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RESTRICTED WEAK TYPE ON MAXIMAL LINEAR AND MULTILINEAR INTEGRAL MAPS

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Abstract. It is shown that multilinear operators of the form $T(f_1, ..., f_k)(x) = \int_{\mathbb{R}^n} K(x, y_1, ..., y_k) f_1(y_1) ... f_k(y_k) dy_1 ... dy_k$ of restricted weak type $(1, ..., 1, q)$ are always of weak type $(1, ..., 1, q)$ whenever the map $x \to K_x$ is a locally integrable $L^1(\mathbb{R}^n)$-valued function.

1. Introduction and the main result

Throughout the paper $0 < q, p_1, ..., p_k < \infty$, $l, k \in \mathbb{N}$, $n_j \in \mathbb{N}$ and $n = n_1 + ... + n_k$. We write $y = (y_1, ..., y_k) \in \mathbb{R}^n = \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$, and $m_n(A)$ denotes the Lebesgue measure in $\mathbb{R}^n$. Given a Banach space X we write $L^0(\mathbb{R}^l, X)$, $L^p(\mathbb{R}^l, X)$ and $L^1_{loc}(\mathbb{R}^l, X)$ for the space of (strongly) measurable functions on $\mathbb{R}^l$ with values in X, Bochner p-integrable functions $(0 < p < \infty)$ and locally Bochner integrable, respectively (we use the notation $L^0(\mathbb{R}^l)$, $L^p(\mathbb{R}^l)$ and $L^1_{loc}(\mathbb{R}^l)$ if $X = \mathbb{C}$).

Let us recall that a multilinear operator $T : L^{p_1}(\mathbb{R}^{n_1}) \times ... \times L^{p_k}(\mathbb{R}^{n_k}) \to L^0(\mathbb{R}^l)$ is continuous if for every measurable set $E \subset \mathbb{R}^l$ of finite measure there exists a function $C_E : (0, \infty) \to \mathbb{R}^+$ with $\lim_{\lambda \to \infty} C_E(\lambda) = 0$ such that

$$m_l(\{x \in E : |T(f_1, ..., f_k)(x)| > \lambda \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\mathbb{R}^{n_i})}\}) \leq C_E(\lambda)$$

for $f_i \in L^{p_i}(\mathbb{R}^{n_i})$, $i = 1, ..., k$.

Particular examples are the operators of weak type $(p_1, ..., p_k, q)$, i.e. those for which there exists $C > 0$ such that

$$m_l^{1/q}(\{x \in \mathbb{R}^l : |T(f_1, ..., f_k)(x)| > \lambda\}) \leq \frac{C}{\lambda} \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\mathbb{R}^{n_i})}.$$

When the previous estimate holds only for characteristic functions of measurable sets, i.e. there exists $C > 0$ such that

$$m_l^{1/q}(\{x \in \mathbb{R}^l : |T(\chi_{E_1}, ..., \chi_{E_k})(x)| > \lambda\}) \leq \frac{C}{\lambda} \prod_{i=1}^{k} m_{n_i}(E_i)^{1/p_i}$$

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for measurable sets E_i in $\mathbb{R}^{n_i}$, the operator is said to be of restricted weak type $(p_1, ..., p_k, q)$.

Lots of examples in Harmonic Analysis turn out to be only of weak type or restricted weak type (see [9]) for some tuples $(p_1, ..., p_k, q)$. It is well known that interpolation techniques allow them to pass from restricted weak type in two different tuples to strong type estimates in intermediate spaces.

In general linear operators of restricted weak type (p, q) need not be of weak type (p, q) (see [8] for the case $p > 1$).

It was first shown by K.H. Moon that convolution and maximal of convolution operators of restricted weak type $(1, 1)$ are always of weak type $(1, 1)$.

Theorem 1.1 ([7])**.** *Let $K_j \in L^1(\mathbb{R}^n)$ for $j \in \mathbb{N}$. Denote $T_j(f) = f * K_j$ and $T^*(f) = \sup_{j\in\mathbb{N}}|T_j(f)|$.*

If T^* is of restricted weak type $(1, q)$ for some $q > 0$, then T^* is also of weak type $(1, q)$ with constant independent of the quantities $\|K_j\|_1$.

Recently Moon's theorem was extended to the multilinear case by L. Grafakos and M. Mastylo.

Theorem 1.2 ([4])**.** *Let $K_j \in L^1((\mathbb{R}^l)^k) \cap L^\infty((\mathbb{R}^l)^k)$ for $j \in \mathbb{N}$. Define*

$$T_j(f_1, ..., f_k)(x) = \int_{(\mathbb{R}^l)^k} K_j(x - y_1, ..., x - y_k) f_1(y_1)...f_k(y_k)dy_1...dy_k$$

and

$$T^*(f_1, ..., f_k)(x) = \sup_{j\in\mathbb{N}}|T_j(f_1, ..., f_k)(x)|$$

for $x \in \mathbb{R}^l$ and $f_i \in L^1(\mathbb{R}^l)$, $i = 1, ..., k$.

If T^* is of restricted weak type $(1, ..., 1, q)$ for some $q > 0$, then T^* is also of weak type $(1, ...1, q)$ with constant independent of the quantities $\|K_j\|_1$ and $\|K_j\|_\infty$.

Although the proofs of the previous theorems work the same for all values of $0 < q < \infty$, I would like to point out that only the case $q \leq 1$ is relevant.

Proposition 1.3. *Let $T_j : L^1(\mathbb{R}^{n_1}) \times ... \times L^1(\mathbb{R}^{n_k}) \to L^0(\mathbb{R}^l)$ be a sequence of continuous multilinear operators and set $T^*(f_1, ..., f_k) = \sup_{j\in\mathbb{N}} |T_j(f_1, ..., f_k)|$.*

If $q > 1$ and T^* is of restricted weak type $(1, ..., 1, q)$, then T^* is of weak type $(1, ..., 1, q)$.

Proof. It is known that *weak-*$L^q(\mathbb{R}^n)$ is a complete normed space for $q > 1$ (see [9]). Hence there exists a norm $\|.\|_{L^{q,\infty}(\mathbb{R}^l)}$ such that

$$\|g\|_{L^{q,\infty}(\mathbb{R}^l)} \approx \sup_{\lambda>0} \lambda m_l^{1/q}(\{x \in \mathbb{R}^l : |g(x)| > \lambda\}).$$

Therefore, if $f_i = \sum_{j=1}^{M_i} \alpha_j^i \chi_{E_j^i}$ for pairwise disjoint measurable sets $E_j^i \subset \mathbb{R}^{n_i}$, $1 \leq i \leq k$, then

$$\begin{aligned}
& \lambda m_l^{1/q}(\{x \in \mathbb{R}^l : \sup_{j\in\mathbb{N}} |T_j(f_1, ..., f_k)(x)| > \lambda\}) \\
\leq\ & C\|\sup_j |\sum_{i=1}^{k}\sum_{j_i=1}^{M_i} T_j(\alpha_{j_1}^1 \chi_{E_{j_1}^1}, ..., \alpha_{j_k}^k \chi_{E_{j_k}^k})|\|_{L^{q,\infty}} \\
\leq\ & C\|\sum_{i=1}^{k}\sum_{j_i=1}^{M_i} \sup_j |T_j(\alpha_{j_1}^1 \chi_{E_{j_1}^1}, ..., \alpha_{j_k}^k \chi_{E_{j_k}^k})|\|_{L^{q,\infty}} \\
\leq\ & C\sum_{i=1}^{k}\sum_{j_i=1}^{M_i} |\alpha_{j_1}^1|...|\alpha_{j_k}^k| \|\sup_j |T_j(\chi_{E_{j_1}^1}, ..., \chi_{E_{j_k}^k})\|_{L^{q,\infty}} \\
\leq\ & C'\sum_{i=1}^{k}\sum_{j_i=1}^{M_i} |\alpha_{j_1}^1|...|\alpha_{j_k}^k| m_{n_1}(E_{j_1}^1)...m_{n_k}(E_{j_k}^k) \\
=\ & C'\prod_{i=1}^{k} \|f_i\|_{L^1(\mathbb{R}^{n_i})}.
\end{aligned}$$

□

On the other hand, it was shown by M. Akcoglu, J. Baxter, A. Bellow and R.L. Jones that, in the linear case, if we replace $\mathbb{R}$ by $\mathbb{Z}$ the Moon's result is no longer true.

Theorem 1.4 ([1])**.** *There exists a countable set $\mathcal{C}$ of probability densities on $\mathbb{Z}$ such that $\mathcal{M}_{\mathcal{C}} f = \sup_{g\in\mathcal{C}} g * f$ for non-negative $f \in \ell^1(\mathbb{Z})$ is of restricted weak type $(1,1)$ but not of weak type $(1,1)$*

Making use of such a construction and the transference principle due to A. Calderón (see [2]), P.H. Hagelstein and R.L. Jones have recently shown the following.

Theorem 1.5 ([6])**.** *There exists a sequence of translation invariant operators T_j acting on $L^1(\mathbb{T})$ such that $T^*(f) = \sup_{j\in\mathbb{N}} |T_j(f)|$ is of restricted weak type $(1,1)$ but it is not of weak type $(1,1)$.*

The operators in [6] are given by

$$T_j(f)(e^{i\theta}) = \sum_{k\in\mathbb{Z}} w_j(k) f(e^{i(\theta+k)})$$

for a sequence $\{w_j\}$ of probability measures on $\mathbb{Z}$ with finite support. In other words, for $K_j = \sum_{k\in\mathbb{Z}} w_j(k)\delta_{-k} \in M(\mathbb{T})$ we have

$$T_j(f)(e^{i\theta}) = K_j * f(e^{i\theta}) = \int_{\mathbb{T}} K_j(e^{i(\theta-\theta')}) f(e^{i(\theta')}) \frac{d\theta'}{2\pi}.$$

The aim of this paper is to exhibit a general class of the continuous multilinear operators $T_j : L^1(\mathbb{R}^{n_1}) \times ... \times L^1(\mathbb{R}^{n_k}) \to L^0(\mathbb{R}^l)$ for which the restricted $(1, ..., 1, q)$-weak type of $T^*(f_1, ..., f_k) = \sup_{j\in\mathbb{N}} |T_j(f_1, ..., f_k)|$ implies the $(1, ..., 1, q)$-weak type of T^*. We shall restrict ourselves to the class of operators T_j given by

$$T_j(f_1, ..., f_k)(x) = \int_{\mathbb{R}^n} K_j(x, y_1, ..., y_k) f_1(y_1)...f_k(y_k) dy_1...dy_k,$$

where $K_j : \mathbb{R}^l \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_1} \to \mathbb{C}$ is measurable.

Let us start by mentioning some weak assumptions for the integral above to be well defined for almost all $x \in \mathbb{R}^l$.

Definition 1.6. Let $T : L^1(\mathbb{R}^{n_1}) \times ... \times L^1(\mathbb{R}^{n_k}) \to L^0(\mathbb{R}^l)$ be a continuous multilinear operator. We shall say that T is an integral operator with kernel K if there exists a measurable function $K : \mathbb{R}^l \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k} \to \mathbb{C}$ such that $K^0(x) = K_x$ given by

$$K_x(y_1, ..., y_k) = K(x, y_1, ..., y_k)$$

is a strongly measurable $L^1(\mathbb{R}^n)$-valued function, i.e. $K^0 \in L^0(\mathbb{R}^l, L^1(\mathbb{R}^n))$, and

$$T(f_1, ..., f_k)(x) = \int_{\mathbb{R}^n} K(x, y_1, ..., y_k) f_1(y_1) ... f_k(y_k) dy_1 ... dy_k$$

for almost all $x \in \mathbb{R}^l$ and $f_i \in L^\infty(\mathbb{R}^{n_i})$ for $i = 1, ..., k$.

We shall write $T = T_K$.

Remark 1.1. If $K : \mathbb{R}^l \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k} \to \mathbb{C}$ is measurable and $K^0 \in L^p(\mathbb{R}^l, L^1(\mathbb{R}^n))$ for some $1 \le p \le \infty$, then $T_K : L^1(\mathbb{R}^{n_1}) \times ... \times L^1(\mathbb{R}^{n_k}) \to L^p(\mathbb{R}^l)$ is bounded and

$$\|T_K(f_1, ..., f_k)\|_{L^p(\mathbb{R}^l)} \le \|K^0\|_{L^p(\mathbb{R}^l, L^1(\mathbb{R}^n))} \prod_{i=1}^k \|f_i\|_{L^1(\mathbb{R}^{n_i})}.$$

Now we state the main result of the paper.

Theorem 1.7. *Let $0 < q \le 1$ and let T_j be a sequence of continuous multilinear operators from $L^1(\mathbb{R}^{n_1}) \times ... \times L^1(\mathbb{R}^{n_k}) \to L^0(\mathbb{R}^l)$ with kernels K_j such that*

$$K_j^0 \in L^1_{loc}(\mathbb{R}^l, L^1(\mathbb{R}^n)). \tag{1.1}$$

Let $T^*(f_1, ..., f_k)(x) = \sup_{j \in \mathbb{N}} |\int_{\mathbb{R}^n} K_j(x, y_1, ..., y_k) f_1(y_1) ... f_k(y_k) dy_1 ... dy_k|$ for $f_j \in L^\infty(\mathbb{R}^{n_j})$, $1 \le j \le k$.

If T^* is of restricted weak type $(1, ..., 1, q)$, then T^* is of weak type $(1, ..., 1, q)$.

Let us mention that our result gives the following corollary (which seems to be new even in the linear case) when applied to a single operator.

Corollary 1.8. *Let $0 < q \le 1$ and let $T : L^1(\mathbb{R}^n) \times ... \times L^1(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ be multilinear with kernel K such that $K^0 \in C(\mathbb{R}^n, L^1(\mathbb{R}^n))$.*

Then $T_K(f_1, ..., f_k)(x) = \int_{(\mathbb{R}^n)^k} K_j(x, y_1, ..., y_k) f_1(y_1) ... f_k(y_k) dy_1 ... dy_k$ is of restricted weak type $(1, ..., 1, q)$ if and only if it is of weak type $(1, ..., 1, q)$.

Let us mention some particular examples where Corollary 1.8 or its maximal formulation can be applied.

Proposition 1.9. *Let $k \ge 1$, let $n_1 = ... = n_k = l$ (hence $n = kl$), let $\phi \in L^1(\mathbb{R}^n)$ and let Φ be a real-valued function uniformly continuous on $\mathbb{R}^l \times \mathbb{R}^n$. Define $K : \mathbb{R}^l \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k} \to \mathbb{C}$ by*

$$K(x, y_1, ..., y_k) = e^{i\Phi(x, y_1, ..., y_k)} \phi(x - y_1, ..., x - y_k).$$

Then $K^0 : \mathbb{R}^l \to L^1(\mathbb{R}^n)$ is uniformly continuous and bounded.

Proof. Clearly $\|K_x\|_{L^1(\mathbb{R}^n)} = \|\phi\|_{L^1(\mathbb{R}^n)}$ for all $x \in \mathbb{R}^n$.

Given $\epsilon > 0$ take $\delta > 0$ so that $|e^{i\Phi(x, y_1, ..., y_k)} - e^{i\Phi(x', y_1', ..., y_k')}| < \epsilon$ whenever $|x - x'| + \sum_{i=1}^k |y_i - y_i'| < \delta$.

Denoting $\tau'_x\phi(y_1,...,y_k) = \phi(x-y_1,...,x-y_k)$, if for $|x-x'|<\delta$, then

$$\begin{aligned}\|K_x - K_{x'}\|_1 &\leq \int_{\mathbb{R}^l} |\phi(x-y_1,...,x-y_k) - \phi(x'-y_1,...,x'-y_k)|dy_1...dy_k \\ &+ \int_{\mathbb{R}^l} |e^{i\Phi(x,y_1,...,y_k)} - e^{i\Phi(x',y_1,...,y_k)}||\phi(x-y_1,...,x-y_k)|dy_1...dy_k \\ &\leq \|\tau'_{x-x'}\phi - \phi\|_1 + \epsilon\|\phi\|_1.\end{aligned}$$

Now use the fact that $x \to \tau'_x\phi$ is uniformly continuous from $\mathbb{R}^l$ into $L^1(\mathbb{R}^n)$ to finish the proof. □

In particular one obtains Theorems 1.1 and 1.2 as particular cases of Theorem 1.7.

2. Proof of the main theorem

Let us first establish the approximation lemmas to be used in the proof. Denote, as usual, $\varphi_t(u) = \frac{1}{t^n}\varphi(\frac{u}{t})$ for $u \in \mathbb{R}^j (j \in \mathbb{N})$ and $t > 0$.

The proof of the following result is the same as in the scalar-valued case, and it is left to the reader (see [9]).

Lemma 2.1. *Let X be a Banach space and let $\Phi \in L^1(\mathbb{R}^l, X)$. Let P_t denote the Poisson kernel in $\mathbb{R}^l$, that is, $P_t(x) = \frac{t}{(t^2+|x|^2)^{\frac{l+1}{2}}}$. Then*

$$\Phi_t = P_t * \Phi(x) = \int_{\mathbb{R}^l} \Phi(x-u)P_t(u)du \in C_0(\mathbb{R}^l, X) \cap L^1(\mathbb{R}^l, X), \tag{2.1}$$

$$\sup_{t>0} \|\Phi_t\|_{L^1(\mathbb{R}^l,X)} = \|\Phi\|_{L^1(\mathbb{R}^l,X)}, \tag{2.2}$$

$$\lim_{t\to 0} \|\Phi_t - \Phi\|_{L^1(\mathbb{R}^l,X)} = 0, \tag{2.3}$$

$$\lim_{t\to 0} \|\Phi_t(x) - \Phi(x)\|_X = 0 \text{ for almost all } x \in \mathbb{R}^l. \tag{2.4}$$

Lemma 2.2. *Let $\phi \in C_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $\phi \geq 0$, and $\int_{\mathbb{R}^n} \phi(y)dy = 1$.*

If $\mathcal{K}$ is a relatively compact set in $L^1(\mathbb{R}^n)$, then

$$\lim_{t\to 0} \sup_{F\in\mathcal{K}} \|\phi_t * F - F\|_{L^1(\mathbb{R}^n)} = 0. \tag{2.5}$$

For each $t > 0$ the family $\{\phi_t * F : F \in \mathcal{K}\}$ is equicontinuous, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{F\in\mathcal{K}} |\phi_t * F(y') - \phi_t * F(y)| < \epsilon, \qquad |y'-y| < \delta. \tag{2.6}$$

Proof. It is known (see [3], Theorem 4.8.20) that a set $\mathcal{K} \subset L^1(\mathbb{R}^n)$ is relatively compact if and only if $\mathcal{K}$ is bounded,

$$\lim_{y\to 0} \sup_{F\in\mathcal{K}} \|\tau_y F - F\|_{L^1(\mathbb{R}^n)} = 0, \tag{2.7}$$

where $\tau_y F(y') = F(y'-y)$ and

$$\lim_{M\to\infty} \sup_{F\in\mathcal{K}} \int_{|y'|>M} |F(y')|dy' = 0. \tag{2.8}$$

Using the standard approach one obtains the estimate

$$\begin{aligned} |\phi_t * F(y') - F(y')| &\leq \int_{|y|<\delta} |F(y'-y) - F(y')|\phi_t(y)dy \\ &+ \int_{|y|\geq\delta} |F(y'-y) - F(y')|\phi_t(y)dy. \end{aligned}$$

This leads to

$$\begin{aligned} \|\phi_t * F - F\|_{L^1(\mathbb{R}^n)} &\leq \int_{|y|<\delta} \|\tau_y F - F\|_{L^1(\mathbb{R}^n)}\phi_t(y)dy \\ &+ 2\|F\|_{L^1(\mathbb{R}^n)} \int_{|y|\geq\delta} \phi_t(y)dy \\ &\leq \sup_{|y|<\delta} \|\tau_y F - F\|_{L^1(\mathbb{R}^n)} + 2\|F\|_{L^1(\mathbb{R}^n)} \int_{|y|\geq\frac{\delta}{t}} \phi(y)dy. \end{aligned}$$

Given $\epsilon > 0$, using (2.7) there exists $\delta > 0$ so that

$$\sup_{|y|<\delta} \sup_{F\in\mathcal{F}} \|\tau_y F - F\|_{L^1(\mathbb{R}^n)} < \epsilon/2.$$

For such a δ one has

$$\sup_{F\in\mathcal{F}} \|\phi_t * F - F\|_{L^1(\mathbb{R}^n)} \leq \epsilon/2 + 2\sup_{F\in\mathcal{F}} \|F\|_{L^1(\mathbb{R}^n)} \int_{|y|\geq\frac{\delta}{t}} \phi(y)dy.$$

Taking limits as $t \to 0$ one gets (2.5).

To obtain (2.6) use (2.7) and the estimate

$$|\phi_t * F(y') - \phi_t * F(y)| \leq \frac{\|\phi\|_\infty}{t} \int_{\mathbb{R}^n} |F(y'-u) - F(y-u)|du.$$

□

Proof of Theorem 1.7. Assume T^* is of restricted weak type $(1, ..., 1, q)$. Let $N \in \mathbb{N}, \lambda > 0$ and let $f_i \neq 0$ be a non-negative simple function on $\mathbb{R}^{n_i}$ for $1 \leq i \leq k$ and denote $f(y) = f_1(y_1)...f_k(y_k)$.

Let us show that there exists $C > 0$ (independent of N)

$$m_l^{1/q}(\{|x| \leq N : \sup_{1\leq j\leq N} |T_{K_j}(f_1, ..., f_k)(x)| > \lambda\}) \leq \frac{C}{\lambda} \prod_{i=1}^k \|f_i\|_{L^1(\mathbb{R}^{n_i})}. \tag{2.9}$$

Let $t > 0$ and $1 \leq j \leq N$ and let us use the notation

$$K_{j,N}(x, y) = K_j(x, y)\chi_{\{|x|\leq N\}}(x),$$

$$\tilde{K}_{t,j,N}(x, y_1, ..., y_k) = \int_{\mathbb{R}^l} P_t(x-u)K_{j,N}(u, y_1, ..., y_k)du.$$

Consider the Banach space

$$X_N = L^1(\mathbb{R}^n, \ell_\infty^N) = \{(g_j)_{j=1}^N : \int_{\mathbb{R}^n} \sup_{1\leq j\leq N} |g_j(y)|dy < \infty\}.$$

Now define $\Phi_N : \mathbb{R}^l \to X_N$ given by

$$\Phi_N(x) = (K_j^0(x)\chi_{\{|x|\leq N\}})_{j=1}^N. \tag{2.10}$$

From the assumption (1.1) one has $K_{j,N}^0 \in L^1(\mathbb{R}^l, L^1(\mathbb{R}^n))$. Hence $\Phi_N \in L^1(\mathbb{R}^l, X_N)$ and $(\tilde{K}_{t,j,N}^0)_{j=1}^N = P_t * \Phi_N$.

Taking into account (2.4) in Lemma 2.1 one obtains that there exists $A \subset \mathbb{R}^l$ with $m_l(A) = 0$, and if $x \notin A$, then

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \sup_{1\le j\le N} |P_t * K_{j,N}(x, y_1, ..., y_k) - K_{j,N}(x, y_1, ..., y_k)| dy_1...dy_k = 0.$$

Therefore, if $x \notin A$, then

$$\lim_{t\to 0} \sup_{1\le j\le N} |T_{\tilde{K}_{t,j,N}}(f_1, ..., f_k)(x)| = \sup_{1\le j\le N} |T_{K_{j,N}}(f_1, ..., f_k)(x)|.$$

Now, for any $\eta > 0$,

$$\begin{aligned} & m_l(\{|x| \le N : \sup_{1\le j\le N} |T_{K_j}(f_1, ..., f_k)(x)| > \eta\}) \\ = \quad & m_l(\{x \notin A : \sup_{1\le j\le N} |T_{K_{j,N}}(f_1, ..., f_k)(x)| > \eta\}) \\ \le \quad & \liminf_{M\to\infty} m_l(\{|x| \le N : \sup_{1\le j\le N} |T_{\tilde{K}_{1/M,j,N}}(f_1, ..., f_k)(x)| > \eta\}). \end{aligned}$$

Let $M \in \mathbb{N}$ be fixed. Using (2.1) in Lemma 2.1 one has that

$$\tilde{K}^0_{1/M,j,N} : \{|x| \le N\} \to L^1(\mathbb{R}^n)$$

is continuous for all $1 \le j \le N$. Hence $\mathcal{F}_{i,j,N} = \{(\tilde{K}_{1/M,j,N})_x : |x| \le N\}$ is relatively compact in $L^1(\mathbb{R}^n)$ for each $1 \le j \le N$. Select, for instance, $\phi(y) = \frac{1}{(1+|y|^2)^{(n+1)/2}}$ in Lemma 2.2 and define $H^t_{M,j}(x, y) = \phi_t * (\tilde{K}_{1/M,j,N})_x$ for $1 \le j \le N$. Let us denote

$$T_{H^t_{M,j}}(f_1, ..., f_k)(x) = \int_{\mathbb{R}^n} H^t_{M,j}(x, y_1, ..., y_k) f_1(y_1)...f_k(y_k) dy_1...dy_k.$$

If $1 \le j \le N$ and $|x| \le N$, then

$$\begin{aligned} & |T_{\tilde{K}_{1/M,j,N}}(f_1, ..., f_k)(x) - T_{H^t_{M,j}}(f_1, ..., f_k)(x)| \\ \le \quad & \int_{\mathbb{R}^n} |\tilde{K}_{1/M,j,N}(x, y) - H^t_{M,j}(x, y)||f(y)|dy \\ \le \quad & \|f\|_{L^\infty(\mathbb{R}^n)} \|\phi_t * (\tilde{K}_{1/M,j,N})_x - (\tilde{K}_{1/M,j,N})_x\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

For a given $\epsilon > 0$, from (2.5), there exists $t = t(M) > 0$ such that

$$\sup_{1\le j\le N, |x|\le N} \|(\tilde{K}_{1/M,j,N})_x - \phi_t * (\tilde{K}_{1/M,j,N})_x\|_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{\|f\|_{L^\infty(\mathbb{R}^n)}}.$$

Therefore

$$|T_{\tilde{K}_{1/M,j,N}}(f_1, ..., f_k)(x) - T_{H^t_{M,j}}(f_1, ..., f_k)(x)| < \epsilon. \tag{2.11}$$

From (2.6) there exists $\delta > 0$ such that

$$\sup_{1\le j\le N, |x|\le N} |H^t_{M,j}(x, y) - H^t_{M,j}(x, y')| < \frac{\epsilon}{\|f\|_{L^1(\mathbb{R}^n)}}, \qquad |y - y'| < \sqrt{n}\delta. \tag{2.12}$$

Now consider, for $1 \le i \le k$, $\mathbb{R}^{n_i} = \bigcup_{s\in\mathbb{N}} I^{(i)}_s$, where $I^{(i)}_s$ are disjoint cubes with length side δ (in particular, $m_{n_i}(I^{(i)}_s) = \delta^{n_i}$ and $diam(I^{(i)}_s) < \sqrt{n_i}\delta$) and write $f_i = \sum_{s=1}^{M_i} \alpha^{(i)}_s \chi_{I^{(i)}_s}$ for some $\alpha^{(i)}_s > 0$. Denote $\alpha^{(i)} = \|f_i\|_\infty$. Since $\alpha^{(i)}_s \le \alpha^{(i)}$ and the Lebesgue measure is non-atomic we can then find $J^{(i)}_s \subset I^{(i)}_s$ such that $\alpha^{(i)} m_{n_i}(J^{(i)}_s) = \alpha^{(i)}_s m_{n_i}(I^{(i)}_s) = \alpha^{(i)}_s \delta^{n_i}$. Hence if $E^{(i)} = \bigcup_{s=1}^{M_i} J^{(i)}_s$ one gets $\|f_i\|_1 = \alpha^{(i)} m(E^{(i)})$. Denote $E = E^{(1)} \times ... \times E^{(k)}$.

Therefore, denoting $I_{(j_1,...,j_k)} = I^{(1)}_{j_1} \times ... \times I^{(k)}_{j_k}$ and $J_{(j_1,...,j_k)} = J^{(1)}_{j_1} \times ... \times J^{(k)}_{j_k}$ for $1 \leq j_l \leq M_l$ and $1 \leq l \leq k$, one gets

$$\begin{aligned}
& T_{H^t_{M,j}}(f_1,...,f_k)(x) - T_{H^t_{M,j}}(\alpha^{(1)}\chi_{E^{(1)}},...,\alpha^{(k)}\chi_{E^{(k)}})(x) \\
= & \sum_{j_1=1}^{M_1} ... \sum_{j_k=1}^{M_k} \Big(\alpha^{(1)}_{j_1} ... \alpha^{(k)}_{j_k} T_{H^t_{M,j}}(\chi_{I^{(1)}_{j_1}},...,\chi_{I^{(k)}_{j_k}})(x) \\
- & \alpha^{(1)} ... \alpha^{(k)} T_{H^t_{M,j}}(\chi_{J^{(1)}_{j_1}},...,\chi_{J^{(k)}_{j_k}})(x) \Big) \\
= & \sum_{j_1=1}^{M_1} ... \sum_{j_k=1}^{M_k} \Big(\alpha^{(1)}_{j_1} ... \alpha^{(k)}_{j_k} \int_{I_{(j_1,...,j_k)}} H^t_{M,j}(x,y_1,...,y_k)dy \\
- & \alpha^{(1)} ... \alpha^{(k)} \int_{J_{(j_1,...,j_k)}} H^t_{M,j}(x,y_1,...,y_k)dy \Big).
\end{aligned}$$

Now, denoting $\alpha_{(j_1,...,j_k)} = \alpha^{(1)}_{j_1} ... \alpha^{(k)}_{j_k}$ and $\alpha = \alpha^{(1)} ... \alpha^{(k)}$ one has that $\alpha_{(j_1,...,j_k)}\delta^n = \alpha_{(j_1,...,j_k)} m_n(I_{(j_1,...,j_k)}) = \alpha m_n(J_{(j_1,...,j_k)})$. Therefore

$$\begin{aligned}
& T_{H^t_{M,j}}(f_1,...,f_k)(x) - T_{H^t_{M,j}}(\alpha^{(1)}\chi_{E^{(1)}},...,\alpha^{(k)}\chi_{E^{(k)}})(x) \\
= & \sum_{j_1=1}^{M_1} ... \sum_{j_k=1}^{M_k} \alpha_{(j_1,...,j_k)} m_n(I_{(j_1,...,j_k)}) \\
\times & \Big(\frac{1}{m_n(I_{(j_1,...,j_k)})} \int_{I_{(j_1,...,j_k)}} H^t_{M,j}(x,y_1,...,y_k)dy \\
- & \frac{1}{m_n(J_{(j_1,...,j_k)})} \int_{J_{(j_1,...,j_k)}} H^t_{M,j}(x,y_1,...,y_k)dy \Big).
\end{aligned}$$

Now observe that $y \in I_{(j_1,...,j_k)}$ and $y' \in J_{(j_1,...,j_k)}$; then $|y - y'| < \sqrt{n}\delta$. Hence (2.12) shows that, for $1 \leq j \leq N$ and $|x| \leq N$,

$$\begin{aligned}
& |T_{H^t_{M,j}}(f_1,...,f_k)(x) - \alpha T_{H^t_{M,j}}(\chi_{E^{(1)}},...,\chi_{E^{(k)}})(x)| \leq \\
\leq & \sum_{j_1=1}^{M_1} ... \sum_{j_k=1}^{M_k} \alpha_{(j_1,...,j_k)} m_n(I_{(j_1,...,j_k)}) \Big(\frac{1}{m_n(I_{(j_1,...,j_k)}) m_n(J_{(j_1,...,j_k)})} \\
\times & \int_{I_{(j_1,...,j_k)}} \int_{J_{(j_1,...,j_k)}} |H^t_{M,j}(x,y) - H^t_{M,j}(x,y')| dy dy' \Big) \\
\leq & \frac{\epsilon}{\|f\|_1} \prod_{i=1}^{k} \Big(\sum_{j_i=1}^{M_i} \alpha^{(i)}_{j_i} m(I^{(i)}_{j_i}) \Big) \\
= & \frac{\epsilon}{\|f\|_1} \prod_{i=1}^{k} \|f_i\|_1 = \epsilon.
\end{aligned}$$

Therefore, using (2.11) and the previous estimate one gets

$$\begin{aligned}
& m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{\tilde{K}_{1/M,j,N}}(f_1,...,f_k)(x)|>\lambda+3\epsilon\}) \\
\le\ & m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{H^t_{M,j}}(f_1,...,f_k)(x)|>\lambda+2\epsilon\}) \\
\le\ & m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{H^t_{M,j}}(\alpha^{(1)}\chi_{E^{(1)}},...,\alpha^{(k)}\chi_{E^{(k)}})(x)|>\lambda+\epsilon\}) \\
\le\ & m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{\tilde{K}_{1/M,j,N}}(\chi_{E^{(1)}},...,\chi_{E^{(k)}})(x)|>\frac{\lambda}{\alpha}\}).
\end{aligned}$$

From the restricted weak type assumption we conclude that

$$\begin{aligned}
& m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{\tilde{K}_{1/M,j,N}}(\chi_{E^{(1)}},...,\chi_{E^{(k)}})(x)|>\frac{\lambda}{\alpha}\}) \\
\le\ & m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{\tilde{K}_j}(\chi_{E^{(1)}},...,\chi_{E^{(k)}})(x)|>\frac{\lambda}{2\alpha}\}) \\
+\ & m_l(\{|x|\le N: \|P_{1/M}*\Phi_N(x)-\Phi_N(x)\|_{X_N}>\frac{\lambda}{2\alpha}\}) \\
\le\ & m_l(\{|x|\le N: \sup_{1\le j\le N}|T_{\tilde{K}_j}(\chi_{E^{(1)}},...,\chi_{E^{(k)}})(x)|>\frac{\lambda}{2\alpha}\}) \\
+\ & \frac{2\alpha}{\lambda}\|P_{1/M}*\Phi_N-\Phi_N\|_{L^1(\mathbb{R}^l,X_N)} \\
\le\ & C\frac{\alpha^q}{\lambda^q}m_n^q(E)+\frac{2\alpha}{\lambda}\|P_{1/M}*\Phi_N-\Phi_N\|_{L^1(\mathbb{R}^l,X_N)} \\
=\ & C\frac{\|f\|^q_{L^1(\mathbb{R}^n)}}{\lambda^q}+\frac{2\alpha}{\lambda}\|P_{1/M}*\Phi_N-\Phi_N\|_{L^1(\mathbb{R}^l,X_N)}.
\end{aligned}$$

Taking $\liminf_{M\to\infty}$ and combining all the previous estimates one gets

$$m_l^{1/q}(\{|x|\le N: \sup_{1\le j\le N}|T_{K_j}(f_1,...,f_k)(x)|>\lambda+3\epsilon\})\le C\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda^q}.$$

Finally, since $\epsilon>0$ is arbitrary one gets (2.9).

Using the fact that $\{|x|\le N: \sup_{1\le j\le N}|T_{K_j}(f_1,...,f_k)(x)|>\lambda\}$ is an increasing sequence, one concludes that

$$m_l^{1/q}(\{x\in\mathbb{R}^l: |T^*(f_1,...,f_k)(x)|>\lambda\})\le C\frac{\prod_{i=1}^k\|f_i\|_{L^1(\mathbb{R}^{n_i})}}{\lambda} \tag{2.13}$$

for all simple functions $f_i\ge 0$, $1\le i\le k$.

Let us now extend (2.13) for integrable functions f_i. Let $f_i\ge 0$ be an arbitrary integrable function in $L^1(\mathbb{R}^{n_i})$ with $\|f_i\|_1=1$ for $i=1,...,k$.

For each $N,j\in\mathbb{N}$ denote

$$C_{j,N}(\lambda)=\sup\{m_l(\{|x|\le N: |T_{K_j}(g_1,...,g_k)(x)|>\lambda\}): \|g_i\|_{L^1(\mathbb{R}^{n_i})}\le 1\}.$$

Given $\epsilon>0$ there exists $\lambda_0>0$ such that $C_{j,N}(\eta)<\frac{\epsilon}{kN}$ for $\eta>\lambda_0$ and $1\le j\le N$.

On the other hand, for each $1\le i\le k$, $N\in\mathbb{N}$ we can find a simple $f_N^{(i)}\ge 0$ such that $f_N^{(i)}\le f_i$ and $\|f_i-f_N^{(i)}\|_{L^1(\mathbb{R}^{n_i})}<\frac{\epsilon}{\lambda_0}$.

Denote

$$B_j^{(1)}(N)=\{x\in\mathbb{R}^l: |T_{K_j}(f_1-f_N^{(1)},f_2,...,f_k)|>\epsilon\},$$

and, for $2 \leq i \leq N$,

$$B_j^{(i)}(N) = \{x \in \mathbb{R}^l : |T_{K_j}(f_N^{(1)}, ..., f_N^{(i-1)}, f_i - f_N^{(i)}, f_{i+1}, ..., f_k)| > \epsilon\}.$$

Set $B^{(i)}(N) = \bigcup_{j=1}^N B_j^{(i)}(N)$ and $B(N) = \bigcup_{i=1}^k B^{(i)}(N)$. Note that

$$\begin{aligned} m_l(B(N)) &\leq \sum_{j=1}^N \sum_{i=1}^k m_l(B_j^{(i)}(N)) \\ &\leq \sum_{j=1}^N \sum_{i=1}^k C_{j,N}(\epsilon/\|f_i - f_N^{(i)}\|_{L^1(\mathbb{R}^{n_i})}) < \epsilon. \end{aligned}$$

Since

$$\begin{aligned} T_{K_j}(f_1, ..., f_k) &= T_{K_j}(f_1 - f_N^{(1)}, f_2, ..., f_k) \\ &+ \sum_{i=2}^{k-1} T_{K_j}(f_N^{(1)}, ..., f_N^{(i-1)}, f_i - f_N^{(i)}, f_{i+1}, ..., f_k) \\ &+ T_{K_j}(f_N^{(1)}, ..., f_N^{(k-1)}, f_k - f_N^{(k)}) \\ &+ T_{K_j}(f_N^{(1)}, ..., f_N^{(k)}), \end{aligned}$$

then, for each $x \notin B(N)$, one has

$$\begin{aligned} \sup_{1 \leq j \leq N} |T_{K_j}(f_1, ..., f_k)(x)| &\leq \sup_{1 \leq j \leq N} |T_{K_j}(f_N^{(1)}, ..., f_N^{(k)})(x)| + k\epsilon \\ &\leq T^*(f_N^{(1)}, ..., f_N^{(k)})(x) + k\epsilon. \end{aligned}$$

Therefore

$$\begin{aligned} & m_l(\{x \in \mathbb{R}^l : \sup_{1 \leq j \leq N} |T_{K_j}(f_1, ..., f_k)(x)| > \lambda + k\epsilon\}) \\ \leq\ & m_l(\{x \notin B(N) : \sup_{1 \leq j \leq N} |T_{K_j}(f_1, ..., f_k)(x)| > \lambda\}) + m_l(B(N)) \\ \leq\ & m_l(\{x \notin B(N) : T^*(f_N^{(1)}, ..., f_N^{(k)})(x) > \lambda\}) + \epsilon \\ \leq\ & (\frac{C}{\lambda} \prod_{i=1}^k \|f_N^{(i)}\|_{L^1(\mathbb{R}^{n_i})})^q + \epsilon \\ \leq\ & \frac{C^q}{\lambda^q}(1 + \frac{\epsilon}{\lambda_0})^{qk} + \epsilon. \end{aligned}$$

From this, the fact $\sup_{1 \leq j \leq N} |T_{K_j}(f_1, ..., f_k)| \leq \sup_{1 \leq j \leq N+1} |T_{K_j}(f_1, ..., f_k)|$ and multilinearity we conclude that, for non-negative integrable functions f_i,

$$m_l^{1/q}(\{x \in \mathbb{R}^l : |T^*(f_1, ..., f_k)(x)| > \lambda\}) \leq \frac{C}{\lambda} \prod_{i=1}^k \|f_i\|_{L^1(\mathbb{R}^{n_i})}.$$

The case of complex-valued functions in now immediate using the multilinearity of the operators. □

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