HANKEL OPERATORS WITH UNBOUNDED SYMBOLS

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Abstract. We prove that there are holomorphic functions $f$ in the Hardy space of the unit ball or the bidisc such that the big Hankel operator with symbol $\bar{f}$ is bounded and for any holomorphic function $g$ the function $\bar{f} + g$ cannot be bounded.

1. Introduction

Throughout this note $\Omega = \Omega_n$ will denote the domain in $\mathbb{C}^n$ which is either the unit ball $\mathbb{B}_n$ or the the unit polydisc $\mathbb{D}_n$, where $\mathbb{D}^n$ is the unit disc in $\mathbb{C}$. We let $\sigma$ denote the normalized Lebesgue measure on the Shilov boundary $\partial_s \Omega$ of $\Omega$. For the unit ball then $\partial_s \mathbb{B}_n := S$ is the unit sphere and for the unit polydisc, $\partial_s \mathbb{D}_n = T^n$ where $T$ is the unit circle. Let $H^2(\Omega)$ be the Hardy space on $\Omega$ with respect to $\sigma$.

The Berezin transform of a function $f \in L^2(\sigma)$ is defined by

$$\tilde{f}(z) := \int_{\partial_s \Omega} f(\xi) P(z, \xi) d\sigma(\xi), \quad z \in \Omega,$$

where $P(z, \xi)$ is the Poisson kernel of $\Omega$ with respect to $\sigma$. We follow Coburn [C] to define the space $\tilde{BMO}(\partial \Omega)$ of functions with bounded mean oscillation with respect to $\sigma$ as the subspace of functions $f \in L^2(\sigma)$ that satisfy

$$\sup_{z \in \Omega} \left[ |f(z)|^2 - |\tilde{f}(z)|^2 \right] < \infty.$$

We point out that in the one-dimensional case this definition gives the classical $BMO$ space. See [Z1]. The subspace of $\tilde{BMO}(\partial \Omega)$ consisting of those analytic functions on $\Omega$ will be denoted by $\tilde{BMOA}(\partial \Omega)$. In the case of the unit ball the space $\tilde{BMOA}(\mathbb{B})$ was shown to be the dual of the Hardy space $H^1(\mathbb{B})$. See Theorem 5.13 of [Z2]. For the polydisc case we show in Proposition 3.5 below that $\tilde{BMO}$ coincides with the small $bmo$ defined by Cotlar and Sadosky [CS]. It is not the dual of the real $H^1(\mathbb{T}^2)$, and its predual is given by Theorem 1.5 of [CS].

The big Hankel operator $H_f$ with symbol $f \in L^2(\sigma)$ is defined by

$$H_f(g) := (I - P)(fg), \quad g \in H^\infty(\Omega).$$

We recall that in the one variable setting $\Omega = \mathbb{D}$, if a function $f$ is in $H^2(\mathbb{D})$, then the big Hankel operator $H_f$ is bounded on $H^2(\mathbb{T})$ if and only if there is another $g \in H^2(\mathbb{T})$ such that $f + g$ is bounded. See [Z1]. The goal of this note is to show this...
is no longer true in higher dimensions. In particular, this answers in the negative a question by E. Strouse [S] which she raised for the case of the bidisc. We shall establish the following.

**Theorem A.** There are functions \( f \in H^2(\Omega_2) \) such that the big Hankel operator \( H_f \) is bounded on \( H^2(\Omega_2) \) but \( f + g \) is unbounded for any holomorphic function \( g \) on \( \Omega_2 \).

We point out that the existence of big Hankel operators with not necessarily antianalytic unbounded symbols was established in [CS] and [BT] for the case of the bidisc.

It is not hard to see [C] that if \( f \in L^2(\partial \Omega) \) and both operators \( H_f \) and \( H_{\bar{f}} \) are bounded, then \( f \in \text{BMO}(\partial \Omega) \). The converse is true for the bidisc. Indeed,

**Theorem B.** Let \( f \in L^2(\mathbb{T}^2) \). Then \( f \in \text{BMO}(\mathbb{T}^2) \) if and only if the big Hankel operators \( H_f \) and \( H_{\bar{f}} \) are bounded on \( H^2(\mathbb{D}^2) \).

The analogue of Theorem B for the unit ball in \( \mathbb{C}^2 \) remains open.

2. **The ball case**

We recall that the Bergman distance \( \beta(\zeta, z) \) between two points \( \zeta \) and \( z \) in the unit disc \( \mathbb{D} \) in the complex plane is given by

\[
\beta(\zeta, z) = \frac{1}{2} \log \frac{1 + |\varphi_z(\zeta)|}{1 - |\varphi_z(\zeta)|}, \quad \text{where} \quad \varphi_z(\zeta) := \frac{\zeta - z}{1 - \bar{z}\zeta}.
\]

**Lemma 2.1.** There is a positive constant \( C \) independent of \( n > 0 \) such that

\[
I_n(z) := \int_{\mathbb{D}} \frac{dA(w)}{(1 - |w|^2)^{\frac{n}{2}}} \leq \frac{C}{(n + 1)^{\frac{n}{2}}} (1 - |z|^2)^{-\frac{2n}{n+1}}
\]

for all \( z \in \mathbb{D} \).

**Proof.** Setting \( \zeta = \varphi_z(w) \) we see that

\[
I_n(z) = \int_{\mathbb{D}} \frac{\beta(0, w)(1 - |w|^2)^{\frac{n}{2}}}{|1 - \zeta w|^2} dA(w)
\]

\[
= \frac{(1 - |z|^2)^{-\frac{2n}{n+1}}}{2} \int_0^1 r \log \frac{1 + r}{1 - r} (1 - r^2)^{\frac{n}{2}} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 - re^{i\theta}|z|^2} dr
\]

\[
= \frac{(1 - |z|^2)^{-\frac{n}{n+1}}}{2} \int_0^1 r \left( \frac{1 - r^2}{1 - r^2|z|^2} \log \frac{1 + r}{1 - r} \right) dr
\]

\[
\leq \frac{(1 - |z|^2)^{-\frac{n}{n+1}}}{2} \left( \int_0^1 r \log(1 + r)(1 - r^2)^{\frac{n}{n+1}} dr + \int_0^1 r(1 - r^2)^{\frac{n}{n+1}} \log \frac{1}{1 - r} dr \right).
\]

Integration by parts and the change of variable \( t = 1 - r \) give that the second integral in the last equality is equal to \( 4n^{-2} \). To estimate the first integral in terms
of $n$, we have
\[
\int_0^1 r \log(1 + r)(1 - r^2)^{\frac{1}{2} - 1} dr = \frac{1}{n} \int_0^1 \frac{(1 - r^2)^{\frac{1}{2}}}{1 + r} dr \\
\leq \frac{1}{2n} \int_0^1 (1 - r^2)^{\frac{1}{2}} dr \\
= \frac{1}{2n} \int_0^\pi (\sin t)^n dt \leq \frac{C}{n^\frac{3}{2}}. \tag*{□}
\]

Let $\mathcal{B}(\mathbb{D})$ be the Bloch space of $\mathbb{D}$ and if $F \in \mathcal{B}(\mathbb{D})$, we extend $F$ to the ball $\mathbb{B}_2$ by setting $F(z, w) := F(z)$. Then we have

**Lemma 2.2.** If $F \in \mathcal{B}(\mathbb{D})$, then the big Hankel operator $H_{TF}$ is bounded on the Hardy space $H^2(\mathbb{B}_2)$.

We write each $g \in H^2(\mathbb{B})$ in the form $g(z, w) = \sum_n g_n(z) w^n$, where the functions $g_n$ are holomorphic in $\mathbb{D}$. In addition, we have the identity
\[
\int_S g_n(z) w^n g_m(z) w^m d\sigma(z, w) = \begin{cases} 0, & n \neq m, \\
\int_D |g_n(z)|^2 (1 - |z|^2)^n dA(z), & n = m. 
\end{cases}
\]
Hence
\[
\|g\|^2_{L^2(S)} = \sum_n \int_D |g_n(z)|^2 (1 - |z|^2)^n dA(z).
\]

On the other hand, for each $n$, we have that
\[
(PFg_n)(z, w) = \int_D \frac{F(\zeta) g_n(\zeta) n^n}{(1 - \zeta z - \bar{\eta} w)^2} d\sigma(\zeta, \eta) \\
= \int_D \frac{F(\zeta) g_n(\zeta)}{(1 - \zeta z - \bar{\eta} w)^2} \int_0^{2\pi} \frac{(1 - |z|^2)^{\frac{1}{2}} e^{i n \theta}}{(1 - \zeta z - \sqrt{1 - |z|^2} e^{i \theta})^2} \frac{d\theta}{2\pi} dA(\zeta) \\
= \int_D \frac{F(\zeta) g_n(\zeta)}{(1 - \zeta z)^2} \int_0^{2\pi} \frac{e^{i n \theta}}{(1 - \sqrt{1 - |z|^2} e^{i \theta})^2} \frac{d\theta}{2\pi} dA(\zeta) \\
= (n + 1) w^n \int_D \frac{F(\zeta) g_n(\zeta)(1 - |\zeta|^2)^n}{(1 - \zeta z)^{n+2}} dA(\zeta) w^n.
\]
Since $(n + 1)(1 - |\zeta|^2)^n$ reproduces holomorphic functions on the unit disc, it follows that
\[
(H_{TF}g)(z, w) = \sum_n (n + 1) w^n \int_D \frac{F(\zeta)}{(1 - \zeta z)^{n+2}} g_n(\zeta)(1 - |\zeta|^2)^n dA(\zeta) \\
= \sum_n H_n(z) w^n,
\]
where
\[
H_n(z) := (n + 1) \int_D \frac{F(\zeta)}{(1 - \zeta z)^{n+2}} g_n(\zeta)(1 - |\zeta|^2)^n dA(\zeta).
\]
Since $F \in \mathcal{B}(\mathbb{D})$, we see that $|F(z) - F(\zeta)| \leq \|F\|_{\mathcal{B}(\mathbb{D})} |z - \bar{\zeta}|$. Hence
\[
|H_n(z)| \leq (n + 1) \|F\|_{\mathcal{B}(\mathbb{D})} \int_D \frac{\beta(z, \zeta) |g_n(\zeta)|}{|1 - \zeta z|^{n+2}} (1 - |\zeta|^2)^n dA(\zeta)
\]
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so that by Hölder’s inequality and Lemma 2.1 we see that

$$|H_n(z)|^2 \leq (n + 1)^2 \|F\|_{B^2(B)}^2 \langle I_n(z) \rangle^2 \int_{\mathbb{D}} \frac{\beta(z, \zeta)|g_n(\zeta)|^2(1 - |\zeta|^2)^{\frac{n}{2}}}{|1 - \zeta|^n + 2} dA(\zeta)$$

$$\leq C \sqrt{n} \|F\|_{B^2(B)}^2 (1 - |z|^2)^{-\frac{n}{2}} \int_{\mathbb{D}} \frac{\beta(z, \zeta)|g_n(\zeta)|^2(1 - |\zeta|^2)^{\frac{n}{2}}}{|1 - \zeta|^n + 2} (1 - |\zeta|^2)^n dA(\zeta).$$

Therefore, again applying Lemma 2.1, we have

$$\int_{\mathbb{D}} |H_n(z)|^2 (1 - |z|^2)^n dA(\zeta)$$

$$\leq 2 \sqrt{n} \|F\|_{B^2(B)}^2 \int_{\mathbb{D}} |g_n(\zeta)|^2 (1 - |\zeta|^2)^{\frac{n}{2}} \int_{\mathbb{D}} \frac{\beta(z, \zeta)(1 - |\zeta|^2)^{\frac{n}{2}}}{|1 - \zeta|^n + 2} dA(z) dA(\zeta)$$

$$\leq 2 \frac{\|F\|_{B^2(B)}^2}{n} \int_{\mathbb{D}} |g_n(\zeta)|^2 (1 - |\zeta|^2)^n dA(\zeta).$$

Thus

$$\|(H_T g)\|_{H^2(B)}^2 = \sum_n \int_{\mathbb{D}} |H_n(z)|^2 (1 - |z|^2)^n dA(\zeta)$$

$$\leq \sum_n \frac{\|F\|_{B^2(B)}^2}{n} \int_{\mathbb{D}} |g_n(\zeta)|^2 (1 - |\zeta|^2)^n dA(\zeta)$$

$$\leq \|F\|_{B^2(B)}^2 \|g\|_{H^2(B)}^2.$$

**Proof of Theorem A in the ball case** $\Omega_2 = \mathbb{B}_2$. Let $F \in B(B) \setminus BMOA(\mathbb{T})$ be extended to $\mathbb{B}_2$ as the function $(z, w) \mapsto F(z)$. By Lemma 2.2 the big Hankel operator $H_T$ is bounded on the Hardy space $H^2(\mathbb{B}_2)$. Assume that there is a holomorphic function $g$ on $\mathbb{B}_2$ such that the function $G := F + g$ is bounded. Then the function $z \mapsto G(z, 0)$ is bounded on the unit disc and hence in $BMOA(\mathbb{T})$. This shows that both functions $F$ and $z \mapsto g(z, 0)$ are in $BMOA(\mathbb{T})$. This contradicts the fact that $F \notin BMOA(\mathbb{T})$. \qed

### 3. The bidisc case

Let $bmo(\mathbb{T}^2)$ be the subspace of functions $f \in L^1(\mathbb{T}^2)$ such that $f$ is in $BMO(\mathbb{T})$ in each variable and

$$\sup_{z \in \mathbb{T}} \|f(z, \cdot)\|_{BMO} + \sup_{w \in \mathbb{T}} \|f(\cdot, w)\|_{BMO} < +\infty.$$ 

In particular, if $f \in L^1(\mathbb{T}^2)$, then $f$ is in $bmo(\mathbb{T}^2)$ if and only if

$$\sup_{z \in \mathbb{T}} \|Mf(z, \cdot)\|_\infty + \sup_{w \in \mathbb{T}} \|Mf(\cdot, w)\|_\infty < +\infty,$$

where

$$(M\varphi)(\zeta) := \sqrt{|\varphi|^2(\zeta) - |\zeta|^2(\zeta)}, \quad \zeta \in \mathbb{D}, \varphi \in L^2(\mathbb{T}).$$

The subspace of $bmo(\mathbb{T}^2)$ consisting of holomorphic functions on $\mathbb{D}^2$ will be denoted by $bmoa(\mathbb{T}^2)$

**Theorem 3.1.** If $\varphi \in L^\infty(\mathbb{D})$, then

$$(T\varphi)(z, w) := \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1 - z\zeta)(1 - w\zeta)} dA(\zeta), \quad (z, w) \in \mathbb{T}^2,$$
is in bmoa(T^2). In addition, the operator T is bounded from L^∞(D) into bmoa(T^2).

To prove this theorem we need some auxiliary results. The following lemma can be found in [3].

**Lemma 3.2.** There is a positive constant C such that

\[
\int_{\mathbb{D}} \frac{dA(\zeta)d}{|1 - z\zeta|^{\alpha}|1 - w\zeta|^{\beta}} \leq \begin{cases} 
C, & \alpha + \beta < 2, \\
\frac{C}{\log|1 - z\bar{w}|}, & \alpha + \beta = 2, \\
\frac{C}{1 - z\bar{w},} & \alpha + \beta > 2,
\end{cases}
\]

for all z, w ∈ D.

**Lemma 3.3.** There is a positive constant C such that

\[
I(z, w) := \int_{\mathbb{D} \times \mathbb{D}} \frac{dA(\zeta)dA(\eta)}{|1 - z\zeta||1 - z\bar{\eta}||1 - w\zeta||1 - w\bar{\eta}|1 - \zeta\bar{\eta}|} \leq \frac{C}{1 - z\bar{w}}
\]

for all z, w ∈ D.

**Proof.** We observe that I(z, w) ≤ I_1(z, w) + I_2(z, w) + I_3(z, w), where

\[
I_j(z, w) := \int_{E_j} \frac{dA(\zeta)dA(\eta)}{|1 - z\zeta||1 - z\bar{\eta}||1 - w\zeta||1 - w\bar{\eta}|1 - \zeta\bar{\eta}|}, \quad j = 1, 2, 3,
\]

\[
E_1 := \{ (\zeta, \eta) \in \mathbb{D} \times \mathbb{D} : |1 - \zeta\bar{\eta}| \leq \frac{1}{2} |1 - z\bar{\zeta}| \},
\]

\[
E_2 := \{ (\zeta, \eta) \in \mathbb{D} \times \mathbb{D} : |1 - \zeta\bar{\eta}| \leq \frac{1}{2} |1 - w\bar{\zeta}| \}
\]

and E_3 is the complement of E_1 ∪ E_2 in D × D. By symmetry the integrals I_1(z, w) and I_2(z, w) can be estimated similarly. Since by Lemma 3.2

\[
I_3(z, w) \leq 2 \int_{\mathbb{D} \times \mathbb{D}} \frac{dA(\zeta)dA(\eta)}{|1 - z\zeta||1 - z\bar{\eta}||1 - w\zeta||1 - w\bar{\eta}|1 - \zeta\bar{\eta}|^{2}} \leq \frac{C}{|1 - z\bar{w}|},
\]

we only need to estimate I_1(z, w). To do so, we write I_1(z, w) = I_{11}(z, w) + I_{12}(z, w), where

\[
I_{1j}(z, w) := \int_{E_{1j}} \frac{dA(\zeta)dA(\eta)}{|1 - z\zeta||1 - z\bar{\eta}||1 - w\zeta||1 - w\bar{\eta}|1 - \zeta\bar{\eta}|}, \quad j = 1, 2,
\]

\[
E_{11} := \{ (\zeta, \eta) \in E_1 : |1 - \zeta\bar{\eta}| \leq \frac{1}{2} |1 - w\bar{\zeta}| \} \text{ and } E_{12} \text{ is the complement of } E_{11} \text{ in } E_1.
\]

Set R(z, w, \zeta) := \min(|1 - z\bar{\zeta}|, |1 - w\zeta|). Since on E_{11} we have |1 - z\bar{\zeta}| ≈ |1 - z\bar{\zeta}| and |1 - w\zeta| ≈ |1 - w\zeta|, it follows that

\[
I_{11}(z, w) \approx \int_{E_{11}} \frac{dA(\zeta)dA(\eta)}{|1 - z\zeta||1 - w\zeta||1 - \zeta\bar{\eta}|} \leq \int_{\mathbb{D}} \int_{\eta \in \mathbb{D} : |1 - \zeta\bar{\eta}| \leq \frac{1}{2} R(z, w, \zeta)} \frac{dA(\eta)}{|1 - \zeta\bar{\eta}|} \frac{dA(\zeta)}{|1 - z\zeta||1 - w\zeta|} \leq \frac{C}{|1 - z\bar{w}|}.
\]
where the latter inequality holds in view of Lemma 3.2. It now remains to establish the estimate of \( I_{12}(z, w) \). We write \( I_{12}(z, w) = \sum_{j=1}^{3} I_{12j}(z, w) \), where

\[
I_{12j}(z, w) := \int_{E_{12j}} \frac{dA(\zeta)dA(\eta)}{|1 - z\zeta||1 - z\bar{\eta}||1 - w\zeta||1 - w\bar{\eta}||1 - \zeta\bar{\eta}|}, \quad j = 1, 2, 3,
\]

\[
E_{121} := \{ (\zeta, \eta) \in E_{12} : |1 - w\bar{\eta}| \leq \frac{1}{2}|1 - \zeta\bar{\eta}| \},
\]

\[
E_{122} := \{ (\zeta, \eta) \in E_{12} : |1 - \zeta\bar{\eta}| \leq \frac{1}{2}|1 - w\bar{\eta}| \},
\]

and \( E_{123} \) is the complement of \( E_{121} \cup E_{122} \) in \( E_{12} \). On \( E_{121} \) we have \( |1 - w\bar{\zeta}| \approx |1 - \zeta\bar{\eta}| \) and \( |1 - \zeta\bar{\eta}| \leq CR(z, w, \zeta) \). Hence, by Lemma 3.2,

\[
I_{121}(z, w) \approx \int_D \int_{\{ \eta \in D : |1 - w\bar{\eta}| \leq CR(z, w, \zeta) \}} \frac{A(\eta)}{|1 - w\bar{\eta}|} \frac{dA(\zeta)}{|1 - z\zeta||1 - w\zeta|^2} \leq C \int_D \frac{R(z, w, \zeta)}{|1 - z\zeta||1 - w\zeta|^2} dA(\zeta) \leq \frac{C}{|1 - zw|}.
\]

On \( E_{122} \) we have \( |1 - w\bar{\zeta}| \approx |1 - w\bar{\eta}| \) and \( |1 - \zeta\bar{\eta}| \leq CR(z, w, \zeta) \). Hence as in the previous case we have

\[
I_{122}(z, w) \approx \int_D \int_{\{ \eta \in D : |1 - w\bar{\eta}| \leq CR(z, w, \zeta) \}} \frac{A(\eta)}{|1 - w\bar{\eta}|} \frac{dA(\zeta)}{|1 - z\zeta||1 - w\zeta|^2} \leq \frac{C}{|1 - zw|}.
\]

Finally, on \( E_{123} \) we have \( |1 - w\bar{\eta}| \approx |1 - \zeta\bar{\eta}| \) and \( |1 - w\zeta| \leq 2|1 - \zeta\bar{\eta}| \). Thus

\[
I_{123}(z, w) \approx \int_D \int_{\{ \eta \in D : |1 - \zeta\bar{\eta}| \leq \frac{1}{2}|1 - z\zeta| \}} \frac{A(\eta)}{|1 - \zeta\bar{\eta}|} \frac{dA(\zeta)}{|1 - z\zeta||1 - w\zeta|^2} \leq C \int_D \frac{dA(\zeta)}{|1 - z\zeta||1 - w\zeta|^2} \leq \frac{C}{|1 - zw|}.
\]

\[\square\]

**Proof of Theorem 3.1.** Let

\[
f(z, w) = (T\varphi)(z, w) = \int_D \frac{\varphi(\zeta)}{1 - z\zeta(1 - w\zeta)} dA(\zeta), \quad (z, w) \in D^2.
\]

Then \( f \) is in the Hardy space \( H^2(D^2) \) and satisfies \( f(z, w)(z) = f(z, w) \) and

\[
|f(z, w)|^2(z) = \frac{1}{2\pi} \int_T \int_{D \times D} \frac{\varphi(\zeta)\varphi(\eta)(1 - |z|^2)dA(\zeta)dA(\eta)}{(1 - \tau\zeta)(1 - w\zeta)(1 - \tau\eta)(1 - w\eta)|1 - z\zeta|^2|1 - z\zeta|^2} d\tau d\zeta
\]

\[
= \int_{D \times D} \frac{\varphi(\zeta)\varphi(\eta)(1 - |\zeta|^2)dA(\zeta)dA(\eta)}{(1 - \zeta(1 - w\zeta)(1 - \zeta)(1 - w\eta)(1 - \zeta)}
\]

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by the Cauchy formula. Therefore, by Lemma 3.3 we see that
\[
(Mf(z, \cdot))^2(z) = |\overline{f(z)}|^2(z) - |f(z, w)|^2
\]
\[
= \int_{\mathbb{D} \times \mathbb{D}} \frac{\zeta \overline{\varphi(\zeta)} \varphi(\eta)(1 - |z|^2) dA(\zeta) dA(\eta)}{(1 - \zeta \zeta)(1 - w \zeta)(1 - \bar{\zeta} \eta)(1 - \bar{\eta} \eta)(1 - \zeta \eta)}
\]
\[
\leq \|\varphi\|^2_{\infty} \int_{\mathbb{D} \times \mathbb{D}} \frac{dA(\zeta) dA(\eta)}{(1 - \zeta \zeta)(1 - w \zeta)(1 - \bar{\zeta} \eta)(1 - \bar{\eta} \eta)(1 - \zeta \eta)}
\]
\[
\leq C\|\varphi\|^2_{\infty}.
\]
Similarly we have \[\|Mf(z, \cdot)\|_\infty \leq C\|\varphi\|_\infty\], showing that \(T\varphi \in \text{bmoa}(\mathbb{T}^2)\) and
\[\|T\varphi\|_{\text{bmoa}} \leq C\|\varphi\|_\infty.\]

\[\square\]

**Theorem 3.4.** Suppose that \(h \in \mathcal{B}(\mathbb{D})\) and \(h \notin \text{BMOA}(\mathbb{T})\) and let
\[
f(z, w) := \int_{\mathbb{D}} \frac{h(\zeta)}{(1 - z \zeta)(1 - w \zeta)} dA(\zeta), \ (z, w) \in \mathbb{T}^2.
\]
Then the big Hankel operator \(H_f\) is bounded on \(H^2(\mathbb{D}^2)\), and for any holomorphic function \(g\) on \(\mathbb{D} \times \mathbb{D}\), the function \(f + g\) cannot be bounded.

**Proof.** Since \(h \in \mathcal{B}(\mathbb{D})\), there is \(\varphi \in L^\infty(\mathbb{D})\) such that
\[
h(\lambda) = \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1 - \lambda \zeta)^2} dA(\zeta), \ \lambda \in \mathbb{D}.
\]
By the reproducing formula of the Bergman kernel and Theorem 3.1 we see that
\[
f(z, w) := \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1 - z \zeta)(1 - w \zeta)} dA(\zeta) = (T\varphi)(z, w) \in \text{bmoa}(\mathbb{T}^2).
\]
By \([FS]\) we see that \(H_f\) is bounded on \(H^2(\mathbb{D}^2)\). Now suppose that there is a holomorphic function \(g\) on \(\mathbb{D}^2\) such that the function \(F := f + g\) is bounded. Then the restriction of \(F\) to the diagonal is also bounded and hence in \(\text{BMOA}(\mathbb{T})\). Since if \(\lambda \in \mathbb{D}\), we have \(F(\lambda, \lambda) = h(\lambda) + g(\lambda, \lambda)\), it follows that both functions \(h\) and \(g(\lambda, \lambda)\) are in \(\text{BMOA}(\mathbb{T})\). This contradicts the fact that \(h \notin \text{BMOA}(\mathbb{T})\). \[\square\]

Now, Theorem A in the bidisc case follows from Theorem 3.4.

**Proposition 3.5.** \(\text{bmoa}(\mathbb{T}^2) = \widehat{\text{BMO}}(\mathbb{T}^2)\).

**Proof.** For \(z, w \in \mathbb{D}\) and \(\theta \in [0, 2\pi]\), set \(P_z(\theta) := \frac{1 - |z|^2}{1 - ze^{-\theta}}\), and define the measure \(\mu_{z,w}\) on \([0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]\) by
\[
d\mu_{z,w}(\theta, s, \psi, t) := \frac{1}{(2\pi)^4} P_z(\theta) P_z(\psi) P_w(s) P_w(t) d\theta ds d\psi dt.
\]
A little computing shows that if \(f \in L^2(\mathbb{T}^2)\), then we have the following key equality:
\[
2 \left( |\overline{f}|^2 - |f|^2 \right)(z, w) = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |f(e^{i\theta}, e^{i\psi}) - f(e^{i\psi}, e^{i\theta})|^2 d\mu_{z,w}(\theta, s, \psi, t),
\]
which will be used to prove both inclusions. On the one hand, suppose that \( f \in bmo(T^2) \). Then the key inequality and the triangle inequality imply that
\[
\sqrt{2 \left( \left| f \right|^2 - \left| \tilde{f} \right|^2 \right)} (z, w) \leq \sqrt{A(z, w)} + \sqrt{B(z, w)},
\]
where
\[
A(z, w) := \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| f(e^{i\theta}, e^{i\psi}) - f(e^{i\psi}, e^{i\epsilon}) \right|^2 d\mu_{z,w}(\theta, \psi, t) \\
= \frac{1}{(2\pi)^3} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| f(e^{i\theta}, e^{i\epsilon}) - f(e^{i\epsilon}, e^{i\psi}) \right|^2 P_z(\theta)P_w(s) P_w(t) d\theta ds dt \\
= \frac{1}{\pi} \int_{0}^{2\pi} (M[f(e^{i\theta}, \cdot)](w))^2 P_z(\theta) d\theta
\]
and similarly
\[
B(z, w) := \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| f(e^{i\theta}, e^{i\epsilon}) - f(e^{i\psi}, e^{i\epsilon}) \right|^2 d\mu_{z,w}(\theta, \psi, t) \\
= \frac{1}{\pi} \int_{0}^{2\pi} (M[f(\cdot, e^{i\epsilon})](z))^2 P_w(t), dt
\]
showing that \( bmo(T^2) \subseteq BMO(T^2) \).

On the other hand, suppose that \( f \in \widehat{BMO}(T^2) \). By Proposition 1.3 of [CS], in order to show that \( f \in bmo(T^2) \), we only need prove that \( f \) is of bounded mean oscillation on rectangles; that is, for some constant \( C > 0 \) we have
\[
\frac{1}{|I|} \frac{1}{|J|} \int_{I \times J} \left| f(e^{i\theta}, e^{i\epsilon}) - f_{I \times J} \right|^2 d\theta ds \leq C
\]
for all intervals \( I, J \subset [0, 2\pi] \), where \( |I| \) is the length of an interval \( I \) and
\[
f_{I \times J} := \frac{1}{|I||J|} \int_{I} \int_{J} f(e^{i\theta}, e^{i\epsilon}) d\theta ds.
\]
To do so, let \( z = \sqrt{1 - \epsilon} e^{i\theta_0} \) and \( w = \sqrt{1 - \epsilon} e^{i\epsilon_0} \) where \( \theta_0, s_0 \in [0, 2\pi] \) and \( \epsilon \in [0, 1] \), then for some constant \( C > 0 \) independent of all parameters we have
\[
P_z(\theta) \geq \frac{C}{1 - |\epsilon|^2} \quad \text{and} \quad P_w(\theta) \geq \frac{C}{1 - |\epsilon|^2}.
\]
Thus by the key equality above we have
\[
2 \left( \left| f \right|^2 - \left| \tilde{f} \right|^2 \right) (z, w) \\
\geq \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} \int_{s_0 - \epsilon}^{s_0 + \epsilon} \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} \int_{s_0 - \epsilon}^{s_0 + \epsilon} \left| f(e^{i\theta}, e^{i\epsilon}) - f(e^{i\epsilon}, e^{i\psi}) \right|^2 d\mu_{z,w}(\theta, \psi, t) \\
\geq \frac{C^4}{(2\pi)^3} \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} \int_{s_0 - \epsilon}^{s_0 + \epsilon} \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} \int_{s_0 - \epsilon}^{s_0 + \epsilon} \left| f(e^{i\theta}, e^{i\epsilon}) - f(e^{i\epsilon}, e^{i\psi}) \right|^2 dt d\psi ds d\theta \\
= \frac{2C^4}{(2\pi)^4} \frac{1}{(2\epsilon)^2} \int_{\theta_0 - \epsilon}^{\theta_0 + \epsilon} \int_{s_0 - \epsilon}^{s_0 + \epsilon} \left| f(e^{i\theta}, e^{i\epsilon}) - f_{[\theta_0 - \epsilon, \theta_0 + \epsilon][s_0 - \epsilon, s_0 + \epsilon]} \right|^2 d\theta.
\]
This shows that
\[
\frac{1}{|I|} \frac{1}{|J|} \int_{I \times J} |f(e^{i\theta}, e^{is}) - f_{I \times J}|^2 \, d\theta ds \leq \frac{(2\pi)^4}{C^4} \sup_{z,w \in \mathbb{D}} \left( \left| \tilde{f} \right|^2 - \left| f \right|^2 \right)(z,w)
\]
for all intervals $I, J$ with length less than or equal to 2. Similar argument shows that
\[
\sup_{|I| \leq 2, |J| > 2} \frac{1}{|I|} \frac{1}{|J|} \int_{I \times J} |f(e^{i\theta}, e^{is}) - f_{I \times J}|^2 \, d\theta ds \leq \frac{(2\pi)^2}{C^2} \sup_{z \in \mathbb{D}} \left( \left| \tilde{f} \right|^2 - \left| f \right|^2 \right)(z,0),
\]
and by symmetry we have
\[
\sup_{|J| \leq 2, |I| > 2} \frac{1}{|I|} \frac{1}{|J|} \int_{I \times J} |f(e^{i\theta}, e^{is}) - f_{I \times J}|^2 \, d\theta ds \leq \frac{(2\pi)^2}{C^2} \sup_{w \in \mathbb{D}} \left( \left| \tilde{f} \right|^2 - \left| f \right|^2 \right)(0,w).
\]
The case where both intervals have length greater than 2 can be handled easily. □

Proposition 3.5, combined with Theorem 2.1 of [FS], proves Theorem B and answers in the affirmative a conjecture due to Berger, Coburn and Zhu [C] for the bidisc case.

References


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