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MODEL THEORY OF PARTIAL DIFFERENTIAL FIELDS: FROM COMMUTING TO NONCOMMUTING DERIVATIONS

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Abstract. McGrail (2000) has shown the existence of a model completion for the universal theory of fields on which a finite number of commuting derivations act and, independently, Yaffe (2001) has shown the existence of a model completion for the univeral theory of fields on which a fixed Lie algebra acts as derivations. We show how to derive the second result from the first.

1. Introduction

In [3], McGrail gives axioms for the model completion of the universal theory of fields of characteristic zero with several commuting derivations. Independently, Yaffe [7] gave axioms for the model completion of the universal theory of a more general class of fields: LDF_0, the universal theory of fields of characteristic zero together with a finite-dimensional Lie algebra acting as derivations. The goal of this short note is to show that starting from McGrail's results one can quickly write down axioms for Yaffe's model completion, that is, one can reduce the noncommutative case to the commutative case.

Another axiomatization of the model completion of a theory of differential fields with noncommuting derivations has been given by Pierce [4] using differential forms and a version of the Frobenius Theorem. In fact, one can state the Frobenius Theorem as a result that allows one to replace noncommuting vector fields with commuting ones: *given an involutive analytic system $\mathcal{L}_y, y \in M$, of tangent spaces of rank p on an analytic manifold M and a point $x \in M$, one can choose analytic coordinates $x_1, \ldots, x_p$ around x such that $\mathcal{L}_y$ is spanned by $(\partial/\partial x_1)_y, \ldots, (\partial/\partial x_p)_y$ for all y in an open set containing x* (*cf.* [6], Theorem 1.3.3). In his introductory remarks, Pierce states that one might be able to reduce the noncommutative case to the commutative case ([4], pp. 924-925) but does not give details,[1] proceeding rather to develop his results *ab initio*. The present note carries through this reduction.

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[1] *Note added in proof*. The proof of Theorem 5.3 of [4] does give more of an indication as to how this can be done. It should also be noted that there are similarities between Lemmas 5.2 and 4.7 of [4] and Lemmas 2.1 and 2.2 of this paper, although the proofs are different.

2. Commuting bases of derivations

In [7], Yaffe defines the universal theory LDF_0 of Lie differential fields. To do this he fixes a field $\mathcal{F}$ of characteristic zero, a finite-dimensional $\mathcal{F}$-vector space $\mathcal{L}$ with a Lie multiplication making it a Lie algebra over a subfield of $\mathcal{F}$, and a vector space homomorphism $\phi_{\mathcal{F}} : \mathcal{L} \to \mathrm{Der}(\mathcal{F})$, the Lie algebra of derivations on $\mathcal{F}$, preserving the Lie multiplication. We fix a basis $\{D_1, \ldots, D_n\}$ of $\mathcal{L}$ and let $[D_k, D_l] = \sum_m \alpha_{kl}^m D_m$ for some $\alpha_{kl}^m \in \mathcal{F}$. The elements α_{kl}^m are called the structure constants of $\mathcal{L}$. The language for LDF_0 is the language of rings together with unary function symbols $D_i, i = 1, \ldots, n$, and constant symbols for each element of $\mathcal{F}$. The theory LDF_0 consists of

- the diagram of $\mathcal{F}$ including the action of the D_i,
- the theory of integral domains of characteristic zero,
- axioms stating that the D_i are derivations,
- for each k, l an axiom of the form $\forall x(D_k D_l x - D_l D_k x = \sum_m \alpha_{kl}^m D_m x)$.

We shall refer to a model of LDF_0 as a *Lie ring over* $\mathcal{L}$ or, more simply, a *Lie ring*, if it is clear which $\mathcal{L}$ is used.

We will need another concept from [7]: the ring of normal polynomials. This is defined as follows. Let $\mathcal{A}$ be a model of LDF_0. For each $I = \langle i_1, \ldots, i_n \rangle \in \mathbb{N}^n$ define a variable X_I and consider the ring $\mathcal{R}_{\mathcal{A}} = \mathcal{A}[\{X_I\}_{I \in \mathbb{N}^n}]$ of polynomials in this infinite set of variables. Yaffe shows ([7], pp. 57-61) how one can define an action of $\mathcal{L}$ on $\mathcal{R}_{\mathcal{A}}$ so that this ring is a model of LDF_0, and one sees from his proof that the D_i act as linearly independent derivations. Although this is not needed in the following lemma, we note that Yaffe also shows that if $\mathcal{B}$ is a model of LDF_0 that extends $\mathcal{A}$ and $b \in \mathcal{B}$, then the map from $\mathcal{R}_{\mathcal{A}}$ to $\mathcal{B}$ given by $X_I \mapsto D^I(b) = D_1^{i_1} D_2^{i_2} \ldots D_n^{i_n}(b)$ for all I is a homomorphism of structures. Yaffe's construction of $\mathcal{R}_{\mathcal{A}}$ allows us to conclude

Lemma 2.1. *If $\mathcal{A}$ is a model of LDF_0 with derivations $D_1^{\mathcal{A}}, \ldots, D_n^{\mathcal{A}}$, then $\mathcal{A}$ embeds in a model $\mathcal{B}$ of LDF_0 with derivations $D_1^{\mathcal{B}}, \ldots, D_n^{\mathcal{B}}$ extending the derivations of $\mathcal{A}$ such that the $D_1^{\mathcal{B}}, \ldots, D_n^{\mathcal{B}}$ are linearly independent over $\mathcal{B}$.*

To show that the axioms in the next section yield a model completion of this theory we need two additional facts that are contained in the next results.

Lemma 2.2. *Let $\mathcal{A}$ with derivations $D_1^{\mathcal{A}}, \ldots, D_n^{\mathcal{A}}$ be a model of LDF_0 and assume that the $D_i^{\mathcal{A}}$ are linearly independent over $\mathcal{A}$. Then there exist $a_{i,j} \in \mathcal{A}$ such that the derivations $\overline{D}_i^{\mathcal{A}} = \sum_j a_{i,j} D_j^{\mathcal{A}}$ are linearly independent over $\mathcal{A}$ and commute.*

Proof. We shall assume $\mathcal{A}$ is a field and follow the proof of Proposition 6 in Chapter 0, §5 of [1], with a few small modifications. As noted in the Introduction, this is an algebraic version of the Frobenius Theorem.

Let $\mathcal{C} = \bigcap_{i=1}^n \mathrm{Ker}\, D_i^{\mathcal{A}}$. Let $\{x_i\}_{i \in I}$ be a transcendence basis of $\mathcal{A}$ over $\mathcal{C}$. Let δ_i be the unique derivation on $\mathcal{A}$ satisfying $\delta_i(c) = 0$ for all $c \in \mathcal{C}$ and $\delta_i(x_j) = 1$ if $i = j$ and 0 if $i \neq j$. Any $x \in \mathcal{A}$ will lie in a subfield algebraic over $\mathcal{C}(x_1, \ldots, x_N)$ for some N, and so $\delta_i(x) = 0$ for all $i > N$. For any derivation D of $\mathcal{A}$ that is trivial on $\mathcal{C}$, consider the sum $\sum_{i \in I} D(x_i)\delta_i$. Although this sum is infinite, the previous remark shows that for any $x \in \mathcal{A}$, the sum $\sum_{i \in I} D(x_i)\delta_i$ makes sense and, using the results of Chapter VII, §5 of [2], one can show that $D = \sum_{i \in I} D(x_i)\delta_i$.

Since the D_i are linearly independent over $\mathcal{A}$ there exist $a_{i,j} \in \mathcal{A}$ such that the derivations $\overline{D}_i^{\mathcal{A}} = \sum_j a_{i,j} D_j^{\mathcal{A}}$ satisfy (after a possible renumbering of the δ_j)

$$\overline{D}_i^{\mathcal{A}} = \delta_i \pmod{\sum_{j>n} \mathcal{A}\delta_j}$$

for $i = 1, \ldots, n$. To see that the $\overline{D}_i^{\mathcal{A}}$ are linearly independent over $\mathcal{A}$ note that if $\sum_j b_j \overline{D}_j^{\mathcal{A}} = 0$, then $0 = \sum_j b_j \overline{D}_j^{\mathcal{A}}(x_i) = b_i$. We now claim that the $\overline{D}_i^{\mathcal{A}}$ commute. Since the δ_i commute, we have that

$$[\overline{D}_r^{\mathcal{A}}, \overline{D}_s^{\mathcal{A}}] = 0 \pmod{\sum_{i>n} \mathcal{A}\delta_i}$$

for any r, s. Since the $\mathcal{A}$-span of the $\overline{D}_i^{\mathcal{A}}$ is closed under $[\ ,\ ]$, we have that there exist $b_{r,s,j} \in \mathcal{A}$ such that

$$\begin{aligned} [\overline{D}_r^{\mathcal{A}}, \overline{D}_s^{\mathcal{A}}] &= \sum_{j=1}^{n} b_{r,s,j} \overline{D}_j^{\mathcal{A}} \\ &= \sum_{j=1}^{n} b_{r,s,j} \delta_j \pmod{\sum_{i>n} \mathcal{A}\delta_i}. \end{aligned}$$

Therefore, we have that $b_{r,s,j} = 0$ for all r, s, and j, and hence that the $\overline{D}_j^{\mathcal{A}}$ commute. □

We shall need to compare Lie rings for two different Lie algebras $\mathcal{L}_1$ and $\mathcal{L}_2$. We will denote by LDF_0^1 (resp. LDF_0^2) the theory of Lie rings based on the action of the Lie algebra $\mathcal{L}_1$ (resp. $\mathcal{L}_2$). We will assume the two algebras are of the same dimension over $\mathcal{F}$.

Lemma 2.3. *Let $\mathcal{A}$ with derivations $D_1^1, \ldots, D_n^1$ be a model of LDF_0^1 and assume that the D_i^1 are linearly independent over $\mathcal{A}$. For $i = 1, \ldots, n$, let $D_i^2 = \sum_j a_{i,j} D_j^1$ for some $a_{i,j} \in \mathcal{A}$ and assume that $\mathcal{A}$ with derivations $D_1^2, \ldots, D_n^2$ is a model of LDF_0^2. If $\mathcal{B}$ is an extension of $\mathcal{A}$ that is a model of LDF_0^1 with respect to the extensions of the derivations $D_1^1, \ldots, D_n^1$, then the formulas $D_i^2 = \sum_j a_{i,j} D_j^1$ define derivations on $\mathcal{B}$ such that $\mathcal{B}$ with these derivations is a model of LDF_0^2.*

Proof. Let $\alpha_{k,l}^m$ be the structure constants of $\mathcal{L}_1$ and let $\beta_{k,l}^m$ be the structure constants of $\mathcal{L}_2$. We have

$$\begin{aligned} [D_l^2, D_k^2] &= \sum_j (\sum_i (a_{l,i} D_i^1(a_{k,j}) - a_{k,i} D_i^1(a_{l,j})) + \sum_{r,s} a_{l,r} a_{k,s} \alpha_{r,s}^j) D_j^1, \text{ and} \\ \sum_m \beta_{l,k}^m D_m^2 &= \sum_j (\sum_m \beta_{l,k}^m a_{m,j}) D_j^1. \end{aligned}$$

First note that since the D_j^1 are linearly independent over $\mathcal{A}$, these derivations are linearly independent over $\mathcal{B}$ as well. We therefore have that $[D_l^2, D_k^2] = \sum_m \beta_{l,k}^m D_m^2$ if and only if

$$\sum_i (a_{l,i} D_i^1(a_{k,j}) - a_{k,i} D_i^1(a_{l,j})) + \sum_{r,s} a_{l,r} a_{k,s} \alpha_{r,s}^j = \sum_m \beta_{l,k}^m a_{m,j}$$

for all j. If this holds in $\mathcal{A}$, then it will hold in $\mathcal{B}$. □

In particular, if the basis elements D_i^2 commute, we have

Corollary 2.4. *Let $\mathcal{A}$ with derivations $D_1^1, \ldots, D_n^1$ be a model of LDF_0 and assume that the D_i^1 are linearly independent over $\mathcal{A}$. For $i = 1, \ldots, n$, let $D_i^2 = \sum_j a_{i,j} D_j^1$ for some $a_{i,j} \in \mathcal{A}$ and assume that the D_i^2 commute as derivations on $\mathcal{A}$. If $\mathcal{B}$ is an extension of $\mathcal{A}$ that is a model of LDF_0, then the D_i^2 commute as derivations on $\mathcal{B}$.*

Note that in model theoretic terms, the statement that the basis elements D_i^2 commute, which *a priori* is a universal statement, is actually (equivalent to) a quantifier-free statement, given that the D_i^1 are linearly independent.

3. Axioms for the model completion of LDF_0

Let us begin by recalling the situation for fields with commuting derivations. In [3], McGrail gave axioms for the model completion m-DCF of the universal theory m-DF of differential fields with m commuting derivations $D_1, \ldots, D_n$. She also showed that this former theory has elimination of quantifiers. In particular, for any field F with m commuting derivations and any system

$$\begin{aligned}\mathcal{S}(u_1, \ldots, u_r, v_1, \ldots, v_s) = \{&f_1(u_1, \ldots, u_r, v_1, \ldots, v_s)\\ &= 0, \ldots, f_t(u_1, \ldots, u_r, v_1, \ldots, v_s)\\ &= 0, g(u_1, \ldots, u_r, v_1, \ldots, v_s) \neq 0\}\end{aligned}$$

of differential polynomials, there exist systems $\mathcal{T}_1(u_1, \ldots, u_r), \ldots, \mathcal{T}_l(u_1, \ldots, u_r)$ such that for any $u_1, \ldots, u_r \in F$ there exists a solution $v_1, \ldots, v_s$ in some differential extension of F if and only if the u_i satisfy one of the systems $\mathcal{T}_i(u_1, \ldots, u_r)$.

Let $\mathcal{F}, \mathcal{L}$ and $\phi_{\mathcal{F}}$ be as above. Let $\mathcal{R}_{\mathcal{F}}$ be the ring of normal polynomials with coefficients in $\mathcal{F}$. For any $t \in \mathbb{N}$, we will denote by $\mathcal{R}_{\mathcal{F}}[x_1, \ldots, x_t]$ the usual (not differential) ring of polynomials in the variables $x_1, \ldots, x_t$ with coefficients in $\mathcal{R}_{\mathcal{F}}$. The axioms for $LDCF_0$, the theory of Lie differentially closed fields of characteristic zero, are the axioms for LDF_0 plus the axioms for fields and the following axioms:

(1) for any t and any polynomial $p(x_1, \ldots, x_t, \{X_I\}) \in \mathcal{R}_{\mathcal{F}}[x_1, \ldots, x_t]$ we have an axiom that states that for any $a_1, \ldots, a_n$ such that $p(a_1, \ldots, a_n, \{X_I\}) \neq 0$ there exists b such that $p(a_1, \ldots, a_n, \{D^I(b)\}) \neq 0$;

(2) there exists an n^2-tuple of elements $\mathbf{x} = (x_{1,1}, \ldots, x_{n,n})$ such that the derivations $D_1^{\mathbf{x}} = \sum_j x_{1,j} D_j, \ldots, D_n^{\mathbf{x}} = \sum_j x_{n,j} D_j$ form a linearly independent set of commuting derivations;

(3) for any $\mathbf{x}$ such that $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$ is a linearly independent set of commuting derivations and any system $S(u_1, \ldots, u_n, v_1, \ldots, v_m)$ involving differential polynomials in the $D_i^{\mathbf{x}}$, we have

$$\exists u_1, \ldots, v_m \mathcal{S}(u_1, \ldots, u_n, v_1, \ldots, v_m)$$

if and only if

$$\bigvee_{j=1}^{l} \mathcal{T}_j(u_1, \ldots, u_n),$$

where the $\mathcal{T}_j$ are as described above.

Note that (3) implies that the models of this theory are algebraically closed.

We will use Blum's criteria ([5], Section 17) to show that these axioms give the model completion of LDF_0. The first step is to show that any model $\mathcal{A}$ of LDF can

be extended to a model of $LDCF_0$. Taking quotient fields of the ring constructed in Lemma 2.1 we extend $\mathcal{A}$ to a model K^0 of LDF_0 were the derivations are linearly independent over K^0. Note that the ring constructed in Lemma 2.1 allows us to conclude that K^0 satisfies axiom scheme (1). Lemma 2.2 implies that axiom (2) holds. Let $\mathcal{N}_0$ be the set of n^2-tuples of elements in K^0 and assume that this set is well ordered. Let $\mathbf{x}$ be the smallest element of $\mathcal{N}_0$ such that $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$ is a linearly independent set of commuting derivations and let K_1^0 be the differential closure of K^0 thought of as a field with commuting derivations $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$. Since the D_i can be expressed a K^0-linear combinations of the $D_i^{\mathbf{x}}$, Lemma 2.3 implies that K_1^0 is still a model of LDF_0. Let $\bar{\mathbf{x}}$ be the next smallest element of $\mathcal{N}_0$ such that $\{D_1^{\bar{\mathbf{x}}}, \ldots, D_n^{\bar{\mathbf{x}}}\}$ is a linearly independent set of commuting derivations and let K_2^0 be the differential closure of K_1^0 thought of as a field with commuting derivations $\{D_1^{\bar{\mathbf{x}}}, \ldots, D_n^{\bar{\mathbf{x}}}\}$. We can continue in this way for all elements of $\mathcal{N}_0$ and, taking unions, form a field $\bar{K}^0$ such that for any $\mathbf{x}$ in $\mathcal{N}_0$ such that $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$ is a linearly independent set of commuting derivations, $\bar{K}^0$ contains the differential closure of K^0 thought of as a field with commuting derivations $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$. Again Lemma 2.3 implies that $\bar{K}^0$ is a model of LDF_0. We let K^1 be the quotient field of $\mathcal{R}_{\bar{K}^0}$. Note that K^1 satisfies axiom scheme (1). We now repeat this process and form a field K^2 such that for any n^2-tuple $\mathbf{x}$ of elements in K^1 such that $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$ is a linearly independent set of commuting derivations, K^2 contains the differential closure of K^1 thought of as a field with commuting derivations $\{D_1^{\mathbf{x}}, \ldots, D_n^{\mathbf{x}}\}$ and also satisfies axiom scheme (1). One now sees that $K^\infty = \bigcup K^i$ is a model of LDF_0 and satisfies the axioms of $LCDF_0$.

We will now show that if $\mathcal{A} \models LDF_0$, $\mathcal{B} \models LDCF_0$, $\mathcal{B}$ being $|\mathcal{A}|^+$-saturated and $\mathcal{A} \subset \mathcal{B}$, then for any simple extension $\mathcal{A}(a)$ there is an $\mathcal{A}$-embedding $f : \mathcal{A}(a) \to \mathcal{B}$. We will first show that we can assume that $\mathcal{A}$ satisfies axioms (1) and (2) above.

Let $\mathcal{R}_{\mathcal{A}(a)} = \mathcal{A}(a)(\{X_I\})$ be the field fractions of normal polynomials over $\mathcal{A}(a)$. Axiom (1) and saturation imply that there is an $\mathcal{A}$-embedding of $\mathcal{A}(\{X_I\})$ into $\mathcal{B}$. Furthermore, $\mathcal{A}(\{X_I\})$ satisfies axiom schemes (1) and (2). If we can extend this embedding to $\mathcal{A}(\{X_I\})(a)$, then restricting to $\mathcal{A}(a)$ will give the desired conclusion. We will therefore assume from the beginning that $\mathcal{A}$ satisfies axiom schemes (1) and (2).

Let $\mathbf{x}$ be elements of $\mathcal{A}$ such that $D_i^{\mathbf{x}}$ are linearly independent commuting derivations. Corollary 2.4 implies that these derivations commute on $\mathcal{A}(a)$ as well. The isomorphism type of a over the field $\mathcal{A}$ with derivations $\{D_i^{\mathbf{x}}\}$ is determined by the set of $\{D_i^{\mathbf{x}}\}$-differential polynomials that a satisfies and by the set that it does not satisfy. By axiom scheme (3) and saturation, we can realize this type in $\mathcal{B}$ and therefore get an embedding of $\mathcal{A}(a)$ into $\mathcal{B}$, considered as $\{D_i^{\mathbf{x}}\}$-differential fields. Since the $\{D_i\}$ are linear combinations of the $\{D_i^{\mathbf{x}}\}$, this is also an embedding as models of LDF_0.

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