EQUIVARIANT CRYSSTALLINE COHOMOLOGY 
AND BASE CHANGE

ELMAR GROSSE-KLÖNNE

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Abstract. Given a perfect field \( k \) of characteristic \( p > 0 \), a smooth proper \( k \)-scheme \( Y \), a crystal \( E \) on \( Y \) relative to \( W(k) \) and a finite group \( G \) acting on \( Y \) and \( E \), we show that, viewed as a virtual \( k[G] \)-module, the reduction modulo \( p \) of the crystalline cohomology of \( E \) is the de Rham cohomology of \( E \) modulo \( p \). On the way we prove a base change theorem for the virtual \( G \)-representations associated with \( G \)-equivariant objects in the derived category of \( W(k) \)-modules.

1. The theorem

Let \( k \) be a perfect field of characteristic \( p > 0 \), let \( W \) denote its ring of Witt vectors, let \( K = \text{Quot}(W) \). Let \( Y \) be a proper and smooth \( k \)-scheme and suppose that the finite group \( G \) acts (from the right) on \( Y \). Let \( E \) be a locally free, finitely generated crystal of \( \mathcal{O}_Y/W \)-modules and suppose that for each \( g \in G \) we are given an isomorphism of crystals \( \tau_g : E \to g^*E \) (where \( g^*E \) denotes the pull-back of \( E \) via \( g : Y \to Y \) such that \( g^2(\tau_{g_1}) \circ \tau_{g_2} = \tau_{g_2 g_1} \) (equality as maps \( E \to (g_2 g_1)^*E = g_2^* g_1^*E \)) for any two \( g_1, g_2 \in G \). For \( s \in \mathbb{Z} \) let \( H^{s}_{\text{cryst}}(Y/W, E) \) denote the \( s \)-th crystalline cohomology group (relative to \( \text{Spf}(W) \)) of the crystal \( E \), a finitely generated \( W \)-module which is zero if \( s \notin [0, 2 \dim(Y)] \) (see [H]). On the other hand, the reduction modulo \( p \) of the crystal \( E \) is equivalent with a locally free \( \mathcal{O}_Y \)-module \( E_k \) with connection \( E_k \to \Omega_{Y/W}^1 \otimes_{\mathcal{O}_Y} E_k \); here \( \Omega_{Y/W}^1 \) denotes the \( \mathcal{O}_Y \)-module of differentials of \( Y/k \). Let \( \Omega_{Y/W}^s \otimes E_k \) denote the corresponding de Rham complex. The cohomology group \( H^s(Y, \Omega_{Y/W}^s \otimes E_k) \) is a finite-dimensional \( k \)-vector space which is zero if \( s \notin [0, 2 \dim(Y)] \). The isomorphisms \( \tau_g \) for \( g \in G \) provide each \( H^s_{\text{cryst}}(Y/W, E) \), each \( H^s(Y, \Omega_{Y/W}^s \otimes E_k) \) and each \( H^t(Y, \Omega_{Y/W}^s \otimes E_k) \) for \( t \geq 0 \) with an action of \( G \) (from the left). By definition, the reduction modulo \( p \) of the \( k[G] \)-module \( H^s_{\text{cryst}}(Y/W, E) \otimes_W K \) is the \( k[G] \)-module obtained by reducing modulo \( p \) the \( G \)-stable \( W \)-lattice \( H^s_{\text{cryst}}(Y/W, E)/\text{(torsion)} \) in \( H^s_{\text{cryst}}(Y/W, E) \otimes_W K \).

Theorem 1.1. For any \( j \), the following three virtual \( k[G] \)-modules are the same:
(i) the reduction modulo \( p \) of the virtual \( k[G] \)-module

\[
\sum_s (-1)^s H^{j+s}_{\text{cryst}}(Y/W, E) \otimes_W K;
\]
(ii) \( \sum_s (-1)^s H^s(Y, \Omega^s_Y \otimes E_k) \);
(iii) \( \sum_{s,t} (-1)^{s+t} H^s(Y, \Omega^t_Y \otimes E_k) \).

An obvious variant of Theorem 1.1 holds in logarithmic crystalline cohomology, for crystals \( E \) on the logarithmic crystalline site of \( Y/W \) with respect to a log structure defined by a normal crossings divisor on \( Y \). Similarly, the proof which we give below also shows the analog of Theorem 1.1 for the \( \ell \)-adic cohomology (\( \ell \neq p \)) of constructible \( \ell \)-adic sheaves on \( Y \), even if \( Y/k \) is not proper. Of course, the result in the \( \ell \)-adic case (even for nonproper \( Y/k \)) is well known; it has been used for investigating the reduction modulo \( \ell \) of the Deligne-Lusztig characters, usually defined via 

In [3] we use the variant of Theorem 1.1 in logarithmic crystalline cohomology to show that these Deligne-Lusztig characters, usually defined via \( \ell \)-adic cohomology of certain \( \mathbb{F} \)-varieties which are nonproper in general, can also be expressed through the log crystalline cohomology of suitable log crystals on suitable proper and smooth \( \mathbb{F} \)-varieties with a normal crossings divisor. Unfortunately, the (more geometric) proof of the \( \ell \)-adic analog of Theorem 1.1 (due to Deligne and Lusztig; see for example [2], Lemma 12.4 and A3.15) breaks down for crystalline cohomology. On the other hand, our proof of Theorem 1.1 contains a result (Theorem 2.1) on

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### 2. The proof

Proof of Theorem 1.1. (ii)=(iii) is clear. By [1] we know that the total crystalline cohomology \( \mathbb{R} \Gamma_{\text{crys}}(Y/W, E) \), as an object in the derived category \( D(W) \) of the category of \( W \)-modules, is represented by a complex of \( W \)-modules of finite torsion-dimension and with finitely generated cohomology; by functoriality, \( G \) acts on \( \mathbb{R} \Gamma_{\text{crys}}(Y/W, E) \). Also from [1] we know that the total crystalline cohomology commutes with base change, i.e. that \( \mathbb{R} \Gamma_{\text{crys}}(Y/W, E) \otimes_k \hat{k} \) is the total crystalline cohomology of the reduction modulo \( p \) of \( E \) (as a crystal relative to \( \text{Spec}(k) \)). But the latter is known (see [1], Corollary 7.4) to be the de Rham cohomology of \( E_k \); i.e., its \( s \)-th cohomology group is \( H^s(Y, \Omega^s_Y \otimes E_k) \). Hence (i)=(ii) follows from Theorem 2.1 below.

Let \( A \) be a complete discrete valuation ring with perfect residue field \( k \) of characteristic \( p > 0 \) and fraction field \( K \) of characteristic 0. Let \( L^\bullet \) be a complex of \( A \)-modules of finite torsion-dimension and with finitely generated cohomology; by [1], Lemma 7.15, this is equivalent with saying that \( L^\bullet \) is quasi-isomorphic to a strictly perfect complex, i.e. a bounded complex of finitely generated projective \( A \)-modules. Suppose the finite group \( G \) acts on \( L^\bullet \) when \( L^\bullet \) is viewed as an object in the derived category \( D(A) \) of the category of \( A \)-modules. Then each cohomology group \( H^i(L^\bullet \otimes_A K) = H^i(L^\bullet) \otimes_A K \) (resp. each cohomology group \( H^i(L^\bullet \otimes_{\mathbb{F}_k} k) \)) becomes a representation of \( G \) on a finite-dimensional \( K \)-vector space (resp. \( k \)-vector space).

**Theorem 2.1.** The virtual \( k[G] \)-module \( \sum_i (-1)^i H^i(L^\bullet \otimes_{\mathbb{F}_k} k) \) is the reduction (modulo the maximal ideal of \( A \)) of the virtual \( K[G] \)-module \( \sum_i (-1)^i H^i(L^\bullet) \otimes_A K \). Equivalently, the restriction of the character of \( \sum_i (-1)^i H^i(L^\bullet) \otimes_A K \) to the subset of \( p \)-regular elements of \( G \) is the Brauer character of \( \sum_i (-1)^i H^i(L^\bullet \otimes_{\mathbb{F}_k} k) \).

We say that the automorphism \( \gamma \) of the finitely generated \( A \)-module \( M \) is *prime to \( p \)* if and only if the following holds. For any finite extension \( A' \supset A \) with a discrete
valuation ring $A'$ and for any two $\gamma \otimes_A A'$-stable submodules $N, N'$ of $M \otimes_A A'$ with $N' \subset N$ and such that $N/N'$ is a cyclic $A'$-module, the endomorphism which $\gamma \otimes_A A'$ induces on $N/N'$ is of finite order prime to $p$.

**Lemma 2.2.** Let $\gamma$ be an automorphism of the finitely generated $A$-module $M$.

(a) If $M$ is free, then $\gamma$ is prime to $p$ if and only if the roots of the characteristic polynomial of $\gamma$ are roots of unity of order prime to $p$. In particular, $\gamma|_N : N \rightarrow N$ is prime to $p$ for each submodule $N$ of $M$ with $\gamma(N) = N$.

(b) Let $M_1 \subset M$ be a submodule with $\gamma(M_1) = M_1$ and such that $M_2 = M/M_1$ is free. Let $\gamma_1$, resp. $\gamma_2$, be the induced automorphism of $M_1$, resp. of $M_2$. If $\gamma_1$ and $\gamma_2$ are prime to $p$, then $\gamma$ is prime to $p$.

**Proof.** Statement (a) is clear. (b) Let $N' \subset N \subset M \otimes_A A'$ be as in the definition. If $N \subset M_1 \otimes_A A'$ the hypothesis on $\gamma_1$ applies. Otherwise, since $M_2 \otimes_A A'$ is free over $A'$ and $N/N'$ is cyclic, $N/N'$ maps injectively to $M_2 \otimes_A A'$ and the hypothesis on $\gamma_2$ applies. \hfill $\square$

**Proof of Theorem 2.1.** The problem is of course that the $H^i(L^\bullet)$ may have torsion, i.e., $H^i(L^\bullet) \otimes_A k \neq H^i(L^\bullet \otimes_A k)$ in general. Similarly, the task would be easy if we knew that there is a strictly perfect complex $K^\bullet$ quasi-isomorphic to $L^\bullet$ such that the action of $G$ on $L^\bullet$ in $D(A)$ is given by the action of $G$ on $K^\bullet$ by true morphisms of complexes (not just by morphisms in $D(A)$). We introduce some notation. For an automorphism $\gamma : L^\bullet \rightarrow L^\bullet$ in $D(A)$ let $\xi^i_1, \ldots, \xi^i_{n(i)}$ (with $n(i) = \dim_k H^i(L^\bullet \otimes_A k)$) denote the roots of the characteristic polynomial of $\gamma$ acting on $H^i(L^\bullet \otimes_A k)$ and let $\bar{\xi}_1, \ldots, \bar{\xi}_{n(i)}$ denote their Teichmüller liftings. On the other hand, let $\xi'^i_1, \ldots, \xi'^i_{n'(i)}$ (with $n'(i) = \dim_K H^i(L^\bullet) \otimes_A K$) denote the roots of the characteristic polynomial of $\gamma$ acting on $H^i(L^\bullet) \otimes_A K$. Then let

$$Br(\gamma, H^\varnothing(L^\bullet \otimes_A k)) = \sum_i (-1)^i \sum_j \xi^i_j,$$

$$Tr(\gamma, H^\varnothing(L^\bullet) \otimes_A K) = \sum_i (-1)^i \sum_j \xi'^i_j.$$

What we must show is that for all $p$-regular elements $g \in G$ (those whose order in $G$ is not divisible by $p$), if $\gamma : L^\bullet \rightarrow L^\bullet$ denotes the corresponding automorphism of $L^\bullet$ in $D(A)$, then

$$Br(\gamma, H^\varnothing(L^\bullet \otimes_A k)) = Tr(\gamma, H^\varnothing(L^\bullet) \otimes_A K).$$

Clearly it is enough to show the following statement. For any strictly perfect complex $L^\bullet$ of $A$-modules (not necessarily endowed with a $G$-action in $D(A)$) and for any automorphism $\gamma : L^\bullet \rightarrow L^\bullet$ in $D(A)$ which on the cohomology modules induces automorphisms prime to $p$ we have

$$Br(\gamma, H^\varnothing(L^\bullet \otimes_A k)) = Tr(\gamma, H^\varnothing(L^\bullet) \otimes_A K).$$

We use induction on the minimal $m \in \mathbb{Z}_{>0}$ with the following property: after a suitable degree shift we have $L^i = 0$ for all $i \notin [0, m]$. For $m = 0$ the statement is clear from Lemma 2.2(a). Now let $m \geq 1$; shifting degrees we may assume $L^i = 0$ for all $i \notin [0, m]$. For $m = 1$ the statement is clear from Lemma 2.2(a), as in the proof of Theorem 1.1. The problem of course is that the $H^i(L^\bullet)$ may have torsion, i.e., $H^i(L^\bullet) \otimes_A k \neq H^i(L^\bullet \otimes_A k)$ in general. Similarly, the task would be easy if we knew that there is a strictly perfect complex $K^\bullet$ quasi-isomorphic to $L^\bullet$ such that the action of $G$ on $L^\bullet$ in $D(A)$ is given by the action of $G$ on $K^\bullet$ by true morphisms of complexes (not just by morphisms in $D(A)$). We introduce some notation. For an automorphism $\gamma : L^\bullet \rightarrow L^\bullet$ in $D(A)$ let $\xi^i_1, \ldots, \xi^i_{n(i)}$ (with $n(i) = \dim_k H^i(L^\bullet \otimes_A k)$) denote the roots of the characteristic polynomial of $\gamma$ acting on $H^i(L^\bullet \otimes_A k)$ and let $\bar{\xi}_1, \ldots, \bar{\xi}_{n(i)}$ denote their Teichmüller liftings. On the other hand, let $\xi'^i_1, \ldots, \xi'^i_{n'(i)}$ (with $n'(i) = \dim_K H^i(L^\bullet) \otimes_A K$) denote the roots of the characteristic polynomial of $\gamma$ acting on $H^i(L^\bullet) \otimes_A K$. Then let

$$Br(\gamma, H^\varnothing(L^\bullet \otimes_A k)) = \sum_i (-1)^i \sum_j \xi^i_j,$$

$$Tr(\gamma, H^\varnothing(L^\bullet) \otimes_A K) = \sum_i (-1)^i \sum_j \xi'^i_j.$$
for all $i \notin [0, m]$. Let $d^m : L^{m-1} \to L^m$ denote the differential. Choose a sub-$k$-vector space $N^m_k$ of $L^{m-1} \otimes k$ which under $d^m \otimes k$ maps isomorphically to the kernel of

$$L^m \otimes k \to H^m(L^\bullet \otimes k) = H^m(L^\bullet) \otimes k.$$  

Then $N^m_k = N^{m-1}_k \otimes k$ for a direct summand $N^{m-1}$ of $L^{m-1}$. By construction, $N^{m-1}$ maps isomorphically to its image $N^m$ in $L^m$. Thus, setting $N^i = 0$ if $i \notin \{m-1, m\}$, the subcomplex $N^\bullet$ of $L^\bullet$ is acyclic. Dividing it out we may therefore assume $L^m \otimes k = H^m(L^\bullet \otimes k)$. Since the functor $K^-(\text{proj} - A) \to D(A)$ from the homotopy category of complexes of projective $A$-modules bounded above to $D(A)$ is fully faithful, the action of $\gamma$ on $L^\bullet$ in $D(A)$ is in fact represented by a true morphism of complexes $\gamma^\bullet : L^\bullet \to L^\bullet$. Base changing to a finite extension of $A$ by a discrete valuation ring (this does not affect the numbers $\text{Br}$ and $\text{Tr}$) we may suppose that the characteristic polynomial of $\gamma^m : L^m \to L^m$ splits in $A$ (we remark that $\gamma^m$ is bijective: this follows from $L^m \otimes k = H^m(L^\bullet \otimes k)$ and the fact that $\gamma$ acts bijectively on $H^m(L^\bullet \otimes k)$). We therefore find a $\gamma^m$-stable filtration

$$F^e = F^0 \subset F^1 \subset \ldots \subset F^s = L^m \quad (s = \text{rk}(L^m))$$

such that $G^e = F^e/F^{e-1}$ is free of rank one, for any $1 \leq e \leq s$. The cyclic $A$-module

$$F^e/(F^e \cap \text{im}(d^m)) + F^{e-1}$$

is a $\gamma^m$-stable subquotient of $H^m(L^\bullet)$ (it is nonzero because of $L^m \otimes k = H^m(L^\bullet \otimes k)$); hence $\gamma^m$ acts on it by multiplication with a root of unity of order prime to $p$. Let $\xi_e \in A^\times$ denote its Teichmüller lifting. Choose $t_e \in F^e$ which represents a basis element of $G^e$; then $t_1, \ldots, t_s$ is a basis of $L^m$. Modulo $F^{e-1}$ the class of $\xi_e t_e - \gamma^m(t_e) \in F^e$ lies in $\text{im}(d^m)$. Choose a $t_e \in L^{m-1}$ with

$$d^m(t_e) = \xi_e t_e - \gamma^m(t_e) \bmod F^{e-1}.$$

Let $t : L^m \to L^{m-1}$ denote the $A$-linear map which sends $t_e$ to $t_e$, for each $1 \leq e \leq s$. Using $t$ we see that we may modify $\gamma^\bullet$ within its homotopy class to achieve that the filtration [1] is still $\gamma^m$-stable and such that $\gamma^m$ acts on each $G^e$ by multiplication with a root of unity of prime-to-$p$ order in $A^\times$. Therefore we may assume that $\gamma^m : L^m \to L^m$ is prime to $p$. Let $L^m_i = L^m$ and $L^m_i = 0$ for $i \neq m$. Then $L^m_1$ is a $\gamma^\bullet$-stable subcomplex of $L^\bullet$ and since $\text{Br}(\gamma)$ and $\text{Tr}(\gamma)$ are additive in exact $\gamma^\bullet$-equivariant sequences of complexes it suffices to show $\text{Br}(\gamma) = \text{Tr}(\gamma)$ for the complexes $L^m_1$ and $L^\bullet/L^m_1$. Since these complexes are shorter than $L^\bullet$ this follows from the induction hypothesis. Indeed, the prime-to-$p$ hypothesis is clearly satisfied for $L^m_1$, so it remains to show that $\gamma^\bullet$ induces automorphisms prime to $p$ on the cohomology modules of $L^\bullet/L^m_1$. In degrees smaller than $m - 1$ this is clear from the corresponding hypothesis on $L^\bullet$: only $H^{m-1}(L^\bullet/L^m_1)$ is critical. But $H^{m-1}(L^\bullet)$ is a submodule of $H^{m-1}(L^\bullet/L^m_1)$ and the quotient

$$Q = H^{m-1}(L^\bullet/L^m_1)/H^{m-1}(L^\bullet)$$

maps isomorphically to a submodule of $L^m_1 = L^m$. By Lemma [2.2(b)] it suffices to show that $\gamma^\bullet$ induces automorphisms prime to $p$ on $H^{m-1}(L^\bullet)$ and on $Q$. For $H^{m-1}(L^\bullet)$ this holds by hypothesis; for $Q$ this follows from Lemma [2.2(a)]. □
References


Mathematisches Institut der Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: klonne@math.uni-muenster.de