A SHORT PROOF OF BING’S CHARACTERIZATION OF $S^3$

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Abstract. We give a short proof of Bing’s characterization of $S^3$: a compact, connected 3-manifold $M$ is $S^3$ if and only if every knot in $M$ is isotopic into a ball.

Let $M$ be a closed orientable 3-manifold. We assume familiarity with the basic notions of irreducible and prime 3-manifolds (see, e.g., [3] or [4]) and the basic results about Heegaard splittings of compact 3-manifolds (see, e.g., [7]). By genus we always mean Heegaard genus. A knot $k \subset M$ (that is, a smooth embedding of the circle into $M$) is called irreducible if its exterior $E(k) = M \setminus N(k)$ is an irreducible 3-manifold. In his own words, Bing’s Theorem [1, Theorem 1] is:

**Theorem 1** (Bing). A compact, connected 3-manifold $M$ is topologically $S^3$ if each simple closed curve in $M$ lies in a topological cube in $M$.

By “topological cube” Bing meant what we usually call a ball. Clearly, any knot in $S^3$ is contained in a ball. If a knot $k$ in a manifold $M \not\cong S^3$ is contained in a ball (say $B$), then by considering the boundary of $B$ we see that $k$ is not irreducible. Thus, Theorem 1 follows from:

**Theorem 2.** Any compact, connected 3-manifold admits an irreducible knot.

In [5, Theorem 8.1] Jaco and Rubinstein gave a very short proof of Theorem 1 for irreducible manifolds, but their proof relies on the existence of 0-efficient triangulations. The purpose of this note is giving a short, elementary proof of Theorem 1.

1. THE PROOF

We prove Theorem 2 as remarked above Theorem 1 follows.

**Case One: $M$ is prime.** First, when $M$ has genus at most one, let $k$ be a knot on a Heegaard torus (in $M$) with $E(k)$ a Seifert fibered space over the disk with 2 exceptional fibers, which is irreducible.

Second, when $M$ has genus two or more, then $M \not\cong S^2 \times S^1$ and hence is irreducible. Let $M = V_1 \cup_S V_2$ be a minimal genus Heegaard splitting of $M$. By Waldhausen [8] (see also [7, Theorem 3.8]) $\Sigma$ is irreducible. Let $k$ be a core of a 1-handle in $V_1$. Then $\Sigma$ is an irreducible Heegaard surface for $E(k)$; Haken [2] (see also [7, Theorem 3.4]) showed that every Heegaard splitting of a reducible manifold is reducible; hence, $E(k)$ is irreducible.
Remark 3. In Case One, $\partial E(k)$ is incompressible. For manifolds of genus one or less, this is so by construction of $k$. For manifold of genus two or more, if $\partial E(k)$ compressed, then (since $E(k)$ is irreducible) $E(k)$ would be a solid torus; but that implies $M$ has genus at most one, contradiction.

Case Two: $M$ is composite. By Kneser [6] $M$ has a prime decomposition as $M \cong M_1 \# \cdots \# M_n$ with $M_i$ prime ($i = 1, \ldots, n$). Let $k_i \subset M_i$ be the knot obtained in Case One, let $k = \#_{i=1}^n k_i \subset M$ be their connected sum, and let $A \subset E(k)$ be a collection of annuli that decomposes $k$ into its summands, that is, the components of $E(k)$ cut open along $A$ are homeomorphic to $E(k_i)$ ($i = 1, \ldots, n$).

Let $S$ be a sphere in $E(k)$; we will prove that $S$ bounds a ball. By isotopy of $S$, minimize $S \cap A$. Assume that $S \cap A \neq \emptyset$. Since $\chi(S) = 2$, $S$ cut open along $A$ has disk components, and let $D$ be such a disk. Then $D$ is contained in some component of $E(k)$ cut open along $A$ (which is homeomorphic to $E(k_i)$, for some $i$). By Remark [3] $\partial E(k_i)$ is incompressible and hence $D$ is boundary parallel, contradicting the minimality assumption. Hence $S \cap A = \emptyset$, and $S$ is contained in a component of $E(k)$ cut open along $A$. By the construction in Case One $S$ bounds a ball. Thus every sphere in $E(k)$ bounds a ball and $k$ is an irreducible knot, completing the proof of Theorems 2 and 1.

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References


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