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MIDDLE POINTS, MEDIANS AND INNER PRODUCTS

CARLOS BENÍTEZ AND DIEGO YÁÑEZ

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ABSTRACT. Let X be a real normed space with unit sphere S. Gurari and Sozonov proved that X is an inner product space if and only if, for any $u, v \in S$, $\inf_{t\in[0,1]} \|tu + (1-t)v\| = \|\frac{1}{2}u + \frac{1}{2}v\|$. We prove that it suffices to consider points $u, v \in S$ such that $\inf_{t\in[0,1]} \|tu + (1-t)v\| = \frac{1}{2}$.

Making use of the above result we also prove that if $\dim X \geq 3$, X is smooth, and 0 is a Fermat-Torricelli median of any three points $u, v, w \in S$ such that $u + v + w = 0$, then X is an inner product space.

1. INTRODUCTION

Let X be a real normed space with unit sphere S. Gurari and Sozonov [8] proved that X is an inner product space (i.p.s.) if and only if, for any $u, v \in S$,

$$\inf_{t\in[0,1]} \|tu + (1-t)v\| = \|\tfrac{1}{2}u + \tfrac{1}{2}v\|$$

(see, e.g., [1], p. 29, where this result is used to establish many characterizations of i.p.s., especially in chapters 12 to 19). We prove in this paper that it suffices to consider pairs of points $u, v \in S$ such that

$$\inf_{t\in[0,1]} \|tu + (1-t)v\| = \tfrac{1}{2},$$

i.e., we prove that X is an i.p.s. if and only if

$$(1) \qquad u, v \in S,\ \inf_{t\in[0,1]} \|tu + (1-t)v\| = \tfrac{1}{2} \ \Rightarrow\ u + v \in S.$$

In geometrical terms, property (1) states that every chord of S that supports $\frac{1}{2}S$ touches $\frac{1}{2}S$ at its middle point.

As a corollary of the above result, we obtain a new characterization of i.p.s. based on the location of the medians of three points.

By definition, the set $Z_L(u, v, w)$ of the Fermat-Torricelli medians of the points $u, v, w \in X$ from the set $L \subset X$ is formed by the points $z \in L$ such that

$$\|u - z\| + \|v - z\| + \|w - z\| = \inf_{x\in L} (\|u - x\| + \|v - x\| + \|w - x\|).$$

It is well known (see, e.g., [4], p. 274, or [6], p. 98) that if X is either an i.p.s. or a two-dimensional space, then, for every $u, v, w \in X$,

$$Z_X(u, v, w) = Z_{\mathrm{aff}(u,v,w)}(u, v, w) = Z_{\mathrm{co}(u,v,w)}(u, v, w),$$

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where $\operatorname{aff}(u,v,w)$ and $\operatorname{co}(u,v,w)$ are the affine and the convex hull, respectively, of the points $u,v,w \in X$.

However, the formally weaker property

$$Z_X(u,v,w) \cap \operatorname{co}(u,v,w) \neq \emptyset, \quad \text{for every } u,v,w \in X,$$

is characteristic of real i.p.s. of dimension ≥ 3 [2].

Also it is known (see, e.g., [4], p. 238, or the proof of Theorem 3.2 of this paper) that if X is an i.p.s., then

$$u,v,w \in S,\ u+v+w=0 \ \Rightarrow\ 0 \in Z_X(u,v,w). \tag{2}$$

We prove in the above-mentioned Theorem 3.2 that property (2), when X is smooth and of dimension ≥ 3, is also characteristic of i.p.s..

2. Preliminary lemmas

It follows from the nature of property (1) and the fact that X is an i.p.s. when its two-dimensional subspaces are, that it suffices to consider the case in which X is two-dimensional, i.e., the space $\mathbb{R}^2$ endowed with a norm with unit sphere S and unit ball B.

For given $x=(x_1,x_2)$ and $y=(y_1,y_2)$ in X we shall use the following notation: $x \prec y$, when x precedes y in the positive orientation of X, i.e.,

$$x \wedge y = x_1y_2 - x_2y_1 > 0.$$

$x \perp y$, when x is orthogonal to y in the sense of Birkhoff, [3], [9], i.e.,

$$\|x\| \leq \|x+\lambda y\| \quad (\lambda \in \mathbb{R}),$$

or, equivalently (see, e.g., [7]),

$$|x \wedge y| = \sup\{z \wedge y : \|z\| = \|x\|\}.$$

Obviously, $x \perp y$ means that the straight line $L=\{x+\lambda y : \lambda \in \mathbb{R}\}$ supports the sphere $S(0,\|x\|)$ at x, i.e., $x \in L \cap S(0,\|x\|)$ and $L \cap \operatorname{int} B(0,\|x\|) = \emptyset$.

Lemma 2.1. (i) *For any $u \in S$, there are unique $u^*, u^{**} \in S$, $u^{**} \prec u \prec u^*$, such that the segments $[u^{**},u]$ and $[u,u^*]$ support $\frac{1}{2}S$.*

(ii) *The map $u \in S \to u^* \in S$ is a homeomorphism whose inverse is $u \in S \to u^{**} \in S$.*

(iii) *If $u,v \in S$ are such that $u \prec v$, then, $u^* \prec v^*$ and $u+u^* \prec v+v^*$.*

Proof. The proof is very intuitive and not difficult. □

Remark 2.2. $[u,u^*]$ supports $\frac{1}{2}S$ means that $[u,u^*] \cap \frac{1}{2}S \neq \emptyset$ and, for any $x \in [u,u^*] \cap \frac{1}{2}S$, $x \perp u^*-u$. In other words, $x'(u^*-u)=0$ when x' is some linear functional which attains its norm at x.

Lemma 2.3. *If X fulfills* (1)*, then it is regular (rotund and smooth).*

Proof. Suppose that X is non-rotund. Then there exist $u \in S$ and $x,y \in \frac{1}{2}S$, $x \prec y$, such that $[u,u^*] \cap \frac{1}{2}S = [x,y]$.

Property (1) states that $[u,u^*]$ supports $\frac{1}{2}S$ at $\frac{1}{2}(u+u^*)$ and, hence, $x \preceq u+u^* \preceq y$. Suppose that $u+u^* \prec y$. It follows from Lemma 2.1 that there exists $v \in S$ such that $u \prec v$ and $u+u^* \prec v+v^* \prec y$. Then the fact that $[v,v^*]$ supports $\frac{1}{2}S$ at $\frac{1}{2}(v+v^*)$ leads to the absurdity $[v,v^*] \cap \frac{1}{2}S = [u,u^*] \cap \frac{1}{2}S = [x,y]$.

Suppose now that X is non-smooth. Then there are $u, v \in S$, $u \prec v$, such that $\frac{1}{2}(u+u^*) = \frac{1}{2}(v+v^*)$ is the only point of $[u,u^*] \cap \frac{1}{2}S = [v,v^*] \cap \frac{1}{2}S$. Hence, the segments $[u,v]$ and $[u^*,v^*]$ must be contained in S, in contradiction with the first part of this lemma. $\square$

Corollary 2.4. *If X fulfills* (1) *and $u, v \in S$ are such that $u \prec v$ and $u+v \in S$, then $v = u^*$.*

Proof. Let (other cases will be analogous) $u^* \prec v$ and $u \wedge u^* \leq u \wedge v$. Since $(u+u^*) \wedge u^* > 0$, $u^* \wedge v > 0$, and $(u+u^*) \wedge v > 0$ (i.e., u^* is between $u+u^*$ and v), there exist $0 < t < 1$ and $\rho > 0$ such that $\rho u^* = t(u+u^*) + (1-t)v$. Then, $\rho u \wedge u^* = tu \wedge u^* + (1-t) u \wedge v \geq u \wedge u^*$ and hence $\rho \geq 1$.

Therefore, it follows from $u+u^*, v \in S$ that $\rho = 1$ and the segment $[u+u^*, v]$ is in S, in contradiction with the rotundity of X. $\square$

Lemma 2.5. *Suppose that X fulfills* (1)*. Then, for every $u \in S$,*

(i) $(u^*)^* = u^{**}$.

(ii) $u + u^* + u^{**} = 0$.

(iii) $u \wedge u^* = u^* \wedge u^{**} = u^{**} \wedge u$.

(iv) $u + u^* \perp u - u^*$, $u^* + u^{**} \perp u^* - u^{**}$, $u^{**} + u \perp u^{**} - u$.

Proof. (i) and (ii). It follows from (1) that $-(u+u^*) \in S$, and it is obvious that $u-(u+u^*), u^*-(u+u^*) \in S$. Then these properties are an immediate consequence of the above corollary.

(iii). The proof follows from $u \wedge (u+u^*+u^{**}) = u^* \wedge (u+u^*+u^{**}) = 0$.

(iv). It suffices to consider Remark 2.2 and the fact that the segment $[u,u^*]$ supports $\frac{1}{2}S$ at $-\frac{1}{2}(u+u^*)$. $\square$

Lemma 2.6. *Suppose that X fulfills* (1)*. Then:*

(i) *For any $u \in S$ there exist unique ${}^{\perp}u, u^{\perp} \in S$, ${}^{\perp}u \prec u \prec u^{\perp}$, such that ${}^{\perp}u \perp u$ and $u \perp u^{\perp}$.*

(ii) *The map $u \in S \to u^{\perp} \in S$ is a homeomorphism with inverse $u \in S \to^{\perp} u \in S$.*

(iii) *If $u, v \in S$ are such that $u \prec v$, then $u^{\perp} \prec v^{\perp}$.*

Proof. (i). It is easy to see and well known [9] that the uniqueness (for every $u \in S$) of ${}^{\perp}u$ and $u^{\perp}$ are equivalent to the rotundity and smoothness of X, respectively.

(ii) and (iii). As in Lemma 2.1, the proof is very intuitive and not difficult. $\square$

In all that follows

$$s : \theta \in [0, 2\pi] \to s(\theta) \in S$$

will be a "natural map" for S, i.e., a map such that $s(\theta) = (s_1(\theta), s_2(\theta))$ is the point of S that makes an angle θ with a given point $(s_1(0), s_2(0))$ of S, measured with the positive orientation of the plane X. In other words, if $s(0) = \|(1,0)\|^{-1}(1,0)$, then

$$s(\theta) = \|(\cos\theta, \sin\theta)\|^{-1}(\cos\theta, \sin\theta).$$

Since S is a convex curve, the above map is continuous and of bounded variation, and, as a consequence of Lemma 2.1, also continuous and of bounded variation are the maps (non-natural, in general)

$$s^* : \theta \in [0, 2\pi] \to s^*(\theta) \in S, \quad s^{**} : \theta \in [0, 2\pi] \to s^{**}(\theta) \in S,$$

where $s^*(\theta)$ and $s^{**}(\theta)$ are the unique points of S such that $s^{**}(\theta) \prec s(\theta) \prec s^*(\theta)$ and the segments $[s^{**}(\theta), s(\theta)]$, $[s(\theta), s^*(\theta)]$ support $\frac{1}{2}S$.

Moreover, by virtue of Lemma 2.6, the same holds for

$$s^\perp : \theta \in [0, 2\pi] \to s^\perp(\theta) \in S, \quad {}^\perp s : \theta \in [0, 2\pi] \to {}^\perp s(\theta) \in S,$$

when X fulfills (1).

Therefore all the Riemann-Stieltjes integrals that we shall write below make sense. For example, the well-known formula

$$A(B_{s(\alpha)s(\beta)}) = \frac{1}{2}\int_\alpha^\beta s(\theta) \wedge ds(\theta) = \frac{1}{2}\int_\alpha^\beta [s_1(\theta)ds_2(\theta) - s_2(\theta)ds_1(\theta)]$$

correctly gives the area of the sector of the unit ball B that is between two points $s(\alpha)$ and $s(\beta)$, with $0 \le \alpha < \beta \le 2\pi$.

Lemma 2.7. *If X fulfills* (1)*, then:*

(i) *For any $u \in S$, $A(B_{uu^*}) = A(B_{u^*u^{**}}) = A(B_{u^{**}u})$.*

(ii) *For any $u, v \in S$, $A(B_{uv}) = A(B_{u^*v^*})$.*

(iii) *The function $u \in S \to u \wedge u^*$ is constant.*

Proof. (i). Let

$$s : \theta \in [0, 2\pi] \to s(\theta) \in S$$

be a natural map for S such that $u = s(\alpha)$, and let $0 \le \alpha < \alpha^* \le 2\pi$ be such that $s^*(\alpha) = s(\alpha^*)$. Then, $s^{**}(\alpha) = s^*(\alpha^*)$ and

$$\begin{gathered} 2s(\alpha) = s(\alpha) + s^*(\alpha) + [s(\alpha) - s^*(\alpha)], \\ 2s^*(\alpha) = s(\alpha) + s^*(\alpha) - [s(\alpha) - s^*(\alpha)] = s(\alpha^*) + s^*(\alpha^*) + [s(\alpha^*) - s^*(\alpha^*)], \\ 2s^{**}(\alpha) = s(\alpha^*) + s^*(\alpha^*) - [s(\alpha^*) - s^*(\alpha^*)]. \end{gathered}$$

Hence,

$$\begin{aligned} &A(B_{uu^*}) - A(B_{u^*u^{**}}) \\ &= \frac{1}{8}\int_\alpha^{\alpha^*} \{s(\theta) + s^*(\theta) + [s(\theta) - s^*(\theta)]\} \wedge d\{s(\theta) + s^*(\theta) + [s(\theta) - s^*(\theta)]\} \\ &-\frac{1}{8}\int_\alpha^{\alpha^*} \{s(\theta) + s^*(\theta) - [s(\theta) - s^*(\theta)]\} \wedge d\{s(\theta) + s^*(\theta) - [s(\theta) - s^*(\theta)]\} \\ &= \frac{1}{4}\int_\alpha^{\alpha^*} \{[s(\theta) + s^*(\theta)] \wedge d[s(\theta) - s^*(\theta)] + [s(\theta) - s^*(\theta)] \wedge d[s(\theta) + s^*(\theta)]\}. \end{aligned}$$

Let $\alpha = \theta_0 < \theta_1 < ... < \theta_n = \alpha^*$ be a partition of $[\alpha, \alpha^*]$. Since $u + u^* \perp u - u^*$, (see Lemma 2.5(iv)), there exist $\theta_0 \le \delta_1 \le \theta_1 \le ... \le \theta_{n-1} \le \delta_n \le \theta_n$ such that, for $k = 1, ..., n$,

$$[s(\delta_k) - s^*(\delta_k)] \wedge \{s(\theta_k) + s^*(\theta_k) - [s(\theta_{k-1}) + s^*(\theta_{k-1})]\} = 0$$

and, hence,

$$\int_\alpha^{\alpha^*} [s(\theta) - s^*(\theta)] \wedge d[s(\theta) + s^*(\theta)] = 0.$$

Therefore,

$$A(B_{uu^*}) - A(B_{u^*u^{**}})$$
$$= \frac{1}{4}\int_{\alpha}^{\alpha^*} d\{[s(\theta)+s^*(\theta)]\wedge[s(\theta)-s^*(\theta)]\} = \frac{1}{2}\int_{\alpha}^{\alpha^*} d[s^*(\theta)\wedge s(\theta)]$$
$$= \frac{1}{2}[s(\alpha)\wedge s^*(\alpha) - s(\alpha^*)\wedge s^*(\alpha^*)] = \frac{1}{2}[s(\alpha)\wedge s^*(\alpha) - s^*(\alpha)\wedge s^{**}(\alpha)] = 0,$$

where the last equality is justified in Lemma 2.5(iii).

(ii). Suppose (other cases are analogous) that $u, v \in S$, $u \prec v \prec u^*$. Then it suffices to consider (i),

$$A(B_{uu^*}) = A(B_{u^*u^{**}}) = A(B_{u^{**}u}) = \tfrac{1}{3}A(B) = A(B_{vv^*}) = A(B_{v^*v^{**}}) = A(B_{v^{**}v}),$$

and the obvious fact that

$$A(B_{uu^*}) = A(B_{uv}) + A(B_{vu^*}), \quad A(B_{vv^*}) = A(B_{vu^*}) + A(B_{u^*v^*}).$$

(iii). We have proved in (ii) that, for any $0 \le \alpha < \beta \le 2\pi$,

$$0 = A(B_{s(\alpha)s(\beta)}) - A(B_{s^*(\alpha)s^*(\beta)})$$
$$= \frac{1}{8}\int_{\alpha}^{\beta}\{s(\theta)+s^*(\theta)+[s(\theta)-s^*(\theta)]\}\wedge d\{s(\theta)+s^*(\theta)+[s(\theta)-s^*(\theta)]\}$$
$$-\frac{1}{8}\int_{\alpha}^{\beta}\{s(\theta)+s^*(\theta)-[s(\theta)-s^*(\theta)]\}\wedge d\{s(\theta)+s^*(\theta)-[s(\theta)-s^*(\theta)]\}$$
$$= \frac{1}{4}\int_{\alpha}^{\beta}\{[s(\theta)+s^*(\theta)]\wedge d[s(\theta)-s^*(\theta)] + [s(\theta)-s^*(\theta)]\wedge d[s(\theta)+s^*(\theta)]\}.$$

The same argument as in (i) shows that

$$\int_{\alpha}^{\beta}[s(\theta)-s^*(\theta)]\wedge d[s(\theta)+s^*(\theta)] = 0,$$

and hence that

$$0 = \int_{\alpha}^{\beta} d\{[(s(\theta)+s^*(\theta)]\wedge[s(\theta)-s^*(\theta)]\} = 2s(\alpha)\wedge s^*(\alpha) - 2s(\beta)\wedge s^*(\beta).$$

□

Lemma 2.8. *Suppose that X fulfills* (1) *and that $s : [0,2\pi] \to S$ is a natural map for S. Then:*

(i) *s is continuously differentiable, and there is a continuous function $p : [0,2\pi] \to \mathbb{R}_+$ such that $s'(\theta) = p(\theta)s^{\perp}(\theta)$.*

(ii) *For every $\theta \in [0,2\pi]$, $s'(\theta)\wedge s^*(\theta) > 0$.*

(iii) *s^* is continuously differentiable, and there is a continuous function $q : [0,2\pi] \to \mathbb{R}_+$ such that $s^{*\prime}(\theta) = q(\theta)s^{*\perp}(\theta)$.*

Proof. (i). Since X fulfills (1) it is smooth, and it is well known that this is equivalent to the continuous differentiability of $x \in X \to \|x\|$ at $X \setminus \{0\}$. Then it follows from

$$s(\theta) = \|(\cos\theta, \sin\theta)\|^{-1}(\cos\theta, \sin\theta)$$

that s is continuously differentiable.

Also, the fact that

$$s(\theta)\wedge s^{\perp}(\theta) = \sup\{s(\lambda)\wedge s^{\perp}(\theta) : \lambda \in [0,2\pi]\}$$

shows that $s'(\theta) \wedge s^{\perp}(\theta) = 0$. Furthermore, it follows from

$$\|(\cos\theta, \sin\theta)\|^2 s_1'(\theta) = -\sin\theta \|(\cos\theta, \sin\theta)\| - \cos\theta \|(\cos\theta, \sin\theta)\|',$$
$$\|(\cos\theta, \sin\theta)\|^2 s_2'(\theta) = \cos\theta \|(\cos\theta, \sin\theta)\| - \sin\theta \|(\cos\theta, \sin\theta)\|',$$

that $s'(\theta) \neq 0$, and it is obvious that $s(\theta) \prec s'(\theta)$. I.e., $s'(\theta) = p(\theta)s^{\perp}(\theta)$, with $p(\theta) > 0$.

Finally, the continuity of p follows from the continuity of s' and $s^{\perp}$.

(ii). We saw in Lemma 2.7 that, for any $0 \leq \alpha < \beta \leq 2\pi$,

$$\int_\alpha^\beta [s^*(\theta) - s^{**}(\theta)] \wedge d[s^*(\theta) + s^{**}(\theta)] = -\int_\alpha^\beta [2s^*(\theta) + s(\theta)] \wedge ds(\theta) = 0,$$

i.e.,

$$2\int_\alpha^\beta s^*(\theta) \wedge ds(\theta) = -\int_\alpha^\beta s(\theta) \wedge ds(\theta) < 0.$$

Then, since s is continuously differentiable,

$$\int_\alpha^\beta s^*(\theta) \wedge s'(\theta) d\theta < 0,$$

as we wished to show.

(iii). It follows from Lemma 2.7(iii) that, for any $\lambda \neq \theta$,

$$0 = \frac{s(\lambda) - s(\theta)}{\lambda - \theta} \wedge s^*(\lambda) + s(\theta) \wedge \frac{s^*(\lambda) - s^*(\theta)}{\lambda - \theta}.$$

Hence,

$$\lim_{\lambda \to \theta} s(\theta) \wedge \frac{s^*(\lambda) - s^*(\theta)}{\lambda - \theta} = -s'(\theta) \wedge s^*(\theta) < 0.$$

Since X is smooth and s^* is continuous, if $(\lambda_n)_{n\in\mathbb{N}}$ and $(\bar{\lambda}_n)_{n\in\mathbb{N}}$ are convergent to θ sequences such that the sequences

$$\left(\frac{s^*(\lambda_n) - s^*(\theta)}{\lambda_n - \theta}\right)_{n\in\mathbb{N}}, \quad \left(\frac{s^*(\bar{\lambda}_n) - s^*(\theta)}{\bar{\lambda}_n - \theta}\right)_{n\in\mathbb{N}},$$

are also convergent, then there exist two positive numbers $q(\theta)$ and $\bar{q}(\theta)$ such that

$$\lim_{n\to\infty} \frac{s^*(\lambda_n) - s^*(\theta)}{\lambda_n - \theta} = q(\theta)s^{*\perp}(\theta), \quad \lim_{n\to\infty} \frac{s^*(\bar{\lambda}_n) - s^*(\theta)}{\bar{\lambda}_n - \theta} = \bar{q}(\theta)s^{*\perp}(\theta),$$

and it follows from the above result that $q(\theta) = \bar{q}(\theta)$, i.e., that

$$\lim_{\lambda\to\theta} \frac{s^*(\lambda) - s^*(\theta)}{\lambda - \theta} = q(\theta)s^{*\perp}(\theta) = -s'(\theta) \wedge s^*(\theta).$$

Finally, the continuity of $s^{*\prime}$ and q follows from the continuity of s', s^*, and $s^{*\perp}$. □

3. Main results

Theorem 3.1. *X is an inner product space if and only if*

$$(1) \qquad u, v \in S, \ \inf_{t\in[0,1]} \|tu + (1-t)v\| = \tfrac{1}{2} \ \Rightarrow \ u + v \in S.$$

Proof. It is easy to see (and well known) that if X is an i.p.s., i.e. $\|x\|^2 = (x|x)$, then it fulfills (1).

It suffices to consider that, for any $u, v \in S$ the convex function

$$F(t) = \|(1-t)u + tv\|^2 = 1 - 2t + 2t^2 + 2t(1-t)(u|v)$$

is such that

$$F'(t) = 2(1-2t)[(u|v) - 1],$$

and hence, when $(u|v) < 1$ (i.e. when u and v are linearly independent), F attains its minimum at $t = \frac{1}{2}$.

By virtue of the above lemmas, to prove the converse we can take X to be the space $\mathbb{R}^2$ endowed with a norm, and we can denote by u and u^* the two points u and v of hypothesis (1).

Let $s : [0, 2\pi] \to S$ be a natural map for S. It follows from Lemma 2.7(ii) that, for any $\alpha \in [0, 2\pi]$,

$$\int_0^\alpha s(\theta) \wedge ds(\theta) = \int_0^\alpha s^*(\theta) \wedge ds^*(\theta),$$

from Lemma 2.5(ii) and (iv) that

$$s(\theta) \perp s(\theta) + 2s^*(\theta), \quad s^*(\theta) \perp -2s(\theta) - s^*(\theta),$$

and from Lemma 2.8 that s and s^* are continuously differentiable and such that

$$s'(\theta) = p(\theta)s^\perp(\theta), \quad s^{*\prime}(\theta) = q(\theta)s^{*\perp}(\theta),$$

where p and q are positive and continuous functions.

Then it follows from the uniqueness of $s^\perp(\theta)$ that there exist two continuous functions $k : [0, 2\pi] \to \mathbb{R}_+$ and $l : [0, 2\pi] \to \mathbb{R}_+$ such that

$$s'(\theta) = k(\theta)[s(\theta) + 2s^*(\theta)], \quad s^{*\prime}(\theta) = l(\theta)[-2s(\theta) - s^*(\theta)],$$

and the first equality between the Riemann-Stieltjes integrals can be reduced to the following equality between ordinary Riemann integrals:

$$\int_0^\alpha k(\theta)s(\theta) \wedge [s(\theta) + 2s^*(\theta)]d\theta = \int_0^\alpha l(\theta)s^*(\theta) \wedge [-2s(\theta) - s^*(\theta)]d\theta,$$

i.e. (see Lemma 2.7(iii)),

$$\int_0^\alpha k(\theta)d\theta = \int_0^\alpha l(\theta)d\theta \quad (\alpha \in [0, 2\pi]),$$

from which it follows (k and l are continuous) that $k = l$.

Hence, we have the following system of differential equations:

$$\begin{aligned} s_1'(\theta) &= k(\theta)[s_1(\theta) + 2s_1^*(\theta)], \\ s_2'(\theta) &= k(\theta)[s_2(\theta) + 2s_2^*(\theta)], \\ s_1^{*\prime}(\theta) &= -k(\theta)[2s_1(\theta) + s_1^*(\theta)], \\ s_2^{*\prime}(\theta) &= -k(\theta)[2s_2(\theta) + s_2^*(\theta)]. \end{aligned}$$

The first and third give that

$$2s_1(\theta)s_1'(\theta) + s_1^* s_1'(\theta) + s_1(\theta)s_1^{*\prime}(\theta) + 2s_1^*(\theta)s_1^{*\prime}(\theta) = 0,$$

i.e. that $s_1^2(\theta) + s_1^{*2}(\theta) + s_1(\theta)s_1^*(\theta)$ is constant.

Analogously, the second and fourth give that $s_2^2(\theta) + s_2^{*2}(\theta) + s_2(\theta)s_2^*(\theta)$ is also constant.

Then, for the (non-restrictive) initial data

$$(s_1(0), s_2(0)) = (1, 0), \quad (s_1^*(0), s_2^*(0)) = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}),$$

we have that

$$s_1^2(\theta) + s_1^{*2}(\theta) + s_1(\theta)s_1^*(\theta) = s_2^2(\theta) + s_2^{*2}(\theta) + s_2(\theta)s_2^*(\theta) = \tfrac{3}{4}.$$

This, together with

$$s_1(\theta)s_2^*(\theta) - s_2(\theta)s_1^*(\theta) = \tfrac{\sqrt{3}}{2}$$

(see Lemma 2.7 (iii)), leads to

$$s_1^2(\theta) + s_2^2(\theta) = 1,$$

i.e., to the fact that S is a circumference (an ellipse), as we wished to show. □

Theorem 3.2. *Suppose that X is smooth and of dimension ≥ 3. Then X is an inner product space if and only if*

$$u, v, w \in S,\ u + v + w = 0 \ \Rightarrow\ 0 \in Z_X(u, v, w). \tag{2}$$

Proof. It is known (see, e.g., [4], p. 238) that if X is an inner product space (of any dimension), i.e. $\|x\|^2 = (x|x)$, then it fulfills (2). Indeed, let $u, v, w \in S$ be such that $u + v + w = 0$. Then for

$$\begin{aligned} F(x) &= \|u - x\| + \|v - x\| + \|w - x\| \\ &= \sqrt{(u - x|u - x)} + \sqrt{(v - x|v - x)} + \sqrt{(u + v + x|u + v + x)}, \end{aligned}$$

we have that

$$F'(x)(t) = \frac{(x - u|t)}{\sqrt{(u - x|u - x)}} + \frac{(x - v|t)}{\sqrt{(v - x|v - x)}} + \frac{(x + u + v|t)}{\sqrt{(u + v + x|u + v + x)}},$$

from which it follows that

$$F'(0)(t) = -(u|t) - (u|t) + (u + v|t) = 0,$$

i.e., the convex function F attains its minimum at 0.

To prove the converse we may assume $\dim X = 3$.

Since X is smooth, for any $u \in S$ there is a unique $u' \in S'$ (unit sphere of the dual space X') such that $u'(u) = 1$.

Let $u, v, w \in S$ be such that $u + v + w = 0$. Then $0 \in Z_X(u, v, w)$, and a corollary of the Hahn-Banach theorem (see, e.g., [2], Proposition 1) says that this is equivalent to $u' + v' + w' = 0$.

So we have that

$$\begin{gathered} u'(u) = v'(v) = w'(w) = 1, \\ u'(u + v + w) = v'(u + v + w) = w'(u + v + w) = 0, \\ (u' + v' + w')(u) = (u' + v' + w')(v) = (u' + v' + w')(w) = 0, \end{gathered}$$

from which it follows that

$$\begin{gathered} u'(v) + u'(w) = v'(u) + v'(w) = w'(u) + w'(v) = -1, \\ v'(u) + w'(u) = u'(v) + w'(v) = u'(w) + v'(w) = -1, \end{gathered}$$

and, hence,

$$u'(v) = v'(w) = w'(u), \quad u'(w) = v'(u) = w'(v).$$

Let L be the 2-dimensional subspace $\operatorname{span}(u,v,w)$, let (L_n) be a sequence of 2-dimensional subspaces of X that contain u and converges (in the obvious sense) to L, and let $v_n, w_n \in L_n \cap S$ be such that

$$u + v_n + w_n = 0.$$

Since the sequence

$$(\tau_n, -\tau_n) = \left(\frac{v - v_n}{\|v - v_n\|}, \frac{w - w_n}{\|w - w_n\|} \right) = \left(\frac{v - v_n}{\|v - v_n\|}, \frac{v_n - v}{\|v_n - v\|} \right)$$

is in the compact set $S \times S$ it has a subsequence that converges to a point $(\tau, -\tau) \in S \times S$ such that $v \perp \tau$ and $w \perp \tau$, i.e. (see Remark 2.2) $v'(\tau) = w'(\tau) = 0$ and thus $u'(\tau) = 0$.

Moreover, since $\dim(\ker u' \cap \ker v' \cap \ker w') = 1$, every convergent subsequence of $(\tau_n, -\tau_n)$ converges to either $(\tau, -\tau)$ or $(-\tau, \tau)$, and hence

$$\lim_{n\to\infty} u' \left(\frac{v - v_n}{\|v - v_n\|} \right) = \lim_{u+\bar{v}\in S, \bar{v}\to v} u' \left(\frac{v - \bar{v}}{\|v - \bar{v}\|} \right) = 0.$$

Since this is valid for every $v \in S$ such that $u + v \in S$, we get that $\{v \in S : u + v \in S\}$ is a differentiable curve that is contained in a plane parallel to $\ker u'$ (τ is a tangent vector at v and $u'(\tau) = 0$). Specifically,

$$\{v \in S : u + v \in S\} = \left(-\frac{1}{2}u + \ker u' \right) \cap S,$$

i.e. $u'(v) = u'(w) = -1/2$, for every $u, v, w \in S$ such that $u + v + w = 0$.

We have, finally, that for every $\lambda \in \mathbb{R}$

$$\|v + w\| = -u'(v + w + \lambda(v - w)) \leq \|v + w + \lambda(v - w)\|,$$

i.e., $v + w \perp v - w$.

In other words, $w = v^*$ and the segment $[v, w]$ supports $\frac{1}{2}S$ at its middle point. Since this is true for every $v \in S$, Theorem 3.1 shows that X is an inner product space. □

Remark 3.3. We presume that in the above theorem the smoothness of X is unnecessary, but the corresponding proof appears to be much more involved.

Remark 3.4. In a first version of this paper we said that $u + v + w = 0$ and $u' + v' + w' = 0$ imply $u'(v) = u'(w) = -1/2$, and then we concluded that if X (of any dimension) fulfils

$$u, v, w \in S,\ u + v + w = 0 \Rightarrow 0 \in Z_{\operatorname{co}(u,v,w)}(u, v, w),$$

then X is an inner product space.

But for $\dim X = 2$ we only have $u'(v) + u'(w) = -1$, and, furthermore, the following counterexample shows that not only our old proof was wrong.

Example 3.5 ([5]). Let X be the space $\mathbb{R}^2$ endowed with a norm whose unit sphere S is a (rectilinear or curvilinear) regular hexagon, i.e. a convex curve that is invariant under rotations of $\pi/3$. Then, it is easy to see that if $u, v, w \in S$ are such that $u + v + w = 0$, then they are vertices of an equilateral triangle inscribed in S.

Furthermore, if, for the above three points, $z \in Z_X(u, v, w)$, then either $z = 0$ or z is a vertex of an equilateral triangle, centered at 0, whose other vertices are also in $Z_X(u, v, w)$, and, since this set is convex, $0 \in Z_X(u, v, w)$.

Note finally that if S is a rectilinear regular hexagon, then X is neither smooth nor rotund, but for other curvilinear regular hexagons X may be smooth and rotund.

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Departamento de Matemáticas, Universidad de Extremadura, 06071 Badajoz, Spain
E-mail address: `cabero@unex.es`

Departamento de Matemáticas, Universidad de Extremadura, 06071 Badajoz, Spain
E-mail address: `dyanez@unex.es`