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ALGEBRAIC REFLEXIVITY OF LINEAR TRANSFORMATIONS

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Abstract. Let $\mathcal{L}(U,V)$ be the set of all linear transformations from U to V, where U and V are vector spaces over a field $\mathbb{F}$. We show that every n-dimensional subspace of $\mathcal{L}(U,V)$ is algebraically $\lfloor\sqrt{2n}\rfloor$-reflexive, where $\lfloor\ t\ \rfloor$ denotes the largest integer not exceeding t, provided n is less than the cardinality of $\mathbb{F}$.

1. Introduction

Let U and V be vector spaces over a field $\mathbb{F}$, let $\mathcal{L}(U,V)$ be the set of all linear transformations from U to V, and let $\mathcal{L}_F(U,V)$ be the set of finite rank transformations in $\mathcal{L}(U,V)$. For any $x \in U$ and subspace $\mathcal{S} \subseteq \mathcal{L}(U,V)$, let $\mathcal{S}x = \{Ax : A \in \mathcal{S}\}$. Define $ref_a(\mathcal{S}) = \{T \in \mathcal{L}(U,V) : Tx \in \mathcal{S}x, \text{ for all } x \in U\}$. $\mathcal{S}$ is called algebraically reflexive if $ref_a(\mathcal{S}) = \mathcal{S}$. Define $\mathcal{S}^{(n)} = \{S^{(n)} \in \mathcal{L}(U^{(n)}, V^{(n)}) : S \in \mathcal{S}\}$, where $U^{(n)}$ is the direct sum of n copies of U, $V^{(n)}$ is the direct sum of n copies of V, and $S^{(n)}$ is the direct sum of n copies of S. $\mathcal{S}$ is called algebraically n-reflexive if $\mathcal{S}^{(n)}$ is algebraically reflexive in $\mathcal{L}(U^{(n)}, V^{(n)})$. Clearly, if $\mathcal{S}$ is algebraically n-reflexive and $n < m$, then $\mathcal{S}$ is algebraically m-reflexive. A vector $x \in U$ is called a separating vector of $\mathcal{S}$ if the evaluation map $E_x : A \to Ax, A \in \mathcal{S}$ is injective. It is well known and easy to prove that if $\mathcal{S}$ has a separating vector, then it is algebraically 2-reflexive. The local dimension of $\mathcal{S}$, denoted by $k(\mathcal{S})$, is defined by $k(\mathcal{S}) = max\{dim(\mathcal{S}x) : x \in U\}$; clearly $k(\mathcal{S}) \leq dim\ \mathcal{S}$. If $dim\ \mathcal{S} < \infty$, then $k(\mathcal{S}) = dim\ \mathcal{S}$ if and only if $\mathcal{S}$ has a separating vector. Thus, the concept of local dimension generalizes that of separating vectors.

If $U = V = H$ are complex Hilbert spaces and $\mathcal{S}$ is a finite-dimensional subspace of $B(H)$, where $B(H)$ denotes the set of all bounded linear operators, then algebraic reflexivity coincides with the reflexivity of $\mathcal{S}$. (For the definition and general results on reflexivity of subspaces of operators, [1] and [3] are excellent references.) The main result in [6] states that if $\mathcal{S}$ is an n-dimensional subspace of $B(H)$, where H is a separable complex Hilbert space, then $\mathcal{S}$ is algebraically $\lfloor\sqrt{2n}\rfloor$-reflexive.

While the above result answered a question raised in [7], the proof given in [6] is quite technical and relies on results from operator theory on Hilbert spaces. The main purpose of this paper is to generalize the above result to linear transformations on vector spaces and provide a simpler and more self-contained proof.

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Let $|\mathbb{F}|$ denote the cardinality of $\mathbb{F}$. The main result of the paper is the following.

Theorem 1. *If $\mathcal{S}$ is an n-dimensional subspace of $\mathcal{L}(U,V)$ with $n < |\mathbb{F}|$, then $\mathcal{S}$ is algebraically $\lfloor\sqrt{2n}\rfloor$-reflexive.*

The notion of algebraic reflexivity was first introduced in [4]. In [5], the following well-known result on algebraic reflexivity is proved: If $\mathcal{S}$ is a finite-dimensional subspace of $\mathcal{L}(U,V)$, then $ref_a(\mathcal{S}) = \mathcal{S} + ref_a(\mathcal{S} \cap \mathcal{L}_F(U,V))$, where $\mathcal{L}_F(U,V)$ denotes the set of all finite rank linear transformations in $\mathcal{L}(U,V)$. It follows that $\mathcal{S}$ is algebraically n-reflexive if and only if $\mathcal{S} \cap \mathcal{L}_F(U,V)$ is algebraically n-reflexive; so, for the remainder of the paper, we may assume that $\mathcal{S}$ is a subspace of $\mathcal{L}(U,V)$, where U and V are finite-dimensional vector spaces over $\mathbb{F}$.

The following is essentially the same as [2, Proposition 1.1].

Lemma 2 [[2]]**.** *Suppose $\mathcal{S}$ is an n-dimensional subspace of $\mathcal{L}(U,V)$ with $n < |\mathbb{F}|$, x is a separating vector of $\mathcal{S}$, and W is a linear subspace of V satisfying $\mathcal{S}x \cap W = (0)$. Then for each vector $y \in U$, with the exception of at most n values of $\lambda \in \mathbb{F}$, $y + \lambda x$ separates $\mathcal{S}$ and $\mathcal{S}(y + \lambda x) \cap W = (0)$.*

Let M be a subspace of V. For $T \in \mathcal{L}(U,V)$, let $R(T)$ denote the range of T. Define $\mathcal{S}_M = \{A \in \mathcal{S} : R(A) \subseteq M\}$. Let P_{M^c} be the projection of V onto any vector space complement of M in V (the projection is done through M) and let $\mathcal{S}_M^c$ be any vector space complement of $\mathcal{S}_M$ in $\mathcal{S}$. Define $P_{M^c}\mathcal{S}_M^c = \{P_{M^c}A : A \in \mathcal{S}_M^c\}$.

Lemma 3. *If $\mathcal{S}$ is an n-dimensional subspace of $\mathcal{L}(U,V)$ with $n < |\mathbb{F}|$, then $k(\mathcal{S}_M) + k(P_{M^c}\mathcal{S}_M^c) \leq k(\mathcal{S})$.*

Proof. If $P_{M^c}\mathcal{S}_M^c = \{0\}$, then clearly $k(\mathcal{S}_M) \leq k(\mathcal{S})$.

If $\mathcal{S}_M = \{0\}$, then $k(P_{M^c}\mathcal{S}_M^c) \leq k(\mathcal{S}_M^c) \leq k(\mathcal{S})$.

Suppose $k(\mathcal{S}_M) = m \neq 0$ and $k(P^{\perp}\mathcal{S}_M^c) = l \neq 0$. Let $\{A_1, ..., A_m\} \subseteq \mathcal{S}_M$ be linearly independent and let $x_0 \in U$ be a separating vector of $span\{A_1, ..., A_m\}$. Suppose $\{B_1, ..., B_l\} \subseteq \mathcal{S}_M^c$ so that $P_{M^c}B_1, ..., P_{M^c}B_l$ are linearly independent and y_0 is a separating vector of $span\{P_{M^c}B_1, ..., P_{M^c}B_l\}$.

First, we show that $span\{A_1, ..., A_m\}$ and $span\{P_{M^c}B_1, ..., P_{M^c}B_l\}$ have a common separating vector: by Lemma 2, there are at most l values of $\lambda \in \mathbb{F}$ such that $x_0 + \lambda y_0$ are not separating vectors of $span\{P_{M^c}B_1, ..., P_{M^c}B_l\}$. We can assume 0 is one of those values; otherwise, x_0 is a common separating vector. Among the other values in $\mathbb{F}$, by Lemma 2 again, there are at most m values λ such that $x_0 + \lambda y_0$ are not separating vectors of $span\{A_1, ..., A_m\}$; otherwise, there would be more than m vectors of the form $\lambda^{-1}x_0 + y_0$ that are not separating vectors of $span\{A_1, ..., A_m\}$. Since $m + l \leq n < |\mathbb{F}|$, there exists a λ_0 such that $x_0 + \lambda_0 y_0$ is a common separating vector.

Let $z_0 = x_0 + \lambda_0 y_0$. It remains to show that $A_1z_0, ..., A_mz_0, B_1z_0, ..., B_lz_0$ are linearly independent.

For any $\lambda_1, ..., \lambda_m, \mu_1, ..., \mu_l \in \mathbb{F}$, suppose

$$\lambda_1 A_1 z_0 + ... + \lambda_m A_m z_0 + \mu_1 B_1 z_0 + ... + \mu_l B_l z_0 = 0. \tag{1}$$

Applying P_{M^c} to both sides of (1), it follows that

$$\mu_1 P_{M^c} B_1 z_0 + ... + \mu_l P_{M^c} B_l z_0 = 0. \tag{2}$$

Since z_0 is a separating vector of $span\{P_{M^c}B_1, ..., P_{M^c}B_l\}$, we have $\mu_1 = ... = \mu_l = 0$. Now equation (1) implies $\lambda_1 = ... = \lambda_m = 0$, since z_0 is a separating vector of $span\{A_1, ..., A_m\}$. Hence $k(\mathcal{S}) \geq k(\mathcal{S}_M) + k(P_{M^c}\mathcal{S}_M^c)$. $\square$

Lemma 4. *Suppose $\mathcal{S}$ is an n-dimensional subspace of $\mathcal{L}(U,V)$ with $n < |\mathbb{F}|$ and $k(\mathcal{S}) = k$. For $x_0 \in U$ and $A_1, ..., A_k \in \mathcal{S}$, let $M = span\{A_1x_0, ..., A_kx_0\}$. If $dimM = k$, then $\mathcal{S}_M$ contains all $T \in \mathcal{S}$ with $Tx_0 = 0$; moreover, $dim(\mathcal{S}_M^c) \leq k$.*

Proof. To prove the first part, we proceed contrapositively. Assume there exist $T \in \mathcal{S}$ with $Tx_0 = 0$ and $y \in U$ such that $Ty \notin span\{A_1x_0, ..., A_kx_0\}$; then necessarily $T \neq 0$. Let W be the one-dimensional subspace $(\mathbb{F}Ty)$ and $\widehat{\mathcal{S}} = span\{A_1, ..., A_k\}$. Then $\widehat{\mathcal{S}}x_0 \cap W = (0)$. By Lemma 2, there exists $\lambda \in \mathbb{F}$ such that $y + \lambda x_0$ separates $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}(y + \lambda x_0) \cap W = (0)$. Since $Tx_0 = 0$, it follows that $\{A_1, ..., A_k, T\}$ is linearly independent. Let $\widetilde{\mathcal{S}} = span\{A_1, ..., A_k, T\}$. Next we prove that $y + \lambda x_0$ separates $\widetilde{\mathcal{S}}$. For any $A \in \widehat{\mathcal{S}}, t \in \mathbb{F}$, if $(A + tT)(y + \lambda x_0) = 0$, then $A(y + \lambda x_0) = -tTy$. By $\widehat{\mathcal{S}}(y + \lambda x_0) \cap W = (0)$, it follows that $t = 0$ and $A(y + \lambda x_0) = 0$. Since $y + \lambda x_0$ is a separating vector of $\widehat{\mathcal{S}}$, we have $A = 0$. Hence $y + \lambda x_0$ separates $\widetilde{\mathcal{S}}$, which implies $k(\mathcal{S}) \geq k + 1$, a contradiction.

To see the second part, take any $B \in \mathcal{S}$; since $M = \mathcal{S}x_0$, there exist $a_1, ..., a_k \in \mathbb{F}$ such that $Bx_0 = a_1A_1x_0 + ... + a_kA_kx_0$. By the first part, $B - (a_1A_1 + ... + a_kA_k) \in \mathcal{S}_M$. □

For any d-dimensional vector space W over a field $\mathbb{F}$, let W^* be the space of linear functionals on W. Note that both W and W^* can be identified naturally with $\mathbb{F}^{(d)}$. We can view W as W^{**} as well. For any $T \in \mathcal{L}(U,V)$, define $T^* \in \mathcal{L}(V^*, U^*)$ as follows: For any $f \in V^*, T^*f$ is defined by $(T^*f)(u) = f(Tu)$, $\forall u \in U$. Letting $\mathcal{S}^* = \{A^* : A \in \mathcal{S}\}$, we have

Lemma 5. *$\mathcal{S}$ is algebraically reflexive in $\mathcal{L}(U,V)$ if and only if $\mathcal{S}^*$ is algebraically reflexive in $\mathcal{L}(V^*, U^*)$.*

Proof. An application of a version of the Hahn-Banach separation theorem yields:

$$\begin{aligned}
&T \in ref_a(\mathcal{S}) \Longleftrightarrow Tu \in \mathcal{S}u,\ \forall\ u \in U\\
&\Longleftrightarrow\ f(Tu) \in \{f(Au) :\ \forall\ A \in \mathcal{S}\},\ \forall\ u \in U,\ \forall\ f \in V^*\\
&\Longleftrightarrow (T^*f)(u) \in \{(A^*f)(u) :\ \forall\ A \in \mathcal{S}\},\ \forall\ u \in U,\ \forall\ f \in V^*\\
&\Longleftrightarrow T^*f \in \{A^*f :\ \forall\ A \in \mathcal{S}\},\ \forall\ f \in V^*\\
&\Longleftrightarrow T^* \in ref_a(\mathcal{S}^*).
\end{aligned}$$

If U is a k-dimensional vector space, then every subspace of $\mathcal{L}(U,V)$ is algebraically k-reflexive. This follows from the fact that two linear transformations agreeing on a basis of U must be identical. □

Lemma 6. *If $dim(ran(\mathcal{S})) = k$, then $\mathcal{S}$ is algebraically k-reflexive.*

Proof. Let $V_1 = ran(\mathcal{S})$. Without loss of generality, we assume $\mathcal{S} \subseteq \mathcal{L}(U, V_1)$. Since $dimV_1 = k$, $\mathcal{S}^*$ is algebraically k-reflexive. Hence $\mathcal{S}$ is algebraically k-reflexive by Lemma 5. □

Lemma 7. *Suppose $\mathcal{S}$ is an n-dimensional subspace of $\mathcal{L}(U,V)$ with $n < |\mathbb{F}|$. If $k(\mathcal{S}) = k$, then $\mathcal{S}$ is algebraically k-reflexive.*

Proof. We will proceed by induction on k.

First suppose $k = 1$ and take any $T \in ref_a(\mathcal{S})$. Let $\widetilde{\mathcal{S}} = span\{T, \mathcal{S}\}$. Suppose $dim(\widetilde{\mathcal{S}}) \geq 2$ (otherwise, there is nothing to prove). Since $T \in ref_a(\mathcal{S})$, $k(\widetilde{\mathcal{S}}) = 1$. By Lemma 4, there exists a subspace M of V with $dimM = 1$ such that $dim(\widetilde{\mathcal{S}})_M^c \leq 1$. Since $k(\widetilde{\mathcal{S}}) = 1$, $(\widetilde{\mathcal{S}})_M^c = (0)$ by Lemma 3; i.e., $dim(ran(\widetilde{\mathcal{S}})) = 1$. In particular, $dim(ran(\mathcal{S})) = 1$. Therefore, $\mathcal{S}$ is algebraically reflexive by Lemma 6.

Suppose the statement is true for all $\mathcal{S}$ with $k(\mathcal{S}) \leq k-1$, where $k \geq 2$.

For any $\mathcal{S}$ with $k(\mathcal{S}) = k$, where $k \geq 2$, there exist $x_1 \in U$ and $\{A_1, ..., A_k\} \subseteq \mathcal{S}$ such that $\{A_i x_1\}_{i=1}^k$ is a basis of $\mathcal{S}x_1$. Suppose $\mathcal{S} = span\{A_1, ..., A_n\}$. There exists a unique $k \times n$ matrix (a_{ij}) with $a_{ij} \in \mathbb{F}$ such that

$$A_j x_1 = \sum_{i=1}^{k} a_{ij} A_i x_1, j = 1, ..., n, \tag{3}$$

where if $j \leq k, a_{jj} = 1$ and $a_{ij} = 0$ for $i \neq j$.

It is not hard to see that any linear transformation in $ref_a(\mathcal{S}^{(k)})$ must be of the form $T^{(k)}$, for some $T \in \mathcal{L}(U, V)$. Given any $T^{(k)} \in ref_a(\mathcal{S}^{(k)})$, then for any $x_2, ..., x_k \in U$, there exist $t_1, ..., t_n \in \mathbb{F}$ such that

$$\begin{pmatrix} Tx_1 \\ \vdots \\ Tx_k \end{pmatrix} = t_1 \begin{pmatrix} A_1 x_1 \\ \vdots \\ A_1 x_k \end{pmatrix} + ... + t_n \begin{pmatrix} A_n x_1 \\ \vdots \\ A_n x_k \end{pmatrix}. \tag{4}$$

Since $Tx_1 \in span\{A_1 x_1, ..., A_k x_1\}$, there exist unique fixed $c_1, ..., c_k \in \mathbb{F}$ such that

$$Tx_1 = \sum_{i=1}^{k} c_i A_i x_1. \tag{5}$$

By (4) and (5), we have

$$Tx_m = \sum_{i=1}^{k} c_i A_i x_m + \sum_{j=1}^{n} t_j (A_j - \sum_{i=1}^{k} a_{ij} A_i) x_m, \ m = 2, ..., k. \tag{6}$$

Let

$$T_1 = T - \sum_{i=1}^{k} c_i A_i, \text{ and } B_j = A_j - \sum_{i=1}^{k} a_{ij} A_i. \tag{7}$$

Note $B_j = 0$ for $j = 1, ..., k$. By (6) and (7), we have

$$\begin{pmatrix} T_1 x_2 \\ \vdots \\ T_1 x_m \end{pmatrix} = t_{k+1} \begin{pmatrix} B_{k+1} x_2 \\ \vdots \\ B_{k+1} x_k \end{pmatrix} + ... + t_n \begin{pmatrix} B_n x_2 \\ \vdots \\ B_n x_k \end{pmatrix}. \tag{8}$$

Let $M = span\{A_1 x_1, \cdots, A_k x_1\}$ and $\widehat{\mathcal{S}} = span\{B_{k+1}, \cdots, B_n\}$. By (3), $B_j x_1 = 0$, for $j = 1, ..., n$. Thus, by Lemma 4, $R(B_j) \subseteq M$, for $j = 1, ..., n$, i.e. $\widehat{\mathcal{S}} \subseteq \mathcal{S}_M$.

If $\mathcal{S} = \mathcal{S}_M$, then $\mathcal{S}$ is k-reflexive by Lemma 6.

If $\mathcal{S} \neq \mathcal{S}_M$, then Lemma 3 implies $k(\mathcal{S}_M) \leq k-1$. Thus $k(\widehat{\mathcal{S}}) \leq k-1$. By equation (8), $T_1^{(k-1)} \in ref_a(\widehat{\mathcal{S}}^{(k-1)})$. By the induction hypothesis, $\widehat{\mathcal{S}}$ is algebraically $(k-1)$-reflexive; thus, $T_1 \in \widehat{\mathcal{S}} \subseteq \mathcal{S}$. Therefore $T \in \mathcal{S}$. □

Proof of Theorem 1. If $n = 1$ or 2, the conclusion follows from Lemma 7. Suppose the result holds for $dim\, \mathcal{S} \leq n-1, n \geq 3$. Let $dim\, \mathcal{S} = n$, $k(\mathcal{S}) = k$, and $j = \lfloor \sqrt{2n} \rfloor$. If $k \leq j$, by Lemma 7, $\mathcal{S}$ is algebraically k-reflexive, thus algebraically j-reflexive.

Suppose $k > j$. If $k = n$, then $\mathcal{S}$ is algebraically 2-reflexive. Hence $\mathcal{S}$ is algebraically j-reflexive. Suppose $j < k \leq n-1$. Since $j = \lfloor \sqrt{2n} \rfloor$, it follows that $2n < (j+1)^2$. Thus, $2(n-k) < (j+1)^2 - 2k \leq (j+1)^2 - 2(j+1) = j^2 - 1$. Hence, $\lfloor \sqrt{2(n-k)} \rfloor \leq j-1$. Using the same argument as that of Lemma 7, we can obtain

an equation similar to (8). Note that $dim\ (span\{B_{k+1}, ..., B_n\}) \leq n-k$; by the induction hypothesis, $span\{B_{k+1}, ..., B_n\}$ is algebraically $\lfloor\sqrt{2(n-k)}\rfloor$-reflexive since $j-1 \geq \lfloor\sqrt{2(n-k)}\rfloor$. Thus $span\{B_{k+1}, ..., B_n\}$ is algebraically $(j-1)$-reflexive, so $\mathcal{S}$ is algebraically j-reflexive. □

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