EQUIVARIANT CRystalline COHOMOLOGY
AND BASE CHANGE

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abstract. Given a perfect field \( k \) of characteristic \( p \) > 0, a smooth proper \( k \)-scheme \( Y \), a crystal \( E \) on \( Y \) relative to \( W(k) \) and a finite group \( G \) acting on \( Y \) and \( E \), we show that, viewed as a virtual \( k[G] \)-module, the reduction modulo \( p \) of the crystalline cohomology of \( E \) is the de Rham cohomology of \( E \) modulo \( p \). On the way we prove a base change theorem for the virtual \( G \)-representations associated with \( G \)-equivariant objects in the derived category of \( W(k) \)-modules.

1. The theorem

Let \( k \) be a perfect field of characteristic \( p \) > 0, let \( W \) denote its ring of Witt vectors, let \( K = \text{Quot}(W) \). Let \( Y \) be a proper and smooth \( k \)-scheme and suppose that the finite group \( G \) acts (from the right) on \( Y \). Let \( E \) be a locally free, finitely generated crystal of \( \mathcal{O}_Y/W \)-modules and suppose that for each \( g \in G \) we are given an isomorphism of crystals \( \tau_g : E \to g^*E \) (where \( g^*E \) denotes the pull-back of \( E \) via \( g : Y \to Y \) such that \( g_1^2(\tau_{g_1}) \circ \tau_{g_2} = \tau_{g_2g_1} \) (equality as maps \( E \to (g_2g_1)^*E = g_2^*g_1^*E \)) for any two \( g_1, g_2 \in G \). For \( s \in \mathbb{Z} \) let \( H^{s}_{\text{cr}}(Y/W, E) \) denote the \( s \)-th crystalline cohomology group (relative to \( \text{Spf}(W) \)) of the crystal \( E \), a finitely generated \( W \)-module which is zero if \( s \not\in [0, 2\dim(Y)] \) (see [1]). On the other hand, the reduction modulo \( p \) of the crystal \( E \) is equivalent with a locally free \( \mathcal{O}_Y \)-module \( E_k \) with connection \( E_k \to \Omega^1_Y \otimes_{\mathcal{O}_Y} E_k \); here \( \Omega^1_Y \) denotes the \( \mathcal{O}_Y \)-module of differentials of \( Y/k \). Let \( \Omega^s_Y \otimes E_k \) denote the corresponding de Rham complex. The cohomology group \( H^s(Y, \Omega^s_Y \otimes E_k) \) is a finite-dimensional \( k \)-vector space which is zero if \( s \not\in [0, 2\dim(Y)] \). The isomorphisms \( \tau_g \) for \( g \in G \) provide each \( H^s_{\text{cr}}(Y/W, E) \), each \( H^s(Y, \Omega^s_Y \otimes E_k) \) and each \( H^t(Y, \Omega^s_Y \otimes E_k) \) for \( t \geq 0 \) with an action of \( G \) (from the left). By definition, the reduction modulo \( p \) of the \( K[G] \)-module \( H^{s}_{\text{cr}}(Y/W, E) \otimes W K \) is the \( k[G] \)-module obtained by reducing modulo \( p \) the \( G \)-stable \( W \)-lattice \( H^{s}_{\text{cr}}(Y/W, E)/(\text{torsion}) \) in \( H^{s}_{\text{cr}}(Y/W, E) \otimes W K \).

Theorem 1.1. For any \( j \), the following three virtual \( k[G] \)-modules are the same: (i) the reduction modulo \( p \) of the virtual \( K[G] \)-module

\[
\sum_{s} (-1)^s H^{s}_{\text{cr}}(Y/W, E) \otimes_W K;
\]

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An obvious variant of Theorem 1.1 holds in logarithmic crystalline cohomology, for crystals $E$ on the logarithmic crystalline site of $Y/W$ with respect to a log structure defined by a normal crossings divisor on $Y$. Similarly, the proof which we give below also shows the analog of Theorem 1.1 for the $\ell$-adic cohomology ($\ell \neq p$) of constructible $\ell$-adic sheaves on $Y$, even if $Y/k$ is not proper. Of course, the result in the $\ell$-adic case (even for nonproper $Y/k$) is well known; it has been used for investigating the reduction modulo $\ell$ of the Deligne-Lusztig characters of groups $G = G(F)$, where $G$ is a reductive group over a finite field $F$ of characteristic $p$.

In [3] we use the variant of Theorem 1.1 in logarithmic crystalline cohomology to show that these Deligne-Lusztig characters, usually defined via $\ell$-adic cohomology of certain $F$-varieties which are nonproper in general, can also be expressed through the log crystalline cohomology of suitable log crystals on suitable proper and smooth $F$-varieties with a normal crossings divisor. Unfortunately, the (more geometric) proof of the $\ell$-adic analog of Theorem 1.1 (due to Deligne and Lusztig; see for example [2], Lemma 12.4 and A3.15) breaks down for crystalline cohomology. On the other hand, our proof of Theorem 1.1 contains a result (Theorem 2.1) on $G$-actions on strictly perfect complexes in the derived category which should be of independent interest.

2. The proof

Proof of Theorem 1.1. (ii)=(iii) is clear. By [1] we know that the total crystalline cohomology $\mathbb{R}\Gamma_{\text{crys}}(Y/W, E)$, as an object in the derived category $D(W)$ of the category of $W$-modules, is represented by a complex of $W$-modules of finite tor-dimension and with finitely generated cohomology; by functoriality, $G$ acts on $\mathbb{R}\Gamma_{\text{crys}}(Y/W, E)$. Also from [1] we know that the total crystalline cohomology commutes with base change, i.e. that $\mathbb{R}\Gamma_{\text{crys}}(Y/W, E) \otimes_{W} k$ is the total crystalline cohomology of the reduction modulo $p$ of $E$ (as a crystal relative to Spec$(k)$). But the latter is known (see [1], Corollary 7.4) to be the de Rham cohomology of $E_{k}$; i.e., its $s$-th cohomology group is $H^{s}(Y, \Omega_{Y}^{s} \otimes E_{k})$. Hence (i)=(ii) follows from Theorem 2.1 below.

Let $A$ be a complete discrete valuation ring with perfect residue field $k$ of characteristic $p > 0$ and fraction field $K$ of characteristic 0. Let $L^{\bullet}$ be a complex of $A$-modules of finite tor-dimension and with finitely generated cohomology; by [1], Lemma 7.15, this is equivalent with saying that $L^{\bullet}$ is quasi-isomorphic to a strictly perfect complex, i.e. a bounded complex of finitely generated projective $A$-modules. Suppose the finite group $G$ acts on $L^{\bullet}$ when $L^{\bullet}$ is viewed as an object in the derived category $D(A)$ of the category of $A$-modules. Then each cohomology group $H^{i}(L^{\bullet} \otimes_{A} K) = H^{i}(L^{\bullet}) \otimes_{A} K$ (resp. each cohomology group $H^{i}(L^{\bullet} \otimes_{A}^{\mathbb{L}} k)$) becomes a representation of $G$ on a finite-dimensional $K$-vector space (resp. $k$-vector space).

Theorem 2.1. The virtual $k[G]$-module $\sum_{i}(-1)^{i}H^{i}(L^{\bullet} \otimes_{A}^{\mathbb{L}} k)$ is the reduction (modulo the maximal ideal of $A$) of the virtual $K[G]$-module $\sum_{i}(-1)^{i}H^{i}(L^{\bullet}) \otimes_{A} K$. Equivalently, the restriction of the character of $\sum_{i}(-1)^{i}H^{i}(L^{\bullet}) \otimes_{A} K$ to the subset of $p$-regular elements of $G$ is the Brauer character of $\sum_{i}(-1)^{i}H^{i}(L^{\bullet} \otimes_{A}^{\mathbb{L}} k)$.

We say that the automorphism $\gamma$ of the finitely generated $A$-module $M$ is prime to $p$ if and only if the following holds. For any finite extension $A' \supset A$ with a discrete
valuation ring $A'$ and for any two $\gamma \otimes_A A'$-stable submodules $N, N'$ of $M \otimes_A A'$ with $N' \subset N$ and such that $N/N'$ is a cyclic $A'$-module, the endomorphism which $\gamma \otimes_A A'$ induces on $N/N'$ is of finite order prime to $p$.

**Lemma 2.2.** Let $\gamma$ be an automorphism of the finitely generated $A$-module $M$.

(a) If $M$ is free, then $\gamma$ is prime to $p$ if and only if the roots of the characteristic polynomial of $\gamma$ are roots of unity of order prime to $p$. In particular, $\gamma|_N : N \to N$ is prime to $p$ for each submodule $N$ of $M$ with $\gamma(N) = N$.

(b) Let $M_1 \subset M$ be a submodule with $\gamma(M_1) = M_1$ and such that $M_2 = M/M_1$ is free. Let $\gamma_1$, resp. $\gamma_2$, be the induced automorphism of $M_1$, resp. of $M_2$. If $\gamma_1$ and $\gamma_2$ are prime to $p$, then $\gamma$ is prime to $p$.

**Proof.** Statement (a) is clear. (b) Let $N' \subset N \subset M \otimes_A A'$ be as in the definition. If $N \subset M_1 \otimes_A A'$ the hypothesis on $\gamma_1$ applies. Otherwise, since $M_2 \otimes_A A'$ is free over $A'$ and $N/N'$ is cyclic, $N/N'$ maps injectively to $M_2 \otimes_A A'$ and the hypothesis on $\gamma_2$ applies.

**Proof of Theorem 2.1.** The problem is of course that the $H^i(L^\bullet)$ may have torsion, i.e., $H^i(L^\bullet) \otimes_A k \neq H^i(L^\bullet \otimes_A^L k)$ in general. Similarly, the task would be easy if we knew that there is a strictly perfect complex $K^\bullet$ quasi-isomorphic to $L^\bullet$ such that the action of $G$ on $L^\bullet$ in $D(A)$ is given by the action of $G$ on $K^\bullet$ by true morphisms of complexes (not just by morphisms in $D(A)$). We introduce some notation. For an automorphism $\gamma : L^\bullet \to L^\bullet$ in $D(A)$ let $\xi_1, \ldots, \xi_{n(i)}$ (with $n(i) = \dim_k H^i(L^\bullet \otimes_A^L k)$) denote the roots of the characteristic polynomial of $\gamma$ acting on $H^i(L^\bullet \otimes_A^L k)$ and let $\xi_1, \ldots, \xi_{n(i)}$ denote their Teichmüller liftings. On the other hand, let $\xi_1, \ldots, \xi_{n'(i)}$ (with $n'(i) = \dim_k H^i(L^\bullet) \otimes_A K$) denote the roots of the characteristic polynomial of $\gamma$ acting on $H^i(L^\bullet) \otimes_A K$. Then let

$$Br(\gamma, H^\varphi(L^\bullet \otimes^L_A k)) = \sum_i (-1)^i \sum_{j=1}^{n(i)} \xi_j,$$

$$Tr(\gamma, H^\varphi(L^\bullet) \otimes_A K) = \sum_i (-1)^i \sum_{j=1}^{n'(i)} \xi_j.$$ 

What we must show is that for all $p$-regular elements $g \in G$ (those whose order in $G$, is not divisible by $p$), if $\gamma : L^\bullet \to L^\bullet$ denotes the corresponding automorphism of $L^\bullet$ in $D(A)$, then

$$Br(\gamma, H^\varphi(L^\bullet \otimes^L_A k)) = Tr(\gamma, H^\varphi(L^\bullet) \otimes_A K).$$

Clearly it is enough to show the following statement. For any strictly perfect complex $L^\bullet$ of $A$-modules (not necessarily endowed with a $G$-action in $D(A)$) and for any automorphism $\gamma : L^\bullet \to L^\bullet$ in $D(A)$ which on the cohomology modules induces automorphisms prime to $p$ we have

$$Br(\gamma, H^\varphi(L^\bullet \otimes^L_A k)) = Tr(\gamma, H^\varphi(L^\bullet) \otimes_A K).$$

We use induction on the minimal $m \in \mathbb{Z}_{>0}$ with the following property: after a suitable degree shift we have $L^i = 0$ for all $i \notin [0, m]$. For $m = 0$ the statement is clear from Lemma 2.2(a). Now let $m \geq 1$; shifting degrees we may assume $L^i = 0$
for all \( i \notin [0, m] \). Let \( d^m : L^{m-1} \to L^m \) denote the differential. Choose a sub-\( k \)-vector space \( N_{k}^{m-1} \) of \( L^{m-1} \otimes k \) which under \( d^m \otimes k \) maps isomorphically to the kernel of

\[
L^m \otimes k \to H^m(L^\bullet \otimes k) = H^m(L^\bullet) \otimes k.
\]

Then \( N_{k}^{m-1} = N^{m-1} \otimes k \) for a direct summand \( N^{m-1} \) of \( L^{m-1} \). By construction, \( N^{m-1} \) maps isomorphically to its image \( N^m \) in \( L^m \). Thus, setting \( N^i = 0 \) if \( i \notin \{m - 1, m\} \), the subcomplex \( N^\bullet \) of \( L^\bullet \) is acyclic. Dividing it out we may therefore assume \( L^m \otimes k = H^m(L^\bullet \otimes k) \). Since the functor \( K^- (\text{proj} - A) \to D(A) \) from the homotopy category of complexes of projective \( A \)-modules bounded above to \( D(A) \) is fully faithful, the action of \( \gamma \) on \( L^\bullet \) in \( D(A) \) is in fact represented by a true morphism of complexes \( \gamma^\bullet : L^\bullet \to L^\bullet \). Base changing to a finite extension of \( A \) by a discrete valuation ring (this does not affect the numbers \( Br \) and \( Tr \)) we may suppose that the characteristic polynomial of \( \gamma^m : L^m \to L^m \) splits in \( A \) (we remark that \( \gamma^m \) is bijective: this follows from \( L^m \otimes k = H^m(L^\bullet \otimes k) \) and the fact that \( \gamma \) acts bijectively on \( H^m(L^\bullet \otimes k) \)). We therefore find a \( \gamma^m\)-stable filtration

\[
(0) = F^0 \subset F^1 \subset \ldots \subset F^s = L^m \quad (s = \text{rk}(L^m))
\]

such that \( G^e = F^e/F^{e-1} \) is free of rank one, for any \( 1 \leq e \leq s \). The cyclic \( A \)-module

\[
\frac{F^e}{(F^e \cap \text{im}(d^m)) + F^{e-1}}
\]

is a \( \gamma^m\)-stable subquotient of \( H^m(L^\bullet) \) (it is nonzero because of \( L^m \otimes k = H^m(L^\bullet \otimes k) \)); hence \( \gamma^m \) acts on it by multiplication with a root of unity of order prime to \( p \). Let \( \xi_e \in A^\times \) denote its Teichmüller lifting. Choose \( \ell_e \in F^e \) which represents a basis element of \( G^e \); then \( \ell_1, \ldots, \ell_s \) is a basis of \( L^m \). Modulo \( F^{e-1} \) the class of \( \xi_e \ell_e - \gamma^m(\ell_e) \in F^e \) lies in \( \text{im}(d^m) \). Choose a \( t_e \in L^{m-1} \) with

\[
d^m(t_e) = \xi_e \ell_e - \gamma^m(\ell_e) \text{ modulo } F^{e-1}.
\]

Let \( t : L^m \to L^{m-1} \) denote the \( A \)-linear map which sends \( \ell_e \) to \( t_e \), for each \( 1 \leq e \leq s \). Using \( t \) we see that we may modify \( \gamma^\bullet \) within its homotopy class to achieve that the filtration \( (1) \) is still \( \gamma^m\)-stable and such that \( \gamma^m \) acts on each \( G^e \) by multiplication with a root of unity of prime-to-\( p \) order in \( A^\times \). Therefore we may assume that \( \gamma^m : L^m \to L^m \) is prime to \( p \). Let \( L^i_{\text{st}} = L^m \) and \( L^i_1 = 0 \) for \( i \neq m \). Then \( L^\bullet_{\text{st}} \) is a \( \gamma^\bullet\)-stable subcomplex of \( L^\bullet \) and since \( Br(\gamma) \) and \( Tr(\gamma) \) are additive in exact \( \gamma^\bullet\)-equivariant sequences of complexes it suffices to show \( Br(\gamma) = Tr(\gamma) \) for the complexes \( L^\bullet_{\text{st}} \) and \( L^\bullet_{\text{st}}/L^\bullet_1 \). Since these complexes are shorter than \( L^\bullet \) this follows from the induction hypothesis. Indeed, the prime-to-\( p \) hypothesis is clearly satisfied for \( L^\bullet_1 \), so it remains to show that \( \gamma^\bullet \) induces automorphisms prime to \( p \) on the cohomology modules of \( L^\bullet_{\text{st}}/L^\bullet_1 \). In degrees smaller than \( m - 1 \) this is clear from the corresponding hypothesis on \( L^\bullet \); only \( H^{m-1}(L^\bullet_{\text{st}}/L^\bullet_1) \) is critical. But \( H^{m-1}(L^\bullet) \) is a submodule of \( H^{m-1}(L^\bullet_{\text{st}}/L^\bullet_1) \) and the quotient

\[
Q = H^{m-1}(L^\bullet_{\text{st}}/L^\bullet_1)/H^{m-1}(L^\bullet)
\]

maps isomorphically to a submodule of \( L^m_{\text{st}} = L^m \). By Lemma 2.2(b) it suffices to show that \( \gamma^\bullet \) induces automorphisms prime to \( p \) on \( H^{m-1}(L^\bullet) \) and on \( Q \). For \( H^{m-1}(L^\bullet) \) this holds by hypothesis; for \( Q \) this follows from Lemma 2.2(a).
References


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