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ON THE REGULARITY OF PRODUCTS AND INTERSECTIONS OF COMPLETE INTERSECTIONS

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ABSTRACT. This paper proves the formulae

$$\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J),$$
$$\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$$

for arbitrary monomial complete intersections I and J, and provides examples showing that these inequalities do not hold for general complete intersections.

1. Introduction

Let S be a polynomial ring over a field k. For a finitely generated graded S-module M let

$$a_i(M) := \max\{\mu \mid H^i_{\mathfrak{m}}(M)_\mu \neq 0\}$$

if $H^i_{\mathfrak{m}}(M) \neq 0$ and $a_i(M) := -\infty$ otherwise, where $H^i_{\mathfrak{m}}(M)$ denotes the ith local cohomology module of M with respect to the graded maximal ideal $\mathfrak{m}$ of S. Then the Castelnuovo-Mumford regularity (or regularity for short) of M is defined as the invariant

$$\operatorname{reg}(M) := \max_i\{a_i(M) + i\}.$$

It is of great interest to have good bounds for the regularity [BaM].

The regularity of products of ideals was first studied by Conca and Herzog [CoH]. They found some special classes of ideals I and J for which the following formula holds:

$$\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$$

(see also [Si]). In particular, they showed that $\operatorname{reg}(I_1 \cdots I_d) = d$ for any set of ideals $I_1, \ldots, I_d$ generated by linear forms. These results led them to raise the question whether the formula

$$\operatorname{reg}(I_1 \cdots I_d) \leq \operatorname{reg}(I_1) + \cdots + \operatorname{reg}(I_d)$$

holds for any set of complete intersections $I_1, \ldots, I_d$ [CoH, Question 3.6]. Note that this formula does not hold for arbitrary monomial ideals. For instance, Terai and Sturmfels (see [St]) gave examples of monomial ideals I such that $\operatorname{reg}(I^2) > 2\operatorname{reg}(I)$.

On the other hand, Sturmfels conjectured that $\operatorname{reg}(I_1 \cap \cdots \cap I_d) \leq d$ for any set of ideals $I_1, \ldots, I_d$ generated by linear forms. This conjecture was settled in the

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affirmative by Derksen and Sidman [DS]. Their proof was inspired by the work of Conca and Herzog. So one might be tempted to ask whether the formula

$$\operatorname{reg}(I_1 \cap \cdots \cap I_d) \leq \operatorname{reg}(I_1) + \cdots + \operatorname{reg}(I_d)$$

holds for any set of complete intersections $I_1, \ldots, I_d$.

The following result shows that these questions have positive answers in the monomial case and we shall see that there are counter-examples in the general case.

Theorem 1.1. *Let I and J be two arbitrary monomial complete intersections. Then*

$$\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J),$$
$$\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J).$$

Both formulae follow from a more general bound for the regularity of a larger class of ideals constructed from I and J (Theorem 3.1). The proof is a bit intricate. It is based on a bound for the regularity of a monomial ideal in terms of the degree of the least common multiple of the monomial generators and the height of the given ideal.

We are not able to extend the first formula to more than two monomial complete intersections. But we find another proof which extends the second formula to any finite set of monomial complete intersections (Theorem 3.3). We would like to mention that the first formula was already proved in the case where one of the ideals I, J is generated by two elements by combinatorial methods in [M].

In the last section, we give a geometric approach for constructing examples of complete intersection ideals for which the inequalities $\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$ and/or $\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$ fail. We show for instance the following.

Theorem 1.2. *Let Y in $\mathbb{P}^3$ be a curve which is defined by at most 4 equations at the generic points of its irreducible components. Consider 4 elements f_1, f_2, f_3, f_4 in I_Y such that $I := (f_1, f_2)$ and $J := (f_3, f_4)$ are complete intersection ideals and I_Y is the unmixed part of $I + J$. Assume that $\varepsilon := \min\{\mu \mid H^0(Y, \mathcal{O}_Y(\mu)) \neq 0\} < 0$. Then*

$$\operatorname{reg}(IJ) = \operatorname{reg}(I) + \operatorname{reg}(J) - \varepsilon - 1.$$

A similar construction is explained for $I \cap J$. As a consequence, many families of curves with sections in negative degrees give rise to counter-examples for the considered inequalities. In the examples we give, I is a monomial ideal and J is either generated by one binomial or by one monomial and one binomial.

2. Preliminaries

Let us first introduce some conventions. For any monomial ideal we can always find a minimal basis consisting of monomials. These monomials will be called the *monomial generators* of the given ideal. Moreover, for a finite set of monomials $A_i = x_1^{a_{i1}} \cdots x_n^{a_{in}}$, we call the monomial $x_1^{\max_i\{a_{i1}\}} \cdots x_n^{\max_i\{a_{in}\}}$ the *least common multiple* of the monomials A_i.

The key point of our approach is the following bound for the regularity of arbitrary monomial ideals.

Lemma 2.1 ([HT, Lemma 3.1])**.** *Let I be a monomial ideal. Let F denote the least common multiple of the monomial generators of I. Then*

$$\operatorname{reg}(I) \leq \deg F - \operatorname{ht}(I) + 1.$$

This bound is an improvement of the bound $\operatorname{reg}(I) \leq \deg F - 1$ given by Bruns and Herzog in [BrH, Theorem 3.1(a)].

If we apply Lemma 2.1 to the product and the intersection of monomial ideals, we get

$$\operatorname{reg}(I_1 \cdots I_d) \leq \sum_{j=1}^{d} \deg F_j - \operatorname{ht}(I_1 \cdots I_d) + 1,$$
$$\operatorname{reg}(I_1 \cap \cdots \cap I_d) \leq \sum_{j=1}^{d} \deg F_j - \operatorname{ht}(I_1 \cap \cdots \cap I_d) + 1,$$

where F_j denotes the least common multiple of the monomial generators of I_j. If $I_1, \ldots, I_d$ are complete intersections, then $\operatorname{reg}(I_j) = \deg F_j - \operatorname{ht}(I_j) + 1$, whence

$$\operatorname{reg}(I_1 \cdots I_d) \leq \sum_{j=1}^{d} \operatorname{reg}(I_j) + \sum_{j=1}^{d} \operatorname{ht}(I_j) - \operatorname{ht}(I_1 \cdots I_d) - d + 1,$$
$$\operatorname{reg}(I_1 \cap \cdots \cap I_d) \leq \sum_{j=1}^{d} \operatorname{reg}(I_j) + \sum_{j=1}^{d} \operatorname{ht}(I_j) - \operatorname{ht}(I_1 \cap \cdots \cap I_d) - d + 1.$$

These bounds are worse than the bounds in the aforementioned questions. However, the difference is not so great.

To get rid of the difference in the case $d = 2$ we need the following consequence of Lemma 2.1.

Corollary 2.2. *Let I be a monomial complete intersection and Q an arbitrary monomial ideal (not necessarily a proper ideal of the polynomial ring S). Then*

$$\operatorname{reg}(I : Q) \leq \operatorname{reg}(I).$$

Proof. Let F denote the product of the monomial generators of I. Since every monomial generator of $I : Q$ divides a monomial generator of I, the least common multiple of the monomial generators of $I : Q$ divides F. Applying Lemma 2.1 we get

$$\operatorname{reg}(I : Q) \leq \deg F - \operatorname{ht}(I : Q) + 1 \leq \deg F - \operatorname{ht} I + 1 = \operatorname{reg}(I).$$

□

We will decompose the product and the intersection of two monomial ideals as a sum of smaller ideals and apply the following lemma to estimate the regularity.

Lemma 2.3. *Let I and J be two arbitrary homogeneous ideals. Then*

$$\operatorname{reg}(I + J) \leq \max\{\operatorname{reg}(I), \operatorname{reg}(J), \operatorname{reg}(I \cap J) - 1\},$$
$$\operatorname{reg}(I \cap J) \leq \max\{\operatorname{reg}(I), \operatorname{reg}(J), \operatorname{reg}(I + J) + 1\}.$$

Moreover, $\operatorname{reg}(I \cap J) = \operatorname{reg}(I + J) + 1$ *if* $\operatorname{reg}(I + J) > \max\{\operatorname{reg}(I), \operatorname{reg}(J)\}$ *or if* $\operatorname{reg}(I \cap J) > \max\{\operatorname{reg}(I), \operatorname{reg}(J)\} + 1$.

Proof. The statements follow from the exact sequence

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0$$

and the well-known relationship between regularities of modules of an exact sequence (see e.g. [E, Corollary 20.19]). □

3. Main results

We will prove the following general result.

Theorem 3.1. *Let I and J be two arbitrary monomial complete intersections. Let $f_1, \dots, f_r$ be the monomial generators of I. Let $Q_1, \dots, Q_r$ be arbitrary monomial ideals. Then*

$$\operatorname{reg}\big(f_1(J : Q_1) + \cdots + f_r(J : Q_r)\big) \le \operatorname{reg}(I) + \operatorname{reg}(J).$$

The formulae of Theorem 1.1 follow from the above result because

$$IJ = f_1 J + \cdots + f_r J = f_1(J : S) + \cdots + f_r(J : S),$$
$$I \cap J = (f_1) \cap J + \cdots + (f_r) \cap J = f_1(J : f_1) + \cdots + f_r(J : f_r).$$

Proof. If $r = 1$, we have to prove that

$$\operatorname{reg}\big(f_1(J : Q_1)\big) \le \deg f_1 + \operatorname{reg}(J).$$

It is obvious that

$$\operatorname{reg}\big(f_1(J : Q_1)\big) \le \deg f_1 + \operatorname{reg}(J : Q_1).$$

By Corollary 2.2 we have $\operatorname{reg}(J : Q_1) \le \operatorname{reg}(J)$, which implies the assertion.

If $r > 1$, using induction we may assume that

$$\operatorname{reg}\Big(\sum_{i=1}^{r-1} f_i(J : Q_i)\Big) \le \operatorname{reg}(f_1, \dots, f_{r-1}) + \operatorname{reg}(J), \tag{1}$$

$$\operatorname{reg}\big(f_r(J : Q_r)\big) \le \deg f_r + \operatorname{reg}(J). \tag{2}$$

Since $f_1(J : Q_1), \dots, f_{r-1}(J : Q_{r-1})$ are monomial ideals, we have

$$\Big(\sum_{i=1}^{r-1} f_i(J : Q_i)\Big) : f_r = \sum_{i=1}^{r-1} \big(f_i(J : Q_i) : f_r\big).$$

Since $f_1, \dots, f_r$ is a regular sequence, $f_i(J : Q_i) : f_r = f_i(J : f_r Q_i)$. Therefore,

$$\begin{aligned}
\Big(\sum_{i=1}^{r-1} f_i(J : Q_i)\Big) \cap f_r(J : Q_r) &= f_r\Big[\Big(\sum_{i=1}^{r-1} f_i(J : Q_i) : f_r\Big) \cap (J : Q_r)\Big] \\
&= f_r\Big[\Big(\sum_{i=1}^{r-1} f_i(J : f_r Q_i)\Big) \cap (J : Q_r)\Big] \\
&= f_r\Big[\sum_{i=1}^{r-1} f_i\big((J : f_r Q_i) \cap (J : f_i Q_r)\big)\Big] \\
&= f_r\Big[\sum_{i=1}^{r-1} f_i\big(J : (f_r Q_i + f_i Q_r)\big)\Big].
\end{aligned}$$

From this it follows that

$$\operatorname{reg}\big(\big(\sum_{i=1}^{r-1} f_i(J:Q_i)\big)\cap f_r(J:Q_r)\big) \le \deg f_r + \operatorname{reg}\big(\sum_{i=1}^{r-1} f_i\big(J:(f_rQ_i+f_iQ_r)\big)\big).$$

Using induction we may assume that

$$\operatorname{reg}\big(\sum_{i=1}^{r-1} f_i\big(J:(f_rQ_i+f_iQ_r)\big)\big) \le \operatorname{reg}(f_1,\dots,f_{r-1})+\operatorname{reg}(J).$$

Since $\operatorname{reg}(I)=\operatorname{reg}(f_1,\dots,f_{r-1})+\deg f_r-1$, this implies

$$\operatorname{reg}\big(\big(\sum_{i=1}^{r-1} f_i(J:Q_i)\big)\cap f_r(J:Q_r)\big) \le \operatorname{reg}(I)+\operatorname{reg}(J)+1. \tag{3}$$

Now, we apply Lemma 2.3 to the decomposition

$$f_1(J:Q_1)+\cdots+f_r(J:Q_r)=\big(\sum_{i=1}^{r-1} f_i(J:Q_i)\big)+f_r(J:Q_r)$$

and obtain

$$\begin{aligned}
&\operatorname{reg}\big(f_1(J:Q_1)+\cdots+f_r(J:Q_r)\big)\\
&\quad\le \max\Big\{\operatorname{reg}\big(\sum_{i=1}^{r-1} f_i(J:Q_i)\big)\ ,\\
&\qquad\qquad \operatorname{reg}\big(f_r(J:Q_r)\big), \operatorname{reg}\big(\big(\sum_{i=1}^{r-1} f_i(J:Q_i)\big)\cap f_r(J:Q_r)\big)-1\Big\}\\
&\qquad\qquad\qquad \le \operatorname{reg}(I)+\operatorname{reg}(J)
\end{aligned}$$

by using (1), (2), (3). □

Remark 3.2. The above proof would work in the case of more than two monomial complete intersections if we could prove a similar result to that in Corollary 2.2. For instance, if we can prove

$$\operatorname{reg}(IJ:Q)\le \operatorname{reg}(I)+\operatorname{reg}(J)$$

for two monomial complete intersections I, J and an arbitrary monomial ideal Q, then we can give a positive answer to the question of Conca and Herzog in the monomial case for $d=3$. We are unable to verify the above formula, although computations in concrete cases suggest its validity.

Now we will extend the second formula of Theorem 1.1 for any set of monomial complete intersections.

Theorem 3.3. *Let $I_1,\dots,I_d$ be arbitrary monomial complete intersections. Then*

$$\operatorname{reg}(I_1\cap\cdots\cap I_d)\le \operatorname{reg}(I_1)+\cdots+\operatorname{reg}(I_d).$$

Proof. We will use induction on the number n of variables and the sum

$$s:=\operatorname{reg}(I_1)+\cdots+\operatorname{reg}(I_d).$$

First, we note that the cases $n=1$ and $s=1$ are trivial.

Assume that $n \geq 2$ and $s \geq 2$. Let x be an arbitrary variable of the polynomial ring S. It is easy to see that $(I_1, x), \ldots, (I_d, x)$ are monomial complete intersections and

$$(I_1 \cap \cdots \cap I_d, x) = (I_1, x) \cap \cdots \cap (I_d, x).$$

Therefore, using induction on n we may assume that

$$\operatorname{reg}(I_1 \cap \cdots \cap I_d, x) \leq \operatorname{reg}(I_1, x) + \cdots + \operatorname{reg}(I_d, x).$$

If x is a non-zerodivisor on $I_1 \cap \cdots \cap I_d$ and if we assume that the intersection is irredundant, then $I_j : x = I_j$ and hence $\operatorname{reg}(I_j, x) = \operatorname{reg}(I_j)$ for all $j = 1, \ldots, d$. In this case,

$$\operatorname{reg}(I_1 \cap \cdots \cap I_d) = \operatorname{reg}(I_1 \cap \cdots \cap I_d, x) \leq \operatorname{reg}(I_1) + \cdots + \operatorname{reg}(I_d).$$

If x is a zerodivisor on $I_1 \cap \cdots \cap I_d$, we involve the ideal

$$(I_1 \cap \cdots \cap I_d) : x = (I_1 : x) \cap \cdots \cap (I_d : x).$$

If $I_j : x \neq I_j$, either $I_j : x = S$ ($x \in I_j : x$) or $I_j : x$ is a monomial complete intersection generated by the monomials obtained from the generators of I_j by replacing the monomial divisible by x by its quotient by x. In the latter case, we have $\operatorname{reg}(I_j : x) = \operatorname{reg}(I_j) - 1$. Since there exists at least one ideal I_j with $I_j : x \neq I_j$, the ideal $(I_1 \cap \cdots \cap I_d) : x$ is an intersection of monomial complete intersections such that the sum of their regularities is less than s. Using induction on s we may assume that

$$\begin{aligned}\operatorname{reg}((I_1 \cap \cdots \cap I_d) : x) &\leq \operatorname{reg}(I_1 : x) + \cdots + \operatorname{reg}(I_d : x)\\ &\leq \operatorname{reg}(I_1) + \cdots + \operatorname{reg}(I_d) - 1.\end{aligned}$$

Now, from the exact sequence

$$0 \longrightarrow S/(I_1 \cap \cdots \cap I_d) : x \xrightarrow{x} S/I_1 \cap \cdots \cap I_d \longrightarrow S/(I_1 \cap \cdots \cap I_d, x) \longrightarrow 0$$

we can deduce that

$$\begin{aligned}\operatorname{reg}(I_1 \cap \cdots \cap I_d) &\leq \max\{\operatorname{reg}((I_1 \cap \cdots \cap I_d) : x) + 1, \operatorname{reg}(I_1 \cap \cdots \cap I_d, x)\}\\ &\leq \operatorname{reg}(I_1) + \cdots + \operatorname{reg}(I_d).\end{aligned}$$

□

4. Counter-examples

From now on we set $S := k[x, y, z, t]$ for the homogeneous coordinate ring of $\mathbb{P}^3$. First, we will use the theory of liaison to give curves in $\mathbb{P}^3$ with sections in negative degrees. Recall that two curves X and Y in $\mathbb{P}^3$ are said to be directly linked by a complete intersection Z if $I_X = I_Z : I_Y$, or equivalently $I_Y = I_Z : I_X$.

For any curve Y we define

$$\varepsilon(Y) := \min\{\mu \mid H^0(Y, \mathcal{O}_Y(\mu)) \neq 0\}.$$

Note that if we denote the initial degree of a module M by $\operatorname{indeg}(M)$ with the convention $\operatorname{indeg}(0) = +\infty$, then

$$\varepsilon(Y) = \min\{0, \operatorname{indeg}(H^1_{\mathfrak{m}}(S/I_Y))\}.$$

Lemma 4.1. *Let X and Y be two curves in $\mathbb{P}^3$, directly linked by a complete intersection Z. Then, $\operatorname{reg}(S/I_X) \geq \operatorname{reg}(S/I_Z)$ if and only if $\varepsilon(Y) < 0$. In this case,*

$$\operatorname{reg}(S/I_X) = \operatorname{reg}(S/I_Z) - \varepsilon(Y) - 1.$$

Proof. Put $r := \operatorname{reg}(S/I_Z)$. Then $a_2(S/I_X) \leq a_2(S/I_Z) = r-2$ because X is strictly contained in Z. Hence $\operatorname{reg}(S/I_X) \geq r$ if and only if $a_1(S/I_X)+1 \geq r$. In this case, $\operatorname{reg}(S/I_X) = a_1(S/I_X)+1$, By liaison,

$$H^1_{\mathfrak{m}}(S/I_X)_\mu \simeq H^1_{\mathfrak{m}}(S/I_Y)_{r-\mu-2}.$$

Therefore, $a_1(S/I_X) = r - \operatorname{indeg}(H^1_{\mathfrak{m}}(S/I_Y)) - 2$. Thus, $\operatorname{reg}(S/I_X) \geq r$ if and only if $\operatorname{indeg}(H^1_{\mathfrak{m}}(S/I_Y)) < 0$. In this case, $\operatorname{reg}(S/I_X) = r - \operatorname{indeg}(H^1_{\mathfrak{m}}(S/I_Y)) - 1$. □

For instance, we may take for X a reduced irreducible curve whose regularity is at least equal to the sum of the two smallest degrees d_1, d_2 of the minimal generators of its defining ideal. By Lemma 4.1, the residual of X in the complete intersection of degrees d_1, d_2 is a nonreduced curve Y with

$$\varepsilon(Y) = d_1 + d_2 - \operatorname{reg}(I_X) - 2 < 0.$$

We will use curves with sections in negative degrees to construct complete intersection ideals I, J for which $\operatorname{reg}(I \cap J)$ or $\operatorname{reg}(IJ)$ is greater than $\operatorname{reg}(I) + \operatorname{reg}(J)$. The construction is based on the following theorems.

Theorem 4.2. *Let Y be a curve in $\mathbb{P}^3$ with $\varepsilon(Y) < 0$ which is defined by at most 3 equations at the generic points of its irreducible components. Consider 3 elements f_1, f_2, f_3 in I_Y such that I_Y is the unmixed part of (f_1, f_2, f_3) and f_1, f_2 is a regular sequence. Put $I = (f_1, f_2)$ and $J = (f_3)$. Then*

$$\operatorname{reg}(I \cap J) = \operatorname{reg}(I) + \operatorname{reg}(J) - \varepsilon(Y) - 1.$$

Proof. Let $K = (f_1, f_2, f_3)$ and $\sigma = \deg f_1 + \deg f_2 + \deg f_3$. By [Ch, 0.6] we have

$$\begin{aligned} a_0(S/K) &= \sigma - \varepsilon(Y) - 4, \\ a_1(S/K) &= \sigma - \operatorname{indeg}(I_Y/K) - 4, \\ a_2(S/K) &\leq \sigma - \operatorname{indeg}(K) - 5. \end{aligned}$$

Since $\varepsilon(Y) < 0$, $a_0(S/K) \geq a_i(S/K) + i$ for $i = 1, 2$. Therefore,

$$\operatorname{reg}(S/K) = a_0(S/K) = \sigma - \varepsilon(Y) - 4.$$

Applying Lemma 2.3 we get

$$\operatorname{reg}(I \cap J) = \operatorname{reg}(K) + 1 = \sigma - \varepsilon(Y) - 2 = \operatorname{reg}(I) + \operatorname{reg}(J) - \varepsilon(Y) - 1.$$

□

Theorem 4.3. *Let Y be a curve in $\mathbb{P}^3$ with $\varepsilon(Y) < 0$ which is defined by at most 4 equations at the generic points of its irreducible components. Consider 4 elements f_1, f_2, f_3, f_4 in I_Y such that $I := (f_1, f_2)$ and $J := (f_3, f_4)$ are complete intersection ideals and I_Y is the unmixed part of $I + J$. Then*

$$\operatorname{reg}(IJ) = \operatorname{reg}(I) + \operatorname{reg}(J) - \varepsilon(Y) - 1.$$

Proof. Consider the exact sequence

$$0 \to (I \cap J)/IJ \to S/IJ \to S/(I \cap J) \to 0.$$

As $\operatorname{Tor}_1^S(S/I, S/J) \simeq (I \cap J)/IJ$ and $\operatorname{depth}(S/(I \cap J)) > 0$ one has

$$H^0_{\mathfrak{m}}(S/IJ) \simeq H^0_{\mathfrak{m}}(\operatorname{Tor}_1^S(S/I, S/J)).$$

Put $\sigma = \deg f_1 + \deg f_2 + \deg f_3 + \deg f_4$. By [Ch, 5.9], $H^0_{\mathfrak{m}}(\mathrm{Tor}^S_1(S/I, S/J))$ is the graded k-dual of $H^1_{\mathfrak{m}}(S/I_Y)$ up to a shift in degrees by $\sigma - 4$. Therefore, $H^0_{\mathfrak{m}}(S/IJ)_\mu \simeq H^1_{\mathfrak{m}}(S/I_Y)_{\sigma-4-\mu}$. This implies

$$a_0(S/IJ) = \sigma - \mathrm{indeg}(H^1_{\mathfrak{m}}(S/I_Y)) - 4 = \sigma - \varepsilon(Y) - 4.$$

Modifying the generators of I and J, we may assume that f_1f_3, f_2f_4 is a regular sequence, which shows that

$$a_2(S/IJ) \leq a_2(S/(f_1f_3, f_2f_4)) - 1 = \sigma - 5.$$

By [Ch, 3.1 (iii)] one has $H^1_{\mathfrak{m}}((I \cap J)/IJ)_\mu = 0$ for $\mu > \sigma - 4$. Therefore, if $\mu > \sigma - 4$, one has an exact sequence

$$0 \longrightarrow H^1_{\mathfrak{m}}(S/IJ)_\mu \longrightarrow H^1_{\mathfrak{m}}(S/I \cap J)_\mu \longrightarrow H^2_{\mathfrak{m}}((I \cap J)/IJ)_\mu \longrightarrow 0.$$

By [Ch, 3.1 (ii)], $H^2_{\mathfrak{m}}((I \cap J)/IJ)_\mu \simeq H^0_{\mathfrak{m}}(S/I + J)_\mu$ for $\mu > \sigma - 4$. Since S/I and S/J are Cohen-Macaulay rings, using the exact sequence

$$0 \longrightarrow S/I \cap J \longrightarrow S/I \oplus S/J \longrightarrow S/I + J \longrightarrow 0$$

we also have $H^1_{\mathfrak{m}}(S/I \cap J)_\mu \simeq H^0_{\mathfrak{m}}(S/I + J)_\mu$. Therefore, $H^1_{\mathfrak{m}}(S/IJ)_\mu = 0$ for $\mu > \sigma - 4$. Hence

$$a_1(S/IJ) \leq \sigma - 4.$$

As $\varepsilon(Y) < 0$, $a_i(S/IJ) + i \leq a_0(S/IJ)$ for $i = 1, 2$. So we get

$$\mathrm{reg}(IJ) = a_0(S/IJ) + 1 = \sigma - \varepsilon(Y) - 4 = \mathrm{reg}(I) + \mathrm{reg}(J) - \varepsilon(Y) - 1.$$

□

We will now study a specific class of curves with sections in negative degrees.

Consider first a monomial curve C parameterized by $(1 : \theta : \theta^{mn} : \theta^{m(n+1)})$ for $m, n \geq 2$. Note that $\mathrm{reg}(I_C) = mn$ by [BCFH, Corollary 5.1]. Let X be the union of C and the line $\{x = z = 0\}$. On one hand, $\mathrm{reg}(I_X) \geq mn + 1$ because $xy^{mn} - x^{mn}z$ is a minimal generator of $I_X = I_C \cap (x, z)$. On the other hand, it is easy to check that

$$I_C + (x, z) = (x, z, y^m t^{n-1}, y^{2m} t^{n-2}, \ldots, y^{nm})$$

and hence $\mathrm{reg}(I_C + (x, z)) = mn$. By Lemma 2.3, this implies $\mathrm{reg}(I_X) \leq \mathrm{reg}(I_C) + 1 = mn + 1$ (one can also use [Si, 1.8] to show $\mathrm{reg}(I_X) \leq mn + 1$). Therefore, $\mathrm{reg}(I_X) = mn + 1$.

Let Y be the direct link of X by the complete intersection defined by the ideal $(x^m t - y^m z, z^{n+1} - xt^n)$. By Lemma 4.1,

$$\varepsilon(Y) = (m + 1) + (n + 1) - \mathrm{reg}(I_X) - 2 = -(m - 1)(n - 1).$$

One has $I_Y = (x^m t - y^m z) + (z, t)^n$. To prove this note that $(x^m t - y^m z) + (z, t)^n$ defines a locally complete intersection scheme supported on the line $z = t = 0$, has positive depth and degree (or multiplicity) at least n. Then the containment

$$I_Z = I_X \cap I_Y \subseteq I_X \cap ((x^m t - y^m z) + (z, t)^n)$$

and the fact that

$$\deg(I_Z) = (m + 1)(n + 1) = \deg(I_X) + n$$

forces I_Y to coincide with the unmixed ideal $(x^m t - y^m z) + (z, t)^n$. It is also easy to provide a minimal free S-resolution of the ideal $(x^m t - y^m z) + (z, t)^n$, and show these facts along the same line as in the proof of [CD, 2.4].

The above curve Y can be used to construct counter-examples to the inequalities raised in the introduction.

Example 4.4. Let $I = (t^n, z^n)$ and $J = (x^m t - y^m z)$. It is easy to check that $I_Y = (x^m t - y^m z) + (z, t)^n$ is the saturation of $(t^n, z^n, x^m t - y^m z)$. Therefore, we may apply Theorem 4.2 and obtain

$$\operatorname{reg}(I \cap J) = \operatorname{reg}(I) + \operatorname{reg}(J) + (m-1)(n-1) - 1.$$

Thus, $\operatorname{reg}(I \cap J) > \operatorname{reg}(I) + \operatorname{reg}(J)$ if and only if $mn > m + n$ or equivalently $(m, n) \neq (2, 2)$.

Example 4.5. Let $I = (t^n, z^n)$ and $J = (x^m t - y^m z, t^n)$. Then I_Y is the saturation of $I + J$. By Theorem 4.3 we have

$$\operatorname{reg}(IJ) = \operatorname{reg}(I) + \operatorname{reg}(J) + (m-1)(n-1) - 1.$$

Therefore, $\operatorname{reg}(IJ) > \operatorname{reg}(I) + \operatorname{reg}(J)$ if and only if $(m, n) \neq (2, 2)$.

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