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HANKEL OPERATORS WITH UNBOUNDED SYMBOLS

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Abstract. We prove that there are holomorphic functions f in the Hardy space of the unit ball or the bidisc such that the big Hankel operator with symbol $\bar{f}$ is bounded and for any holomorphic function g the function $\bar{f}+g$ cannot be bounded.

1. Introduction

Throughout this note $\Omega = \Omega_n$ will denote the domain in $\mathbb{C}^n$ which is either the unit ball $\mathbb{B}_n$ or the the unit polydisc $\mathbb{D}^n$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$. We let σ denote the normalized Lebesgue measure on the Shilov boundary $\partial_s\Omega$ of Ω. For the unit ball then $\partial_s\mathbb{B}_n := \mathbb{S}$ is the unit sphere and for the unit polydisc, $\partial_s\mathbb{D}^n = \mathbb{T}^n$ where $\mathbb{T}$ is the unit circle. Let $H^2(\Omega)$ be the Hardy space on Ω with respect to σ. The Berezin transform of a function $f \in L^2(\sigma)$ is defined by

$$\widetilde{f}(z) := \int_{\partial_s\Omega} f(\xi)P(z,\xi)d\sigma(\xi),\ z \in \Omega,$$

where $P(z,\xi)$ is the Poisson kernel of Ω with respect to σ. We follow Coburn [C] to define the space $\widetilde{BMO}(\partial\Omega)$ of functions with bounded mean oscillation with respect to σ as the subspace of functions $f \in L^2(\sigma)$ that satisfy

$$\sup_{z\in\Omega}\left[\widetilde{|f|^2}(z) - |\widetilde{f}|^2(z)\right] < \infty.$$

We point out that in the one-dimensional case this definition gives the classical BMO space. See [Z1]. The subspace of $\widetilde{BMO}(\partial\Omega)$ consisting of those analytic functions on Ω will be denoted by $\widetilde{BMOA}(\partial\Omega)$. In the case of the unit ball the space $\widetilde{BMOA}(\mathbb{B})$ was shown to be the dual of the Hardy space $H^1(\mathbb{B})$. See Theorem 5.13 of [Z2]. For the polydisc case we show in Proposition 3.5 below that $\widetilde{BMO}$ coincides with the small *bmo* defined by Cotlar and Sadosky [CS]. It is not the dual of the real $H^1(\mathbb{T}^2)$, and its predual is given by Theorem 1.5 of [CS].

The big Hankel operator H_f with symbol $f \in L^2(\sigma)$ is defined by

$$H_f(g) := (I-P)(fg), \quad g \in H^\infty(\Omega).$$

We recall that in the one variable setting $\Omega = \mathbb{D}$, if a function f is in $H^2(\mathbb{D})$, then the big Hankel operator $H_{\bar{f}}$ is bounded on $H^2(\mathbb{T})$ if and only if there is another $g \in H^2(\mathbb{T})$ such that $\bar{f}+g$ is bounded. See [Z1]. The goal of this note is to show this

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is no longer true in higher dimensions. In particular, this answers in the negative a question by E. Strouse [S] which she raised for the case of the bidisc. We shall establish the following.

Theorem A. *There are functions $f \in H^2(\Omega_2)$ such that the big Hankel operator $H_{\bar f}$ is bounded on $H^2(\Omega_2)$ but $\bar f + g$ is unbounded for any holomorphic function g on Ω_2.*

We point out that the existence of big Hankel operators with not necessarily antianalytic unbounded symbols was estabished in [CS] and [BT] for the case of the bidisc.

It is not hard to see [C] that if $f \in L^2(\partial_s\Omega)$ and both operators H_f and $H_{\bar f}$ are bounded, then $f \in \widetilde{BMO}(\partial_s\Omega)$. The converse is true for the bidisc. Indeed,

Theorem B. *Let $f \in L^2(\mathbb{T}^2)$. Then $f \in \widetilde{BMO}(\mathbb{T}^2)$ if and only if the big Hankel operators H_f and $H_{\bar f}$ are bounded on $H^2(\mathbb{D}^2)$.*

The analogue of Theorem B for the unit ball in $\mathbb{C}^2$ remains open.

2. The ball case

We recall that the Bergman distance $\beta(\zeta, z)$ between two points ζ and z in the unit disc $\mathbb{D}$ in the complex plane is given by

$$\beta(\zeta, z) = \frac{1}{2}\log\frac{1+|\varphi_z(\zeta)|}{1-|\varphi_z(\zeta)|}, \text{ where } \varphi_z(\zeta) := \frac{\zeta - z}{1-\zeta\bar z}.$$

Lemma 2.1. *There is a positive constant C independent of $n > 0$ such that*

$$I_n(z) := \int_{\mathbb{D}} (1-|\zeta|^2)^{\frac{n}{2}} \frac{\beta(\zeta, z)}{|1-\bar\zeta z|^{n+2}} dA(\zeta) \le \frac{C}{(n+1)^{\frac{3}{2}}}(1-|z|^2)^{-\frac{n}{2}}$$

for all $z \in \mathbb{D}$.

Proof. Setting $\zeta = \varphi_z(w)$ we see that

$$\begin{aligned}
I_n(z) &= \int_{\mathbb{D}} \beta(0, w)\frac{(1-|w|^2)^{\frac{n}{2}}}{|1-\bar\zeta w|^2} dA(w) \\
&= \frac{(1-|z|^2)^{-\frac{n}{2}}}{2}\int_0^1 r\log\frac{1+r}{1-r}(1-r^2)^{\frac{n}{2}}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-r\bar z e^{i\theta}|^2}dr \\
&= \frac{(1-|z|^2)^{-\frac{n}{2}}}{2}\int_0^1 r\frac{(1-r^2)^{\frac{n}{2}}}{1-r^2|z|^2}\log\frac{1+r}{1-r}dr \\
&\le \frac{(1-|z|^2)^{-\frac{n}{2}}}{2}\int_0^1 r(1-r^2)^{\frac{n}{2}-1}\log\frac{1+r}{1-r}dr \\
&= \frac{(1-|z|^2)^{-\frac{n}{2}}}{2}\left(\int_0^1 r\log(1+r)(1-r^2)^{\frac{n}{2}-1}dr\right. \\
&\qquad\qquad \left. + \int_0^1 r(1-r^2)^{\frac{n}{2}-1}\log\frac{1}{1-r}dr\right).
\end{aligned}$$

Integration by parts and the change of variable $t = 1-r$ give that the second integral in the last equality is equal to $4n^{-2}$. To estimate the first integral in terms

of n, we have

$$\begin{aligned}\int_0^1 r\log(1+r)(1-r^2)^{\frac{n}{2}-1}dr &= \frac{1}{n}\int_0^1 \frac{(1-r^2)^{\frac{n}{2}}}{1+r}dr \\ &\le \frac{1}{2n}\int_0^1 (1-r^2)^{\frac{n}{2}}dr \\ &= \frac{1}{2n}\int_0^{\frac{\pi}{2}} (\sin t)^n dt \le \frac{C}{n^{\frac{3}{2}}}. \qquad \square\end{aligned}$$

Let $\mathcal{B}(\mathbb{D})$ be the Bloch space of $\mathbb{D}$ and if $F \in \mathcal{B}(\mathbb{D})$, we extend F to the ball $\mathbb{B}_2$ by setting $F(z,w) := F(z)$. Then we have

Lemma 2.2. *If $F \in \mathcal{B}(\mathbb{D})$, then the big Hankel operator $H_{\overline{F}}$ is bounded on the Hardy space $H^2(\mathbb{B}_2)$.*

We write each $g \in H^2(\mathbb{B})$ in the form $g(z,w) = \sum_n g_n(z)w^n$, where the functions g_n are holomorphic in $\mathbb{D}$. In addition, we have the identity

$$\int_{\mathbb{S}} g_n(z)w^n\overline{g_m(z)w^m}d\sigma(z,w) = \begin{cases} 0, & n \ne m, \\ \int_{\mathbb{D}} |g_n(z)|^2(1-|z|^2)^n dA(z), & n = m. \end{cases}$$

Hence

$$\|g\|^2_{L^2(\mathbb{S})} = \sum_n \int_{\mathbb{D}} |g_n(z)|^2(1-|z|^2)^n dA(z).$$

On the other hand, for each n, we have that

$$\begin{aligned}(P(\overline{F}g_n)(z,w) &= \int_{\mathbb{S}} \frac{\overline{F}(\zeta)g_n(\zeta)\eta^n}{(1-\bar{\zeta}z-\bar{\eta}w)^2}d\sigma(\zeta,\eta) \\ &= \int_{\mathbb{D}} \frac{\overline{F}(\zeta)g_n(\zeta)}{(1-\bar{\zeta}z-\bar{\eta}w)^2}\int_0^{2\pi} \frac{(1-|\zeta|^2)^{\frac{n}{2}}e^{in\theta}}{(1-\bar{\zeta}z-\sqrt{1-|\zeta|^2}we^{i\theta})^2}\frac{d\theta}{2\pi}dA(\zeta) \\ &= \int_{\mathbb{D}} \frac{\overline{F}(\zeta)g_n(\zeta)(1-|\zeta|^2)^{\frac{n}{2}}}{(1-\bar{\zeta}z)^2}\int_0^{2\pi} \frac{e^{in\theta}}{\left(1-\frac{\sqrt{1-|\zeta|^2}w}{1-\bar{\zeta}z}e^{-i\theta}\right)^2}\frac{d\theta}{2\pi}dA(\zeta) \\ &= (n+1)w^n\int_{\mathbb{D}} \frac{\overline{F}(\zeta)g_n(\zeta)(1-|\zeta|^2)^n}{(1-\bar{\zeta}z)^{n+2}}dA(\zeta)w^n.\end{aligned}$$

Since $(n+1)\frac{(1-|\zeta|^2)^n}{(1-\bar{\zeta}z)^{n+2}}$ reproduces holomorphic functions on the unit disc, it follows that

$$\begin{aligned}(H_{\overline{F}}g)(z,w) &= \sum_n (n+1)w^n\int_{\mathbb{D}} \frac{\overline{F}(z)-\overline{F}(\zeta)}{(1-\bar{\zeta}z)^{n+2}}g_n(\zeta)(1-|\zeta|^2)^n dA(\zeta) \\ &= \sum_n H_n(z)w^n,\end{aligned}$$

where

$$H_n(z) := (n+1)\int_{\mathbb{D}} \frac{\overline{F}(z)-\overline{F}(\zeta)}{(1-\bar{\zeta}z)^{n+2}}g_n(\zeta)(1-|\zeta|^2)^n dA(\zeta).$$

Since $F \in \mathcal{B}(\mathbb{D})$, we see that $|F(z)-F(\zeta)| \le \|F\|_{\mathcal{B}(\mathbb{D})}\beta(z,\zeta)$. Hence

$$|H_n(z)| \le (n+1)\|F\|_{\mathcal{B}(\mathbb{D})}\int_{\mathbb{D}} \frac{\beta(z,\zeta)|g_n(\zeta)|}{|1-\bar{\zeta}z|^{n+2}}(1-|\zeta|^2)^n dA(\zeta)$$

so that by Hölder's inequality and Lemma 2.1 we see that

$$|H_n(z)|^2 \leq (n+1)^2 \|F\|^2_{\mathcal{B}(\mathbb{D})} (I_n(z))^2 \int_{\mathbb{D}} \frac{\beta(z,\zeta)|g_n(\zeta)|^2(1-|\zeta|^2)^{\frac{3n}{2}}}{|1-\bar{\zeta}z|^{n+2}} dA(\zeta)$$
$$\leq C\sqrt{n}\|F\|^2_{\mathcal{B}(\mathbb{D})}(1-|z|^2)^{-\frac{n}{2}} \int_{\mathbb{D}} \frac{\beta(z,\zeta)|g_n(\zeta)|^2(1-|\zeta|^2)^{\frac{n}{2}}}{|1-\bar{\zeta}z|^{n+2}}(1-|\zeta|^2)^n dA(\zeta).$$

Therefore, again applying Lemma 2.1, we have

$$\int_{\mathbb{D}} |H_n(z)|^2(1-|z|^2)^n dA(\zeta)$$
$$\leq 2\sqrt{n}\|F\|^2_{\mathcal{B}(\mathbb{D})} \int_{\mathbb{D}} |g_n(\zeta)|^2(1-|\zeta|^2)^{\frac{3n}{2}} \int_{\mathbb{D}} \frac{\beta(z,\zeta)(1-|z|^2)^{\frac{n}{2}}}{|1-\bar{\zeta}z|^{n+2}} dA(z)dA(\zeta)$$
$$\leq 2\frac{\|F\|^2_{\mathcal{B}(\mathbb{D})}}{n} \int_{\mathbb{D}} |g_n(\zeta)|^2(1-|\zeta|^2)^n dA(\zeta).$$

Thus

$$\|(H_{\overline{F}}g)\|^2_{H^2(\mathbb{B})} = \sum_n \int_{\mathbb{D}} |H_n(z)|^2(1-|z|^2)^n dA(\zeta)$$
$$\leq \sum_n \frac{\|F\|^2_{\mathcal{B}(\mathbb{D})}}{n} \int_{\mathbb{D}} |g_n(\zeta)|^2(1-|\zeta|^2)^n dA(\zeta)$$
$$\leq \|F\|^2_{\mathcal{B}(\mathbb{D})}\|g\|^2_{H^2(\mathbb{B})}.$$

Proof of Theorem A *in the ball case* $\Omega_2 = \mathbb{B}_2$. Let $F \in \mathcal{B}(\mathbb{D}) \setminus BMOA(\mathbb{T})$ be extended to $\mathbb{B}_2$ as the function $(z,w) \mapsto F(z)$. By Lemma 2.2 the big Hankel operator $H_{\overline{F}}$ is bounded on the Hardy space $H^2(\mathbb{B}_2)$. Assume that there is a holomorphic function g on $\mathbb{B}_2$ such that the function $G := \bar{F} + g$ is bounded. Then the function $z \mapsto G(z,0)$ is bounded on the unit disc and hence in $BMO(\mathbb{T})$. This shows that both functions F and $z \mapsto g(z,0)$ are in $BMOA(\mathbb{T})$. This contradicts the fact that $F \notin BMOA(\mathbb{T})$. □

3. The bidisc case

Let $bmo(\mathbb{T}^2)$ be the subspace of functions $f \in L^1(\mathbb{T}^2)$ such that f is in $BMO(\mathbb{T})$ in each variable and

$$\sup_{z\in\mathbb{T}} \|f(z,\cdot)\|_{BMO} + \sup_{w\in\mathbb{T}} \|f(\cdot,w)\|_{BMO} < +\infty.$$

In particular, if $f \in L^1(\mathbb{T}^2)$, then f is in $bmo(\mathbb{T}^2)$ if and only if

$$\sup_{z\in\mathbb{T}} \|Mf(z,\cdot)\|_\infty + \sup_{w\in\mathbb{T}} \|Mf(\cdot,w)\|_\infty < +\infty,$$

where

$$(M\varphi)(\zeta) := \sqrt{\widetilde{|\varphi|^2}(\zeta) - |\widetilde{\varphi}|^2(\zeta)},\ \zeta \in \mathbb{D}, \varphi \in L^2(\mathbb{T}).$$

The subspace of $bmo(\mathbb{T}^2)$ consisting of holmorphic functions on $\mathbb{D}^2$ will be denoted by $bmoa(\mathbb{T}^2)$

Theorem 3.1. *If* $\varphi \in L^\infty(\mathbb{D})$, *then*

$$(T\varphi)(z,w) := \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1-z\bar{\zeta})(1-w\bar{\zeta})} dA(\zeta),\ (z,w) \in \mathbb{T}^2,$$

is in bmoa$(\mathbb{T}^2)$. *In addition, the operator* T *is bounded from* $L^\infty(\mathbb{D})$ *into bmoa*$(\mathbb{T}^2)$.

To prove this theorem we need some auxilary results. The following lemma can be found in [J].

Lemma 3.2. *There is a positive constant* C *such that*

$$\int_{\mathbb{D}} \frac{dA(\zeta)d}{|1-z\bar{\zeta}|^\alpha |1-w\bar{\zeta}|^\beta} \le \begin{cases} C, & \alpha+\beta<2, \\ C\log\frac{1}{|1-z\bar{w}|}, & \alpha+\beta=2, \\ \frac{C}{|1-z\bar{w}|^{\alpha+\beta-2}}, & \alpha+\beta>2, \end{cases}$$

for all $z, w \in \mathbb{D}$.

Lemma 3.3. *There is a positive constant* C *such that*

$$I(z,w) := \int_{\mathbb{D}\times\mathbb{D}} \frac{dA(\zeta)dA(\eta)}{|1-z\bar{\zeta}||1-z\bar{\eta}||1-w\bar{\zeta}||1-w\bar{\eta}||1-\zeta\bar{\eta}|} \le \frac{C}{|1-z\bar{w}|}$$

for all $z, w \in \mathbb{D}$.

Proof. We observe that $I(z,w) \le I_1(z,w) + I_2(z,w) + I_3(z,w)$, where

$$I_j(z,w) := \int_{\mathbb{E}_j} \frac{dA(\zeta)dA(\eta)}{|1-z\bar{\zeta}||1-z\bar{\eta}||1-w\bar{\zeta}||1-w\bar{\eta}||1-\zeta\bar{\eta}|}, \ j=1,2,3,$$

$$\mathbb{E}_1 := \{(\zeta,\eta)\in\mathbb{D}\times\mathbb{D} : |1-\zeta\bar{\eta}| \le \frac{1}{2}|1-z\bar{\zeta}|\},$$

$$\mathbb{E}_2 := \{(\zeta,\eta)\in\mathbb{D}\times\mathbb{D} : |1-\zeta\bar{\eta}| \le \frac{1}{2}|1-w\bar{\eta}|\}$$

and $\mathbb{E}_3$ is the complement of $\mathbb{E}_1 \cup \mathbb{E}_2$ in $\mathbb{D}\times\mathbb{D}$. By symmetry the integrals $I_1(z,w)$ and $I_2(z,w)$ can be estimated similarly. Since by Lemma 3.2

$$\begin{aligned} I_3(z,w) &\le 2\int_{\mathbb{D}\times\mathbb{D}} \frac{dA(\zeta)dA(\eta)}{|1-z\bar{\zeta}|^{\frac{3}{2}}|1-z\bar{\eta}||1-w\bar{\zeta}||1-w\bar{\eta}|^{\frac{3}{2}}} \\ &\le \frac{C}{|1-z\bar{w}|}, \end{aligned}$$

we only need to estimate $I_1(z,w)$. To do so, we write $I_1(z,w) = I_{11}(z,w) + I_{12}(z,w)$, where

$$I_{1j}(z,w) := \int_{\mathbb{E}_{1j}} \frac{dA(\zeta)dA(\eta)}{|1-z\bar{\zeta}||1-z\bar{\eta}||1-w\bar{\zeta}||1-w\bar{\eta}||1-\zeta\bar{\eta}|}, \ j=1,2,$$

$\mathbb{E}_{11} := \{(\zeta,\eta)\in\mathbb{E}_1 : |1-\zeta\bar{\eta}| \le \frac{1}{2}|1-w\bar{\zeta}|\}$ and $\mathbb{E}_{12}$ is the complement of $\mathbb{E}_{11}$ in $\mathbb{E}_1$. Set $R(z,w,\zeta) := \min(|1-z\bar{\zeta}|, |1-w\bar{\zeta}|)$. Since on $\mathbb{E}_{11}$ we have $|1-z\bar{\zeta}| \approx |1-z\bar{\eta}|$ and $|1-w\bar{\zeta}| \approx |1-w\bar{\eta}|$, it follows that

$$\begin{aligned} I_{11}(z,w) &\approx \int_{\mathbb{E}_{11}} \frac{dA(\zeta)dA(\eta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^2|1-\zeta\bar{\eta}|} \\ &\le \int_{\mathbb{D}} \int_{\{\eta\in\mathbb{D}:|1-\zeta\bar{\eta}|\le\frac{1}{2}R(z,w,\zeta)\}} \frac{dA(\eta)}{|1-\zeta\bar{\eta}|} \frac{dA(\zeta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^2} \\ &\le \int_{\mathbb{D}} \frac{R(z,w,\zeta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^2} dA(\zeta) \\ &\le \frac{C}{|1-z\bar{w}|}, \end{aligned}$$

where the latter inequality holds in view of Lemma 3.2. It now remains to establish the estimate of $I_{12}(z,w)$. We write $I_{12}(z,w)=\sum_{j=1}^{3} I_{12j}(z,w)$, where

$$I_{12j}(z,w) := \int_{\mathbb{E}_{12j}} \frac{dA(\zeta)dA(\eta)}{|1-z\bar{\zeta}||1-z\bar{\eta}||1-w\bar{\zeta}||1-w\bar{\eta}||1-\zeta\bar{\eta}|},\ j=1,2,3,$$

$$\mathbb{E}_{121} := \{(\zeta,\eta)\in\mathbb{E}_{12} : |1-w\bar{\eta}| \le \frac{1}{2}|1-\zeta\eta|\},$$

$$\mathbb{E}_{122} := \{(\zeta,\eta)\in\mathbb{E}_{12} : |1-\zeta\bar{\eta}| \le \frac{1}{2}|1-w\bar{\eta}|\},$$

and $\mathbb{E}_{123}$ is the complement of $\mathbb{E}_{121}\cup\mathbb{E}_{122}$ in $\mathbb{E}_{12}$. On $\mathbb{E}_{121}$ we have $|1-w\bar{\zeta}|\approx|1-\zeta\bar{\eta}|$ and $|1-w\bar{\eta}|\le CR(z,w,\zeta)$. Hence, by Lemma 3.2,

$$\begin{aligned} I_{121}(z,w) &\approx \int_{\mathbb{D}}\int_{\{\eta\in\mathbb{D}:|1-w\bar{\eta}|\le CR(z,w,\zeta)\}} \frac{A(\eta)}{|1-w\bar{\eta}|}\frac{dA(\zeta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^2} \\ &\le C\int_{\mathbb{D}} \frac{R(z,w,\zeta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^2}dA(\zeta) \\ &\le \frac{C}{|1-z\bar{w}|}. \end{aligned}$$

On $\mathbb{E}_{122}$ we have $|1-w\bar{\zeta}|\approx|1-w\bar{\eta}|$ and $|1-\zeta\bar{\eta}|\le CR(z,w,\zeta)$. Hence as in the previous case we have

$$\begin{aligned} I_{122}(z,w) &\approx \int_{\mathbb{D}}\int_{\{\eta\in\mathbb{D}:|1-w\bar{\eta}|\le CR(z,w,\zeta)\}} \frac{A(\eta)}{|1-w\bar{\eta}|}\frac{dA(\zeta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^2} \\ &\le \frac{C}{|1-z\bar{w}|}. \end{aligned}$$

Finally, on $\mathbb{E}_{123}$ we have $|1-w\bar{\eta}|\approx|1-\zeta\bar{\eta}|$ and $|1-w\bar{\zeta}|\le 2|1-\zeta\bar{\eta}|$. Thus

$$\begin{aligned} I_{123}(z,w) &\approx \int_{\mathbb{D}}\int_{\{\eta\in\mathbb{D}:|1-\zeta\bar{\eta}|\le\frac{1}{2}|1-z\bar{\zeta}|\}} \frac{A(\eta)}{|1-\zeta\bar{\eta}|^{\frac{3}{2}}}\frac{dA(\zeta)}{|1-z\bar{\zeta}|^2||1-w\bar{\zeta}|^{\frac{3}{2}}} \\ &\le C\int_{\mathbb{D}} \frac{dA(\zeta)}{|1-z\bar{\zeta}|^{\frac{3}{2}}||1-w\bar{\zeta}|^{\frac{3}{2}}} \\ &\le \frac{C}{|1-z\bar{w}|}. \end{aligned}$$

□

Proof of Theorem 3.1. Let

$$f(z,w)=(T\varphi)(z,w)=\int_{\mathbb{D}}\frac{\varphi(\zeta)}{(1-z\bar{\zeta})(1-w\bar{\zeta})}dA(\zeta),\ (z,w)\in\mathbb{D}^2.$$

Then f is in the Hardy space $H^2(\mathbb{D}^2)$ and satisfies $\widetilde{f(\cdot,w)}(z)=f(z,w)$ and

$$\begin{aligned} |\widetilde{f(\cdot,w)}|^2(z) &= \frac{1}{2i\pi}\int_{\mathbb{T}}\int_{\mathbb{D}\times\mathbb{D}} \frac{\varphi(\zeta)\overline{\varphi(\eta)}(1-|z|^2)dA(\zeta)dA(\eta)}{(1-\tau\bar{\zeta})(1-w\bar{\zeta})(1-\bar{\tau}\eta)(1-\bar{w}\eta)|1-z\bar{\tau}|^2}\frac{d\tau}{\tau} \\ &= \int_{\mathbb{D}\times\mathbb{D}} \frac{\varphi(\zeta)\overline{\varphi(\eta)}(1-\eta\bar{\zeta}|z|^2)dA(\zeta)dA(\eta)}{(1-z\bar{\zeta})(1-w\bar{\zeta})(1-\bar{z}\eta)(1-\bar{w}\eta)(1-\bar{\zeta}\eta)} \end{aligned}$$

by the Cauchy formula. Therefore, by Lemma 3.3 we see that

$$\begin{aligned}(Mf(\cdot,w))^2(z) &= |\widetilde{f(\cdot,w)}|^2(z) - |\widetilde{f}(z,w)|^2 \\ &= \left|\int_{\mathbb{D}\times\mathbb{D}} \frac{\zeta\bar{\eta}\varphi(\zeta)\overline{\varphi(\eta)}(1-|z|^2)dA(\zeta)dA(\eta)}{(1-z\bar{\zeta})(1-w\bar{\zeta})(1-\bar{z}\eta)(1-\bar{w}\eta)(1-\bar{\zeta}\eta)}\right| \\ &\le \|\varphi\|_\infty^2(1-|z|^2)\int_{\mathbb{D}\times\mathbb{D}} \frac{dA(\zeta)dA(\eta)}{\left|(1-z\bar{\zeta})(1-w\bar{\zeta})(1-z\bar{\eta})(1-w\bar{\eta})(1-\zeta\bar{\eta})\right|} \\ &\le C\|\varphi\|_\infty^2.\end{aligned}$$

Similarly we have $\|Mf(z,\cdot)\|_\infty \le C\|\varphi\|_\infty$, showing that $T\varphi \in bmoa(\mathbb{T}^2)$ and

$$\|T\varphi\|_{bmoa} \le C\|\varphi\|_\infty.$$

□

Theorem 3.4. *Suppose that $h \in \mathcal{B}(\mathbb{D})$ and $h \notin BMOA(\mathbb{T})$ and let*

$$f(z,w) := \int_{\mathbb{D}} \frac{h(\zeta)}{(1-z\bar{\zeta})(1-w\bar{\zeta})}dA(\zeta),\ (z,w) \in \mathbb{T}^2.$$

Then the big Hankel operator $H_{\bar{f}}$ is bounded on $H^2(\mathbb{D}^2)$, and for any holomorphic function g on $\mathbb{D}\times\mathbb{D}$, the function $\bar{f}+g$ cannot be bounded.

Proof. Since $h \in \mathcal{B}(\mathbb{D})$, there is $\varphi \in L^\infty(\mathbb{D})$ such that

$$h(\lambda) = \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1-\lambda\bar{\zeta})^2}dA(\zeta),\ \lambda \in \mathbb{D}.$$

By the reproducing formula of the Bergman kernel and Theorem 3.1 we see that

$$f(z,w) := \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1-z\bar{\zeta})(1-w\bar{\zeta})}dA(\zeta) = (T\varphi)(z,w) \in bmoa(\mathbb{T}^2).$$

By [FS] we see that $H_{\bar{f}}$ is bounded on $H^2(\mathbb{D}^2)$. Now suppose that there is a holomorphic function g on $\mathbb{D}^2$ such that the function $F := \bar{f}+g$ is bounded. Then the restriction of F to the diagonal is also bounded and hence in $BMO(\mathbb{T})$. Since if $\lambda \in \mathbb{D}$, we have $F(\lambda,\lambda) = \bar{h}(\lambda) + g(\lambda,\lambda)$, it follows that both functions h and $g(\lambda,\lambda)$ are in $BMOA(\mathbb{T})$. This contradicts the fact that $h \notin BMOA(\mathbb{T})$. □

Now, Theorem A in the bidisc case follows from Theorem 3.4.

Proposition 3.5. $bmo(\mathbb{T}^2) = \widetilde{BMO}(\mathbb{T}^2)$.

Proof. For $z,w \in \mathbb{D}$ and $\theta \in [0,2\pi]$, set $P_z(\theta) := \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}$ and define the measure $\mu_{z,w}$ on $[0,2\pi]\times[0,2\pi]\times[0,2\pi]\times[0,2\pi]$ by

$$d\mu_{z,w}(\theta,s,\psi,t) := \frac{1}{(2\pi)^4}P_z(\theta)P_z(\psi)P_w(s)P_w(t)d\theta ds d\psi dt.$$

A little computing shows that if $f \in L^2(\mathbb{T}^2)$, then we have the following key equality:

$$\begin{aligned}&2\left(\widetilde{|f|^2} - \left|\widetilde{f}\right|^2\right)(z,w) \\ &\qquad = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|f(e^{i\theta},e^{is}) - f(e^{i\psi},e^{it})\right|^2 d\mu_{z,w}(\theta,s,\psi,t),\end{aligned}$$

which will be used to prove both inclusions. On the one hand, suppose that $f \in bmo(\mathbb{T}^2)$. Then the key inequality and the triangle inequality imply that

$$\sqrt{2\left(\widetilde{|f|^2} - \left|\widetilde{f}\right|^2\right)(z,w)} \leq \sqrt{A(z,w)} + \sqrt{B(z,w)},$$

where

$$\begin{aligned} A(z,w) &:= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|f(e^{i\theta},e^{is}) - f(e^{i\theta},e^{it})\right|^2 d\mu_{z,w}(\theta,s,\psi,t) \\ &= \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|f(e^{i\theta},e^{is}) - f(e^{i\theta},e^{it})\right|^2 P_z(\theta)P_w(s)P_w(t)d\theta ds dt \\ &= \frac{1}{\pi}\int_0^{2\pi} \left(M[f(e^{i\theta},\cdot)](w)\right)^2 P_z(\theta)d\theta \end{aligned}$$

and similarly

$$\begin{aligned} B(z,w) &:= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|f(e^{i\theta},e^{it}) - f(e^{i\psi},e^{it})\right|^2 d\mu_{z,w}(\theta,s,\psi,t) \\ &= \frac{1}{\pi}\int_0^{2\pi} \left(M[f(\cdot,e^{it})](z)\right)^2 P_w(t), dt \end{aligned}$$

showing that $bmo(\mathbb{T}^2) \subseteq \widetilde{BMO}(\mathbb{T}^2)$.

On the other hand, suppose that $f \in \widetilde{BMO}(\mathbb{T}^2)$. By Proposition 1.3 of [CS], in order to show that $f \in bmo(\mathbb{T}^2)$, we only need prove that f is of bounded mean oscillation on rectangles; that is, for some constant $C > 0$ we have

$$\frac{1}{|I|}\frac{1}{|J|}\int_{I\times J} \left|f(e^{i\theta},e^{is}) - f_{I\times J}\right|^2 d\theta ds \leq C$$

for all intervals $I, J \subset [0, 2\pi]$, where $|I|$ is the legnth of an interval I and

$$f_{I\times J} := \frac{1}{|I||J|}\int_I\int_J f(e^{i\theta},e^{is})d\theta ds.$$

To do so, let $z = \sqrt{1-\varepsilon}e^{i\theta_0}$ and $w = \sqrt{1-\varepsilon}e^{is_0}$ where $\theta_0, s_0 \in [0, 2\pi]$ and $\varepsilon \in]0, 1]$, then for some constant $C > 0$ independent of all parameters we have

$$P_z(\theta) \geq \frac{C}{1-|z|^2} \quad \text{and} \quad P_w(\theta) \geq \frac{C}{1-|w|^2}.$$

Thus by the key equality above we have

$$\begin{aligned} &2\left(\widetilde{|f|^2} - \left|\widetilde{f}\right|^2\right)(z,w) \\ &\geq \int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\int_{s_0-\varepsilon}^{s_0+\varepsilon}\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\int_{s_0-\varepsilon}^{s_0+\varepsilon} \left|f(e^{i\theta},e^{is}) - f(e^{i\psi},e^{it})\right|^2 d\mu_{z,w}(\theta,s,\psi,t) \\ &\geq \frac{C^4}{(2\varepsilon\pi)^4}\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\int_{s_0-\varepsilon}^{s_0+\varepsilon}\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\int_{s_0-\varepsilon}^{s_0+\varepsilon} \left|f(e^{i\theta},e^{is}) - f(e^{i\psi},e^{it})\right|^2 dt d\psi ds d\theta \\ &= \frac{2C^4}{(2\pi)^4}\frac{1}{(2\varepsilon)^2}\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\int_{s_0-\varepsilon}^{s_0+\varepsilon} \left|f(e^{i\theta},e^{is}) - f_{]\theta_0-\varepsilon,\theta_0+\varepsilon[\times]s_0-\varepsilon,s_0+\varepsilon[}\right|^2 ds d\theta. \end{aligned}$$

This shows that

$$\frac{1}{|I|}\frac{1}{|J|}\int_{I\times J}\left|f(e^{i\theta},e^{is})-f_{I\times J}\right|^2 d\theta ds \le \frac{(2\pi)^4}{C^4}\sup_{z,w\in\mathbb{D}}\left(\widetilde{|f|^2}-\left|\widetilde{f}\right|^2\right)(z,w)$$

for all intervals I, J with length less than or equal to 2. Similar argument shows that

$$\sup_{|I|\le 2,|J|>2}\frac{1}{|I|}\frac{1}{|J|}\int_{I\times J}\left|f(e^{i\theta},e^{is})-f_{I\times J}\right|^2 d\theta ds \le \frac{(2\pi)^2}{C^2}\sup_{z\in\mathbb{D}}\left(\widetilde{|f|^2}-\left|\widetilde{f}\right|^2\right)(z,0),$$

and by symmetry we have

$$\sup_{|J|\le 2,|I|>2}\frac{1}{|I|}\frac{1}{|J|}\int_{I\times J}\left|f(e^{i\theta},e^{is})-f_{I\times J}\right|^2 d\theta ds \le \frac{(2\pi)^2}{C^2}\sup_{w\in\mathbb{D}}\left(\widetilde{|f|^2}-\left|\widetilde{f}\right|^2\right)(0,w).$$

The case where both intervals have length greater than 2 can be handled easily. □

Proposition 3.5, combined with Theorem 2.1 of [FS], proves Theorem B and answers in the affirmative a conjecture due to Berger, Coburn and Zhu [C] for the bidisc case.

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