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# OPERATORS THAT ADMIT A MOMENT SEQUENCE, II

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Abstract. As the title indicates, this note is a continuation of a paper by Foias, Jung, Ko and Pearcy, in which it was shown that certain classes of operators on a Hilbert space admit moment sequences. Herein we extend these results.

## 1. Introduction

In this note $\mathcal{H}$ will always be a separable, infinite-dimensional, complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$. As usual, $\mathbf{K} = \mathbf{K}(\mathcal{H})$ will denote the ideal of compact operators in $\mathcal{L}(\mathcal{H})$, and we write $\mathbb{N}\,[\mathbb{N}_0]$ for the set of positive [nonnegative] integers. Following [1] and [7], we say that an operator $T$ in $\mathcal{L}(\mathcal{H})$ *admits a moment sequence* if there exist nonzero vectors $x$ and $y$ in $\mathcal{H}$ and a (finite, regular) Borel measure $\mu$ supported on the spectrum $\sigma(T)$ of $T$ such that for every complex polynomial $p$,

$$\langle p(T)\,x, y\rangle = \int_{\sigma(T)} p(\lambda)\ d\mu(\lambda)\,. \tag{1.1}$$

(We use the term *measure* here in the usual sense of a nonnegative-valued set function.)

The motivation for [7] (and this continuation) is the following nice theorem of Atzmon and Godefroy [1].

**Theorem 1.1.** *Suppose $\mathcal{X}$ is a real separable Banach space and $T$ is a bounded linear operator on $\mathcal{X}$ that admits a moment sequence* (*with associated Borel measure $\mu$ supported on $\sigma(T) \subset \mathbb{R}$*). *Then $T$ has a nontrivial invariant subspace.*

It is obvious that every $T$ in $\mathcal{L}(\mathcal{H})$ that has a nontrivial invariant subspace (n.i.s.) admits a moment sequence (associated with the measure $\mu \equiv 0$ on $\sigma(T)$), and Theorem 1.1 points in the direction of the possible equivalence of the two concepts. Thus the authors believe that the question of which operators in $\mathcal{L}(\mathcal{H})$ can be shown to have a moment sequence is worth further exploration.

The basic tool used in [7] was a rather deep theorem of Foias-Pasnicu-Voiculescu [8], together with the theory of quasitriangular operators (cf., e.g., [5]), and its main theorem was the following, where $(\mathbf{N}+\mathbf{K})$ denotes the set of all operators $T$

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in $\mathcal{L}(\mathcal{H})$ that can be written as a sum $T = N + K$, where $N$ is a normal operator and $K$ is compact.

**Theorem 1.2** ([7]). *Every $T \in (\mathbf{N}+\mathbf{K})$ admits a moment sequence.*

In this note we first show that Theorem 1.2 has a somewhat shorter proof that can be based on a theorem of Lomonosov [6], and then we modestly enlarge the class of operators known to admit moment sequences.

## 2. A new proof

We write, as usual, $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ for the Calkin map, and $\sigma_e(T) := \sigma(\pi(T))$, $\|T\|_e := \|\pi(T)\|$. The above-mentioned result of Lomonosov is the following.

**Theorem 2.1** ([6]). *Let $\mathcal{A}$ be a proper (i.e., $\mathcal{A} \neq \mathcal{L}(\mathcal{H})$) subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed in the weak operator topology ($WOT$) and contains the identity operator $1_{\mathcal{H}}$. Then there exist nonzero vectors $x$ and $y$ in $\mathcal{H}$ such that*

*1) $\langle x, y\rangle \geq 0$, and*

*2) the linear functional $\varphi \in \mathcal{A}^*$ defined by $\varphi(A) = \langle Ax, y\rangle$ satisfies $|\varphi(A)| \leq \langle x, y\rangle \|A\|_e$ for every $A \in \mathcal{A}$ (and therefore also satisfies $\varphi(1_{\mathcal{H}}) = \langle x, y\rangle = \|\varphi\|$).*

Following [6], we say that a functional $\varphi \in \mathcal{A}^*$ with the above properties is a *positive, vector functional* on $\mathcal{A}$. Recall that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is called *essentially normal* if $T^*T - TT^* \in \mathbf{K}$, i.e., if $\pi(T)$ is normal. We begin our program with the following new proof of Theorem 1.2.

*New proof of Theorem* 1.2. Let $\mathcal{A}_T$ be the unital, WOT-closed subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $T$. Since $\mathcal{A}_T$ is abelian, $\mathcal{A}_T \neq \mathcal{L}(\mathcal{H})$. We apply Theorem 2.1 to obtain nonzero vectors $x, y$ in $\mathcal{H}$ and the positive vector functional $\varphi$ on $\mathcal{A}_T$ satisfying $\varphi(A) = \langle Ax, y\rangle$ and $|\varphi(A)| \leq \langle x, y\rangle \|A\|_e$ for every $A \in \mathcal{A}_T$. Then, as above, $\varphi(1_{\mathcal{H}}) = \langle x, y\rangle = \|\varphi\|$, and we also have $\varphi(\mathcal{A}_T \cap \mathbf{K}) = 0$. Thus, by standard facts about factoring through quotient algebras, there exists $\widehat{\varphi} \in (\pi(\mathcal{A}_T))^*$ such that $\varphi = \widehat{\varphi} \circ \pi$ and $\widehat{\varphi}(\pi(1_{\mathcal{H}})) = \langle x, y\rangle = \|\widehat{\varphi}\|$. In particular, we have, for every (complex) polynomial $p$,

$$\varphi(p(T)) = \widehat{\varphi}(\pi(p(T))) = \widehat{\varphi}(p(\pi(T))). \tag{2.1}$$

From the hypothesis, we know that $\pi(T)$ is normal, and writing $\mathcal{P}(\pi(T))$ for the polynomial algebra generated by $\pi(T)$, the Hahn-Banach theorem yields the fact that $\widehat{\varphi}|_{\mathcal{P}(\pi(T))}$ has an extension $\widehat{\varphi}_{\text{ext}}$ to the abelian unital $C^*$-algebra $C^*(\pi(T))$ generated by $\pi(T)$ satisfying

$$\|\widehat{\varphi}_{\text{ext}}\| = \|\widehat{\varphi}_{\text{ext}}(\pi(1_{\mathcal{H}}))\| = \langle x, y\rangle.$$

In other words, $\widehat{\varphi}_{\text{ext}} \in (C^*(\pi(T)))^*$, and since this dual space is isometrically isomorphic to the Banach space $\mathcal{M}(\sigma_e(T))$ of all complex, regular, Borel measures on $\sigma(\pi(T)) = \sigma_e(T)$, we obtain, finally, that there exists $\mu \in \mathcal{M}(\sigma_e(T))$ such that for every complex polynomial $p$,

$$\langle p(T)x, y\rangle = \varphi(p(T)) = \widehat{\varphi}(\pi(p(T))) = \widehat{\varphi}_{\text{ext}}(p(\pi(T))) = \int_{\sigma_e(T)} p(\lambda)\, d\mu(\lambda).$$

Moreover, since $\varphi$, $\widehat{\varphi}$, and $\widehat{\varphi}_{\text{ext}}$ are positive linear functionals, the corresponding complex measure $\mu$ is, in fact, a measure, so the proof is complete. $\square$

Using Theorem 1.2, the BDF-theory of essentially normal operators (cf. [3]), and the characterization of quasitriangular operators from [2], as well as the Berger-Shaw theorem [4], we now obtain, as in [7], the following.

**Corollary 2.2** ([7])**.** *Every $T$ in $\mathcal{L}(\mathcal{H})$ that is either nonbiquasitriangular, essentially normal, or hyponormal admits a moment sequence.*

## 3. Some new results

In this section we enlarge the class of operators in $\mathcal{L}(\mathcal{H})$ known to have a moment sequence. We first recall from [9] that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is called *almost hyponormal* if $T^*T - TT^*$ can be written as $P + K$, where $P \geq 0$ and $K \in \mathcal{C}_1(\mathcal{H})$, the ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$. A little-known theorem from [9] is the following.

**Theorem 3.1** ([9])**.** *Suppose $T \in \mathcal{L}(\mathcal{H})$ is almost hyponormal, and let $X$ be any Hilbert-Schmidt operator in $\mathcal{L}(\mathcal{H})$ (i.e., $X \in \mathcal{C}_2(\mathcal{H})$). Then, if $T^*T - TT^* \notin \mathcal{C}_1(\mathcal{H})$, the operator $T + X$ has an n.i.s.*

Our first new result partially generalizes Corollary 2.2.

**Theorem 3.2.** *Every operator in $\mathcal{L}(\mathcal{H})$ of the form $T + X$, where $T$ is almost hyponormal and $X \in \mathcal{C}_2(\mathcal{H})$, admits a moment sequence.*

*Proof.* If $T^*T - TT^* \notin \mathcal{C}_1(\mathcal{H})$, then by Theorem 3.1, $T + X$ has an n.i.s. and thus admits a moment sequence. On the other hand, if $T^*T - TT^* \in \mathcal{C}_1(\mathcal{H})$, then $T + X$ is essentially normal, and thus admits a moment sequence via Corollary 2.2. $\square$

Our second new result shows that, indeed, the properties of having an n.i.s. and admitting a moment sequence are equivalent for a very special class of operators.

**Theorem 3.3.** *Suppose $T \in \mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains at least one isolated point. Then $T$ has an n.i.s. if and only if $T$ admits a moment sequence.*

*Proof.* If $\sigma(T)$ is not a singleton, then $\sigma(T)$ is disconnected, and hence $T$ has an n.i.s. and admits a moment sequence. The case remaining is that in which $\sigma(T) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$. Note first that if $\lambda_0 = 0$ and $T$ admits a moment sequence, say

$$\langle T^n x, y\rangle = \int_{\{\lambda_0\}} \lambda^n \, d\mu, \qquad n \in \mathbb{N}_0,$$

for some nonzero vectors $x$ and $y$ and some (necessarily atomic) measure $\mu$ supported on $\{0\}$, then $\langle T^n x, y\rangle = 0$ for $n \in \mathbb{N}$, and the vector $Tx$ is not cyclic for $T$. Thus

$$\mathcal{M} = \bigvee_{n \in \mathbb{N}} T^n x$$

is an n.i.s. for $T$. Therefore to complete the proof it suffices to show that translation of an arbitrary operator $A$ in $\mathcal{L}(\mathcal{H})$ by an arbitrary scalar preserves the property of admitting a moment sequence. But this is immediate from (1.1) and the obvious fact that $\{p(A) : p \in \mathbb{C}[x]\} = \{q(A + \gamma \cdot 1_{\mathcal{H}}) : q \in \mathbb{C}[x]\}$ after making the appropriate change of variables. $\square$

We next recall that a corollary of the proof of Theorem 1.2 given in [7] was the following.

**Corollary 3.4.** *Suppose $T \in (\mathbf{N}+\mathbf{K})$ and $(0) \neq \mathcal{M}$ is an n.i.s. for $T$. Then $T|_{\mathcal{M}}$ admits a moment sequence.*

It is worthwile to consider the structure of such $T|_{\mathcal{M}}$. If $\dim \mathcal{M} = n \in \mathbb{N}$, then $T|_{\mathcal{M}}$ is essentially an $n \times n$ complex matrix, and nothing more need be said. On the other hand, if $\dim(\mathcal{H} \ominus \mathcal{M}) \in \mathbb{N}$, then a matricial calculation and use of the well-known fact that $T \in (\mathbf{N}+\mathbf{K})$ if and only if $T$ is essentially normal and biquasitriangular yields the fact that $T|_{\mathcal{M}} \in (\mathbf{N}+\mathbf{K})(\mathcal{M})$. Thus we now consider the structure of $T|_{\mathcal{M}}$ when both $\mathcal{M}$ and $\mathcal{H} \ominus \mathcal{M}$ are infinite dimensional. Upon identifying $\mathcal{H} \ominus \mathcal{M}$ with $\mathcal{M}$, we may suppose that $T \in (\mathbf{N}+\mathbf{K})(\mathcal{M} \oplus \mathcal{M})$, and thus may be written matricially as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where the $T_{ij} \in \mathcal{L}(\mathcal{M})$. Let us write $T = N + K$ with $N$ normal and $K \in \mathbf{K}$, and write the corresponding $2 \times 2$ matrices for $N$ and $K$ as

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where again the $N_{ij}$ and $K_{ij} \in \mathcal{L}(\mathcal{M})$. Obviously, $K_{ij} \in \mathbf{K}(\mathcal{M})$ for $i, j = 1, 2$ and $N_{21} = -K_{21} \in \mathbf{K}$. Thus $\pi_{\mathcal{M}}(N_{21}) = 0$, $\pi_{\mathcal{M}\oplus\mathcal{M}}(T)$ is normal, and $\pi_{\mathcal{M}}(T_{11})$ is subnormal. Since $T_{11} = T|_{\mathcal{M}}$ has a moment sequence, some essentially subnormal operators have moment sequences. This raises the following interesting problem.

**Problem 3.5.** Does every essentially subnormal or essentially hyponormal operator in $\mathcal{L}(\mathcal{H})$ admit a moment sequence?

Theorem 3.2 above gives a partial answer to this question, and this next proposition gives another.

**Proposition 3.6.** *Every $T = S + K \in \mathcal{L}(\mathcal{H})$ with $S$ subnormal and $K \in \mathbf{K}$ has a moment sequence.*

*Proof.* Let $N$ be a minimal normal extension of $S$ acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and let $J = K \oplus 0_{\mathcal{K}\ominus\mathcal{H}}$ be a (compact) extension of $K$. Then clearly $N + J \in (\mathbf{N}+\mathbf{K})(\mathcal{K})$, $(N+J)\mathcal{H} \subset \mathcal{H}$, and $(N+J)|_{\mathcal{H}} = S + K$. So the result follows from Corollary 3.4. $\square$

We close this note with some additional interesting problems in this area.

**Problem 3.7.** Let $T$ be an invertible operator in $\mathcal{L}(\mathcal{H})$ admitting a moment sequence. Does $T^{-1}$ admit a moment sequence?

**Problem 3.8.** Suppose that for $n \in \mathbb{N}$, $T_n \in \mathcal{L}(\mathcal{H})$ admits a moment sequence and $\|T_n - T_0\| \to 0$. Does $T_0$ admit a moment sequence?

**Problem 3.9.** Does every quasinilpotent operator admit a moment sequence? In this connection, remember Theorem 3.3.

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