f_k are bounded in the disk.

In [7] we proved the following results, which we will use in the proof of our main theorem.

Proposition 4. *Let ϕ be a nonzero bounded radial function, let p be a positive integer and let $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$. Then*

a) *T_f commutes with $T_{e^{ip\theta}\phi}$ if and only if $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$ for all $k \in \mathbb{Z}$.*
b) *If there exists $k \in \mathbb{Z}_-$ and a bounded radial function f_k such that*
$$T_{e^{ip\theta}\phi}T_{e^{ik\theta}f_k} = T_{e^{ik\theta}f_k}T_{e^{ip\theta}\phi},$$
then f_k must be equal to zero.
c) *If there exists $k \in \mathbb{Z}_+$ and a bounded radial function f_k such that*
$$T_{e^{ip\theta}\phi}T_{e^{ik\theta}f_k} = T_{e^{ik\theta}f_k}T_{e^{ip\theta}\phi},$$
then f_k is unique up to a constant factor. In particular f_0 is a constant.

Thus if $p > 0$, $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r)$ and T_f commutes with $T_{e^{ip\theta}\phi}$, then each f_k is uniquely determined up to multiplication by a constant and is equal to 0 for $k < 0$.

Next we present two technical but easy results which permit us to prove Propositions 7 and 9, the principal results of this section.

Remark 5. Let $(a_l)_{l\in\mathbb{N}}$ and $(b_l)_{l\in\mathbb{N}}$ be two nonvanishing sequences and p and s two positive integers such that

$$a_{l+s} b_l = b_{l+p} a_l \quad \text{for all } l \in \mathbb{N}. \tag{2}$$

Then if

$$A_k = \prod_{j=0}^{s-1} a_{k+jp} \quad \text{and} \quad B_k = \prod_{j=0}^{p-1} b_{k+js},$$

we have

$$A_k B_{k+p} = A_{k+p} B_k \quad \text{for all } k \in \mathbb{N}.$$

(Just multiply the p equations obtained by taking $l = k, k+s, .., k+(p-1)s$ in (2) together to see that, if (2) is true, then

$$\frac{B_{k+p}}{B_k} = \frac{a_{k+ps}}{a_k} = \frac{A_{k+p}}{A_k} \quad \text{for all } k \in \mathbb{N}.)$$

Notation. Let S and T be two functions (resp. two operators). We will say that $S \equiv T$ if there exists a constant $c \neq 0$ such that $S = cT$.

Lemma 6. *Let F and G be two nonzero bounded holomorphic functions on the half plane $\Pi = \{z : \Re z > 2\}$. If there exists $p \in \mathbb{N}$ such that*

$$F(z)G(z+p) = F(z+p)G(z) \text{ for all } z \in \Pi, \tag{3}$$

then $F \equiv G$.

Proof. Suppose that (3) is true. Then, if (as above) we multiply the k equations obtained by taking $z_n = z + np$ for $n = 0, ..., k-1$, we have

$$F(z)G(z+kp) = F(z+kp)G(z) \text{ for all } k \in \mathbb{N}. \tag{4}$$

Now, let $z_0 \in \Pi$ such that $G(z_0) \neq 0$ and let $E = \{k \in \mathbb{N} : G(z_0 + kp) = 0\}$. If $\sum_{k\in E} \Re(\frac{1}{|z_0+kp|}) = \infty$, then Theorem 1 implies that $G = 0$. This contradicts the hypothesis of the lemma. Thus $\sum_{k\in E^c} \Re(\frac{1}{|z_0+kp|}) = \infty$, where E^c is the complement in $\mathbb{N}$ of the set E.

Now, equation (4) implies that

$$\frac{F(z_0+kp)}{G(z_0+kp)} = \frac{F(z_0)}{G(z_0)} \text{ for all } k \in E^c.$$

So, applying Theorem 1 to the function $F - cG$, where $c = \frac{F(z_0)}{G(z_0)}$, completes the proof. □

Let p and s be two positive integers and ψ a bounded radial function.

If $\left(T_{e^{is\theta}\psi}\right)^p$ is a Toeplitz operator, then it is the unique quasihomogeneous Toeplitz operator of degree ps (see Proposition 3 and Proposition 4 of [7]) which commutes with $T_{e^{is\theta}\psi}$. It is natural to ask whether all nonzero Toeplitz operators which are of quasihomogeneous of degree a multiple of s and which commute with $T_{e^{is\theta}\psi}$ are of this form.

Proposition 7. *Let p and s be two positive integers and ϕ and ψ be two nonzero bounded radial functions such that*

$$T_{e^{ip\theta}\phi}T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi}T_{e^{ip\theta}\phi}. \tag{5}$$

Then

$$\left(T_{e^{ip\theta}\phi}\right)^s \equiv \left(T_{e^{is\theta}\psi}\right)^p. \tag{6}$$

Proof. For all $k \in \mathbb{N}$, let

$$a_k = \frac{\widehat{\phi}(2k+p+2)}{\widehat{\mathbb{1}}(2k+2p+2)} \quad \text{and} \quad b_k = \frac{\widehat{\psi}(2k+s+2)}{\widehat{\mathbb{1}}(2k+2s+2)}$$

so that

$$T_{e^{ip\theta}\phi}(\xi^k)(z) = a_k z^{k+p} \quad \text{and} \quad T_{e^{is\theta}\psi}(\xi^k)(z) = b_k z^{k+s}.$$

Then equation (5) shows that $a_{k+s}b_k = b_{k+p}a_k$ for all $k \in \mathbb{Z}_+$, and so Remark 5 implies that

$$\prod_{j=0}^{s-1} a_{k+jp} \prod_{j=0}^{p-1} b_{k+p+js} = \prod_{j=0}^{s-1} a_{k+p+jp} \prod_{j=0}^{p-1} b_{k+js}. \tag{7}$$

Let F and G be the two bounded holomorphic functions defined for all $z \in \Pi$ by

$$F(z) = \prod_{j=0}^{p-1} \widehat{\mathbb{1}}(z+2js+2s) \prod_{j=0}^{s-1} \widehat{\phi}(z+2jp+p)$$

and

$$G(z) = \prod_{j=0}^{s-1} \widehat{\mathbb{1}}(z+2jp+2p) \prod_{j=0}^{p-1} \widehat{\psi}(z+2js+s).$$

Then equation (7) is equivalent to

$$F(2k+2)G(2k+2p+2) = F(2k+2p+2)G(2k+2) \quad \text{for all } k \in \mathbb{Z}_+.$$

Now, applying Theorem 1, in the form of Remark 2, implies that

$$F(z)G(z+2p) = F(z+2p)G(z) \quad \text{for all } z \in \Pi.$$

Finally, using Lemma 6, we obtain that

$$\frac{\prod_{j=0}^{s-1} \widehat{\phi}(z+2jp+p)}{\prod_{j=0}^{s-1} \widehat{\mathbb{1}}(z+2jp+2p)} \equiv \frac{\prod_{j=0}^{p-1} \widehat{\psi}(z+2js+s)}{\prod_{j=0}^{p-1} \widehat{\mathbb{1}}(z+2js+2s)} \quad \text{for all } z \in \Pi,$$

and Lemma 3 completes the proof. □

Remark 8. i) We will assume that $\left(T_{e^{ip\theta}\phi}\right)^0 = I$, where I is the identity operator of L^2_a onto L^2_a.

ii) If p and s are both negative integers and if $T_{e^{ip\theta}\phi}T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi}T_{e^{ip\theta}\phi}$, then by considering the adjoint operators we obtain

$$T_{e^{-is\theta}\psi}T_{e^{-ip\theta}\phi} = T_{e^{-ip\theta}\phi}T_{e^{-is\theta}\psi}$$

and so Proposition 7 implies that $\left(T_{e^{-ip\theta}\phi}\right)^{-s} \equiv \left(T_{e^{-is\theta}\psi}\right)^{-p}$.
Now, by considering once again the adjoint operators we see that

$$\left(T_{e^{ip\theta}\phi}\right)^{-s} \equiv \left(T_{e^{is\theta}\psi}\right)^{-p}.$$

Proposition 9. *Let ϕ and ψ be two nonzero bounded radial functions and n, p and s be positive integers. Then*

$$\left(T_{e^{ips\theta}\phi}\right)^n = \left(T_{e^{is\theta}\psi}\right)^{np} \implies T_{e^{ips\theta}\phi} = \left(T_{e^{is\theta}\psi}\right)^p.$$

Proof. For all $k \in \mathbb{Z}_+$, let

$$a_k = 2(k+ps+1)\widehat{\phi}(2k+ps+2) \quad \text{and} \quad b_k = 2(k+s+1)\widehat{\psi}(2k+s+2),$$

so that

$$\left(T_{e^{ips\theta}\phi}\right)^n = \left(T_{e^{is\theta}\psi}\right)^{np} \Leftrightarrow \prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{np-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+$$

and

$$T_{e^{ips\theta}\phi} = \left(T_{e^{is\theta}\psi}\right)^p \Leftrightarrow a_k = \prod_{j=0}^{p-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+.$$

Suppose that

$$\prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{np-1} b_{k+js} \quad \text{for all } k \in \mathbb{Z}_+. \tag{8}$$

We will prove that

$$a_{knps} = \prod_{j=0}^{p-1} b_{knps+js} \quad \text{for all } k \in \mathbb{Z}_+. \tag{9}$$

We prove (9) by induction on k. If we take $k = 0$ in equation (8), then we obtain

$$\begin{aligned} \prod_{j=0}^{n-1} a_{jps} &= \prod_{j=0}^{np-1} b_{js} \\ &= \prod_{j=0}^{p-1} b_{js} \prod_{j=p}^{np-1} b_{js} \\ &= \prod_{j=0}^{p-1} b_{js} \prod_{j=0}^{np-1} b_{ps+js}. \end{aligned}$$

Otherwise

$$\begin{aligned} \prod_{j=0}^{n-1} a_{jps} &= a_0 \prod_{j=1}^{n-1} a_{jps} \\ &= a_0 \prod_{j=0}^{n-1} a_{ps+jps}. \end{aligned}$$

But equation (8) implies that

$$\prod_{j=0}^{n-1} a_{ps+jps} = \prod_{j=0}^{np-1} b_{ps+js}.$$

Thus

$$a_0 = \prod_{j=0}^{p-1} b_{js}.$$

Now, assume (9) is true for $knps$. We show it is true for $(k+1)nps$. We set k equal to $knps$ in the left-hand side of (8) and obtain

$$\prod_{j=0}^{n-1} a_{knps+jps} = a_{knps} \prod_{j=0}^{n-2} a_{knps+ps+jps}.$$

Then

$$a_{(k+1)nps} \prod_{j=0}^{n-1} a_{knps+jps} = a_{knps} \prod_{j=0}^{n-1} a_{knps+ps+jps}.$$

But

$$\prod_{j=0}^{n-1} a_{knps+ps+jps} = \prod_{j=0}^{np-1} b_{knps+ps+js}$$

and

$$\prod_{j=0}^{np-1} b_{knps+ps+js} \prod_{j=0}^{p-1} b_{knps+js} = \prod_{j=0}^{np-1} b_{knps+js} \prod_{j=0}^{p-1} b_{(k+1)nps+js}.$$

Thus (9) is proved for $(knps)_{k\in\mathbb{Z}_+}$. Hence, for all $k \in \mathbb{Z}_+$, we have

$$\widehat{\phi}(2knps+ps+2) \prod_{j=0}^{p-1} \widehat{\mathbb{1}}(2knps+2js+2s+2) = \widehat{\mathbb{1}}(2knp+p+2) \prod_{j=0}^{p-1} \widehat{\psi}(2knp+1+2)$$

and, using equation (1) and Remark 2, we complete the proof. □

Remark 10. In [7] (Proposition 6) we prove that if $p > 0$ and ϕ is a nonzero bounded radial function and if there exists a bounded radial function ψ such that T_ψ commutes with $T_{e^{ip\theta}\phi}$, then ψ must be a constant. Here is another proof of this proposition. In fact, using Proposition 7, we have $\left(T_\psi\right)^p \equiv I$, so Proposition 9 implies that $T_\psi \equiv I$, and so, that $\psi \equiv \mathbb{1}$ since I is the Toeplitz operator of symbol $\mathbb{1}$.

4. Main result

Let p be a positive integer. We start this section with the definition of the T-p^{th} root of a quasihomogeneous Toeplitz operator of degree p or $-p$. This new notion plays an important role in the remainder of the paper.

Definition 11. Let ϕ be a nonzero bounded radial function and p be a positive integer. We say that the Toeplitz operator $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$ if and only if there exists a nonzero bounded radial function ψ such that

$$T_{e^{ip\theta}\phi} = \left(T_{e^{i\theta}\psi}\right)^p.$$

Remark 12.

i) The T-p^{th} root of a quasihomogeneous Toeplitz operator is unique. In fact, suppose that $T_{e^{ip\theta}\phi}$ has two T-p^{th} roots $T_{e^{i\theta}\psi}$ and $T_{e^{i\theta}\widetilde{\psi}}$. Then $\left(T_{e^{i\theta}\psi}\right)^p = \left(T_{e^{i\theta}\widetilde{\psi}}\right)^p$. Then, by Proposition 9, we have that $T_{e^{i\theta}\psi} = T_{e^{i\theta}\widetilde{\psi}}$, which implies that $\psi = \widetilde{\psi}$.

ii) If the quasihomogeneous degree is negative we have an analogous definition of the T-p^{th} root. Let p be a positive integer and ϕ be a bounded radial function. Then, we say that $T_{e^{-ip\theta}\phi}$ has a T-p^{th} root if there exists a bounded radial function ψ such that $T_{e^{-ip\theta}\phi} = \left(T_{e^{-i\theta}\psi}\right)^p$. It is easy to see,

by taking adjoints, that $T_{e^{-ip\theta}\phi}$ has a T-p^{th} root $T_{e^{-i\theta}\psi}$ if and only if $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$.

Examples. i) $T_{e^{i\theta}(\frac{r+r^5}{2})}$ is the T-2th root of $T_{e^{2i\theta}r^6}$.

ii) $T_{e^{i\theta}(\frac{3r+2r^5+3r^9}{8})}$ is the T-2th root of $T_{e^{2i\theta}r^{10}}$.

Now, if $T_{e^{i\theta}\psi}$ is the T-p^{th} root of $T_{ip\theta}\phi$ and if $\left(T_{e^{i\theta}\psi}\right)^k$ (for k in $\mathbb{N}$) is a Toeplitz operator, then $\left(T_{e^{i\theta}\psi}\right)^k$ is the unique nonzero quasihomogeneous Toeplitz operator of degree k which can commute with $T_{e^{ip\theta}\phi}$. What we prove below is that if $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$, then the *only* nonzero quasihomogeneous Toeplitz operator of degree s which commutes with $T_{e^{ip\theta}\phi}$ is an sth power of $T_{e^{i\theta}\psi}$, extending the result (Propositions 7 and 9) of section 3 in this case.

Theorem 13. *Let ϕ be a nonzero bounded radial function and let p be a positive integer. Assume that $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$. Suppose that*

$$f(re^{i\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$$

is such that

$$T_f T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi} T_f. \tag{10}$$

Then

i) *$f_k = 0$ for $k < 0$.*

ii) *If $k \geq 0$ and $\left(T_{e^{i\theta}\psi}\right)^k$ is a Toeplitz operator, then either $T_{e^{ik\theta}f_k} \equiv \left(T_{e^{i\theta}\psi}\right)^k$ or $f_k = 0$.*

iii) *If $k \geq 0$ and $\left(T_{e^{i\theta}\psi}\right)^k$ is not a Toeplitz operator, then $f_k = 0$.*

Proof. Assertion a) of Proposition 4 implies that if equation (10) is true, then

$$T_{e^{ik\theta}f_k} T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi} T_{e^{ik\theta}f_k}, \text{ for all } k \in \mathbb{Z}.$$

Thus i) is a direct consequence of assertion b) of Proposition 4.

Now, to prove ii), let k be a positive integer such that $\left(T_{e^{i\theta}\psi}\right)^k$ is a Toeplitz operator. Then $\left(T_{e^{i\theta}\psi}\right)^k$ is a quasihomogeneous Toeplitz operator of degree k which commutes with $T_{e^{ip\theta}\phi}$. So, if f_k is not identically equal to zero, then f_k is a bounded nonzero radial function such that $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$. Thus, assertion c) of Proposition 4 implies that $T_{e^{ik\theta}f_k} \equiv \left(T_{e^{i\theta}\psi}\right)^k$.

Finally, let k be a positive integer such that $\left(T_{e^{i\theta}\psi}\right)^k$ is not a Toeplitz operator and suppose that there exists a nonzero bounded radial function f_k such that $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$. Then Proposition 7 implies that

$$\left(T_{e^{ik\theta}f_k}\right)^p \equiv \left(T_{e^{ip\theta}\phi}\right)^k.$$

Thus $\left(T_{e^{ik\theta}f_k}\right)^p \equiv \left(T_{e^{i\theta}\psi}\right)^{kp}$ and Proposition 9 implies that $T_{e^{ik\theta}f_k} \equiv \left(T_{e^{i\theta}\psi}\right)^k$, which contradicts our hypothesis. This proves iii). □

Before starting with corollaries, we state an interesting theorem which follows from [5] and give an idea of its proof. In fact we will apply this theorem to see that if p is any positive integer and m is any nonnegative integer, then the Toeplitz operator $T_{e^{ip\theta}r^m}$ always has a T-p^{th} root.

Theorem 14. *Let $p \geq 1$ and $m \geq 0$ be two integers. For all integers s, such that $1 \leq s < p$, there exists a unique bounded radial function ψ such that*

$$T_{e^{is\theta}\psi} T_{e^{ip\theta}r^m} = T_{e^{ip\theta}r^m} T_{e^{is\theta}\psi}. \tag{11}$$

Proof. (This is a slight variation of the proof found in [5].) If $m \geq 0, p \geq 1$ and $1 \leq s < p$, we define the radial functions f and g by

$$f(r) = 2pr^{2s}(1-r^{2p})^{-\frac{s}{p}} \quad \text{and} \quad g(r) = 2pr^{m+p}(1-r^{2p})^{\frac{s}{p}-1}.$$

Let ψ be the radial function defined by

$$r^s\psi = f *_M g.$$

Čučković and Rao prove, using a long rather technical calculation, that ψ is bounded. Here, we will show that ψ satisfies (11). To do this, we need only verify that for $k \in \mathbb{Z}_+$

$$\frac{2k+2p+2}{2k+m+p+2}\widehat{r^s\psi}(2k+2p+2) = \frac{2k+2s+2}{2k+m+p+2s+2}\widehat{r^s\psi}(2k+2).$$

By (1), we have $\widehat{r^s\psi}(2k+2) = \widehat{f}(2k+2)\widehat{g}(2k+2)$. A simple substitution $t = r^{2p}$ shows that

$$\widehat{f}(2k+2) = B\Big(\frac{2k+2s+2}{2p}, 1-\frac{s}{p}\Big) \quad \text{and} \quad \widehat{g}(2k+2) = B\Big(\frac{2k+m+p+2}{2p}, \frac{s}{p}\Big),$$

where B denotes the beta function. Using the well-known identities $B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$ and $\Gamma(1+z) = z\Gamma(z)$, where Γ is the gamma function, it is easy to see that

$$\widehat{r^s\psi}(2k+2p+2) = \frac{(2k+2s+2)(2k+m+p+2)}{(2k+2p+2)(2k+m+p+2s+2)}\widehat{r^s\psi}(2k+2), \tag{12}$$

which finishes the proof. □

Remark 15. i) It is trivial that $T_{e^{ip\theta}r^m}$ commutes with itself. So, if $p = s$, assertion c) of Proposition 4 implies that $\psi \equiv r^m$.

ii) We wish to highlight the following case. If $m = (2n+1)p$ for $n \in \mathbb{N}$, then the function ψ exists for all $s \in \mathbb{N}$. In fact, if we substitute $m = (2n+1)p$ in (12) and use Theorem 1, we obtain for all $z \in \Pi$,

$$\frac{\widehat{r^s\psi}(z+2p)}{\widehat{r^s\psi}(z)} = \frac{F(z+2p)}{F(z)}, \text{ where } F(z) = \frac{\Gamma(\frac{z+2s}{2p})\Gamma(\frac{z}{2p}+n+1)}{\Gamma(\frac{z}{2p}+1)\Gamma(\frac{z+2s}{2p}+n+1)}.$$

Now, using the identity $\Gamma(1+z) = z\Gamma(z)$ repeatedly, we have

$$F(z) = 2p\frac{\prod_{j=0}^{n-1}(z+2jp+2p)}{\prod_{j=0}^{n}(z+2jp+2s)},$$

which is a proper fraction in z and can be written as

$$F(z) = \sum_{j=0}^{n} \frac{a_j}{z+2jp+2s}. \tag{13}$$

Since $\frac{1}{z+2jp+2s} = \widehat{r^{2jp+2s}}(z)$, it follows by Lemma 6 that

$$\widehat{r^s\psi}(z) \equiv \sum_{j=0}^{n} a_j \widehat{r^{2jp+2s}}(z),$$

where the a_j are defined by (13), and so Theorem 1 implies that

$$\psi(r) \equiv \sum_{j=0}^{n} a_j r^{2jp+s}.$$

Next, we give some easy but interesting consequences of Theorem 14.

Corollary 16. *For all integers $m \geq 0$, $p \geq 1$, and $s \geq 1$ there exists a bounded radial function ψ such that $\left(T_{e^{is\theta}\psi}\right)^p \equiv T_{e^{ips\theta}r^m}$.*

Proof. Let $m \geq 0$, $p \geq 1$, and $s \geq 1$ be integers. Theorem 14 implies that there exists a bounded radial function ψ such that

$$T_{e^{is\theta}\psi} T_{e^{ips\theta}r^m} = T_{e^{ips\theta}r^m} T_{e^{is\theta}\psi}.$$

Using Proposition 7 we have $\left(T_{e^{is\theta}\psi}\right)^{ps} \equiv \left(T_{e^{ips\theta}r^m}\right)^s$ and so, an application of Proposition 9 finishes the proof. □

In [4], Brown and Halmos studied multiplicativity of Toeplitz operators on the Hardy space and showed that the product of two Toeplitz operators T_f and T_g is equal to a third Toeplitz operator T_h for some f, g and h in $L^\infty(\mathbb{T})$ if and only if f is conjugate analytic or g is analytic, that is, hardly ever. The question of when the product of two Toeplitz operators on the Bergman space is equal to a third is much more complicated and still open. Most work on this question shows that it is not often true that the product of two Toeplitz operators is a Toeplitz operator (see [1] and [6]). But, below, we show that, for certain nontrivial Toeplitz operators $T_{e^{i\theta}\psi}$, not only is $\left(T_{e^{i\theta}\psi}\right)^2$ equal to a Toeplitz operator, but there exists a positive integer k such that $\left(T_{e^{i\theta}\psi}\right)^i$ is a Toeplitz operator for all positive integers $i \leq k$.

Corollary 17. *Let $m \geq 0$ and $p \geq 1$ be two integers. If $T_{e^{ip\theta}r^m}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$ then, for all integers k with $1 \leq k \leq p$, the product $\left(T_{e^{i\theta}\psi}\right)^k$ is a Toeplitz operator.*

Proof. Let k be an integer such that $1 \leq k \leq p$. By Theorem 14 we know that there exists a bounded radial function ϕ such that $T_{e^{ik\theta}\phi}$ commutes with $T_{e^{ip\theta}r^m}$. So, Proposition 7 implies that

$$\left(T_{e^{ik\theta}\phi}\right)^p \equiv \left(T_{e^{ip\theta}r^m}\right)^k.$$

Thus $\left(T_{e^{ik\theta}\phi}\right)^p = \left(T_{e^{i\theta}\psi}\right)^{kp}$ since $T_{e^{i\theta}\psi}$ is the T-p^{th} root of $T_{e^{ip\theta}r^m}$, and so Proposition 9 finishes the proof. □

It is easily seen that if f is a bounded analytic function on $\mathbb{D}$, then T_f is just a multiplication operator. Thus for any integer $k \geq 1$, it is clear that $\left(T_f\right)^k$ is a Toeplitz operator of symbol f^k. By taking adjoints, we can see that the powers of a Toeplitz operator with conjugate analytic symbol are also Toeplitz operators. These are the trivial cases. The next corollary says there are nontrivial symbols f such that $\left(T_f\right)^k$ is always a Toeplitz operator for all integers $k \geq 1$.

Corollary 18. *There exist bounded radial functions ψ such that for all integers $k \geq 1$ the product $\left(T_{e^{i\theta}\psi}\right)^k$ is still a Toeplitz operator.*

Proof. Let $n \geq 0$ and $p \geq 1$ be two integers. By Theorem 14 we know that the Toeplitz operator $T_{e^{ip\theta}r^{(2n+1)p}}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$, where ψ is a bounded radial function. Moreover the assertion *ii*) of Remark 15 tells us that, for all integers $k \geq 1$, there exists a bounded radial function ψ_k such that $T_{e^{ik\theta}\psi_k}$ commutes with $T_{e^{ip\theta}r^{(2n+1)p}}$. Thus Proposition 7 implies that $\left(T_{e^{ik\theta}\psi_k}\right)^p \equiv \left(T_{e^{i\theta}\psi}\right)^{kp}$ since $T_{e^{ip\theta}r^{(2n+1)p}} = \left(T_{e^{i\theta}\psi}\right)^p$ and, again, Proposition 9 finishes the proof. $\square$

Acknowledgments

I am thankful to Elizabeth Strouse my thesis supervisor for helpful discussions and suggestions concerning the results of this paper.

References

[1] P. Ahern, Ž. Čučković, A Theorem of Brown-Halmos Type for Bergman Space Toeplitz Operators, *J. Funct. Anal.* **187** (2001), 200-210. MR1867348 (2002h:47040)

[2] S. Axler, Ž. Čučković, Commuting Toeplitz operators with harmonic symbols, *Integral Equations and Operator Theory* **14** (1991), 1-12. MR1079815 (92f:47018)

[3] S. Axler, Ž. Čučković, N. V. Rao, Commutants of analytic Toeplitz operators on the Bergman space, *Proc. Amer. Math. Soc.* **128** (2000), 1951-1953. MR1694299 (2000m:47035)

[4] A. Brown, P. Halmos, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213** (1964), 89-102. MR0160136 (28:3350)

[5] Ž. Čučković, N. V. Rao, Mellin transform, monomial symbols, and commuting Toeplitz operators, *J. Funct. Anal.* **154** (1) (1998), 195-214. MR1616532 (99f:47033)

[6] I. Louhichi, E. Strouse, L. Zakariasy, Products of Toeplitz operators on the Bergman space, *Integral Equations and Operator Theory*. Published on line: 1 October 2005.

[7] I. Louhichi, L. Zakariasy, On Toeplitz operators with quasihomogeneous symbols, *Archiv der Mathematik*. Volume **85**, Number 3 (2005), 248-257. MR2172383 (2006e:47061)

[8] R. Remmert, *Classical Topics in Complex Function Theory,* Graduate Texts in Mathematics, Springer, New York, 1998. MR1483074 (98g:30002)

UFR Mathématiques et Informatique, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence, France

E-mail address: louhichi@math.u-bordeaux1.fr