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FINITE HEAT KERNEL EXPANSIONS ON THE REAL LINE

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ABSTRACT. Let $\mathcal{L} = d^2/dx^2 + u(x)$ be the one-dimensional Schrödinger operator and let $H(x, y, t)$ be the corresponding heat kernel. We prove that the nth Hadamard's coefficient $H_n(x, y)$ is equal to 0 if and only if there exists a differential operator $\mathcal{M}$ of order $2n - 1$ such that $\mathcal{L}^{2n-1} = \mathcal{M}^2$. Thus, the heat expansion is finite if and only if the potential $u(x)$ is a rational solution of the KdV hierarchy decaying at infinity studied by Adler and Moser (1978) and Airault, McKean and Moser (1977). Equivalently, one can characterize the corresponding operators $\mathcal{L}$ as the rank one bispectral family given by Duistermaat and Grünbaum (1986).

1. Introduction and description of the main result

Let $u(x)$ be a smooth function and let $\mathcal{L}$ be the one-dimensional Schrödinger operator

$$\mathcal{L} = \frac{d^2}{dx^2} + u(x). \tag{1.1}$$

The heat kernel $H(x, y, t)$ is the fundamental solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) f = 0. \tag{1.2}$$

It is well known that $H(x, y, t)$ has an asymptotic expansion of the form

$$H(x, y, t) \sim \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \left(1 + \sum_{n=1}^{\infty} H_n(x, y) t^n\right) \text{ as } t \to 0+, \tag{1.3}$$

valid for x close to y. The coefficients H_n are the so-called Hadamard's coefficients [11].

In a series of papers Berest and Veselov studied linear hyperbolic partial differential operators that satisfy the Huygens principle and discovered a beautiful connection with integrable systems and the bispectral problem [8]; see for example [4, 5] and the references therein. Grünbaum [9] observed that a similar connection is present in the context of the heat equation. More precisely, he showed that for the first few potentials $u_i(x) = 2\log(\tau_i(x))''$, with $\tau_0 = 1$, $\tau_1(x) = x$, $\tau_2(x) = x^3/3 - s_3$, formed by the rational solutions of Korteweg-de Vries (KdV) equation, the asymptotic expansion (1.3) gives rise to an exact formula consisting of a finite number of

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terms. The corresponding operators $\mathcal{L}$ belong to the rank one family of solutions to the bispectral problem [8].

The main result of the present paper is to prove that this holds for all rank one bispectral operators and that this completely characterizes the rank one family. More precisely we have

Theorem 1.1. *The following conditions are equivalent:*

(i) $H_n(x,y) = 0$ *for all* x *and* y.

(ii) *There exists a differential operator* $\mathcal{M}$ *of order* $2n-1$ *such that*

$$\mathcal{L}^{2n-1} = \mathcal{M}^2. \tag{1.4}$$

Operators $\mathcal{L}$ and $\mathcal{M}$, satisfying (1.4), were studied in a paper by Burchnall and Chaundy[1] [6]. Their main result essentially means that the operator $\mathcal{L}$ can be obtained by applying a sequence of rational Darboux transformations to the differential operator $\mathcal{L}_0 = d^2/dx^2$. According to a result by Adler and Moser [1] the corresponding potentials $u(x)$ are precisely the rational solutions of the KdV equation decaying at ∞. The latter were discovered by Airault, McKean and Moser [2] and mysteriously connected with the Calogero-Moser system. Finally, the operators $\mathcal{L}$ in (ii) coincide with the rank one bispectral family in the work of Duistermaat and Grünbaum [8]. Thus, the present paper adds one more characterization of the operators $\mathcal{L}$ satisfying condition (ii), namely the finiteness property of the heat kernel expansion.

The proof of this theorem can be obtained using the connection between the heat kernel and the KdV hierarchy. This goes back to the pioneer work of McKean and van Moerbeke [17], where it was shown that, restricted on the diagonal $x = y$, Hadamard's coefficients give the flows of the KdV hierarchy. See also [3] and the references therein for extensions to matrix-valued heat kernel expansions and applications.

In a recent paper [14] we showed that one can also go in the opposite direction and construct the heat kernel using the τ-function (in the sense of Sato; see [18, 13]). We combine this formula for $H_n(x,y)$ in terms of the Baker function, the explicit description of the coordinate ring for the algebraic curves, and the bilinear (Hirota) equations to show that the vanishing of Hadamard's coefficient $H_n(x,y)$ is equivalent to condition (ii) in the theorem.

For related results concerning the finiteness property of the heat kernel expansion on the integers and rational solutions of the Toda lattice hierarchy see [10]. For solitons of the Toda lattice and purely discrete versions of the heat kernel see [12].

2. Baker functions and an infinite Grassmannian

In this section we collect some preliminary facts about the Segal-Wilson Grassmannian [20] and the parametrization of rank one commutative rings of differential operators. We also present the formula from [14] which relates $H_n(x,y)$ with Sato theory for the KdV hierarchy. Finally we prove a simple lemma which allows us to later use the algebro-geometrical data.

Let us denote by S^1 the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let H denote the Hilbert space $L^2(S^1, \mathbb{C})$. We split H as the orthogonal direct sum $H = H_+ \oplus H_-$, where H_+ (resp. H_-) consists of the functions whose Fourier series involves only

[1] They considered a more general situation of operators P and Q of arbitrary orders, satisfying $P^{\deg(Q)} = Q^{\deg(P)}$.

nonnegative (resp. only negative) powers of z. We denote by Gr the Segal-Wilson Grassmannian of all closed subspaces W of H such that

- the projection $W \to H_+$ is a Fredholm operator of index zero;
- the projection $W \to H_-$ is a compact operator.

Let $\mathrm{Gr}^{(2)}$ be the subspace of Gr given by

$$\mathrm{Gr}^{(2)} = \{W \in \mathrm{Gr} : z^2 W \subset W\}.$$

To each space $W \in \mathrm{Gr}$ there is a unique (Baker) function $\Psi_W(x,z)$ characterized by the following properties:

- Ψ_W has the form $\Psi_W(x,z) = \left(1 + \sum_{i=1}^{\infty} \psi_i(x) z^{-i}\right) e^{xz}$;
- $\Psi_W(x,.)$ belongs to W for each x.

We denote by W^{alg} the subspace of elements of finite order, i.e. the elements of the form $\sum_{j \le k} a_j z^j$. This is a dense subspace of W. A very important object connected to the plane W is the ring A_W of analytic functions $f(z)$ on S^1 that leave W^{alg} invariant, i.e.

$$A_W = \{f(z) \text{ analytic on } S^1 : f(z) W^{\mathrm{alg}} \subset W^{\mathrm{alg}}\}.$$

For each $f(z) \in A_W$ one can show that there exists a unique differential operator $L_f(x, \partial/\partial x)$ such that

$$L_f \Psi(x,z) = f(z) \Psi_W(x,z).$$

The order of the operator L_f is equal to the order of f. In general, A_W is trivial, i.e. $A_W = \mathbb{C}$. The spaces W which arise from algebro-geometrical data (which are important for us) are precisely those such that A_W contains an element of each sufficiently large order. The commutative ring of differential operators

$$\mathfrak{A}_W = \{L_f : f \in A_W\}$$

is isomorphic to A_W. Every rank one commutative ring[2] of differential operators can be obtained by this construction for an appropriate $W \in \mathrm{Gr}$. Note that if $W \in \mathrm{Gr}^{(2)}$, then $z^2 \in W$ and $\mathcal{L} = L_{z^2} = d^2/dx^2 + u(x)$ is a second-order operator.

Remark 2.1. In Sato's theory of the Kadomtsev-Petviashvili hierarchy one usually writes the Baker (wave) function

$$\Psi_W(s,z) = \left(1 + \sum_{i=1}^{\infty} \psi_i(s) z^{-i}\right) e^{\sum_{k=1}^{\infty} s_k z^k}$$

depending on all times $s_1 = x, s_2, s_3, \ldots$. Then if $W \in \mathrm{Gr}^{(2)}$ the operator $\mathcal{L} = L_{z^2}$ satisfies the KdV hierarchy

$$\frac{\partial \mathcal{L}}{\partial s_k} = [(\mathcal{L}^{k/2})_+, \mathcal{L}], \qquad k = 1, 3, 5, \ldots, \tag{2.1}$$

where $(\mathcal{L}^{k/2})_+$ is the differential part of the pseudo-differential operator $\mathcal{L}^{k/2}$.

On the Hilbert space H we have the non-degenerate continuous skew-symmetric bilinear form

$$\langle f, g \rangle = \frac{1}{2\pi i} \oint_{S^1} f(z) g(-z) dz. \tag{2.2}$$

[2] A commutative ring of differential operators is called rank one if it contains an operator of every sufficiently large order.

If $W \in \mathrm{Gr}$, then its annihilator

$$W^* = \{f \in H : \langle f, g\rangle = 0 \text{ for all } g \in W\}$$

also belongs to Gr. The adjoint Baker function $\Psi^*(x,z)$ is defined by

$$\Psi^*_W(x,z) = \Psi_{W^*}(x,-z).$$

It follows immediately from the definition that the bilinear identity

$$\oint_{S^1} \Psi_W(x,z)\Psi^*_W(y,z)dz = 0 \tag{2.3}$$

holds (see [7]).

The main result of [14] is the following formula for $H_n(x,y)$:

$$H_n(x,y) = -\frac{1}{\pi i(x-y)^{2n-1}} \oint_{S^1} g_n\left(2(x-y)z\right)\Psi_W(x,z)\Psi^*_W(y,z)dz, \tag{2.4}$$

where

$$g_n(z) = (-1)^{n-1}\left[\sum_{k=0}^{n-1} \frac{(2n-k-2)!}{k!(n-k-1)!} z^k\right] e^{-\frac{z}{2}}. \tag{2.5}$$

It is interesting to note that the function $g_n(z)$ is closely related to a specific Laguerre polynomial. Recall that the Laguerre polynomials $L^\alpha_n(z)$ are defined by the following formula:

$$L^\alpha_n(z) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-z)^k}{k!(n-k)!}, \tag{2.6}$$

where $\Gamma(z)$ is the usual gamma function; see for example [16, p. 77]. Thus we can write $g_n(z)$ as

$$g_n(z) = (n-1)!L^{-2n+1}_{n-1}(z)e^{-\frac{z}{2}}. \tag{2.7}$$

Using the above formula we prove the following lemma.

Lemma 2.2. *Let*

$$g_n(z) = \sum_{k=0}^{\infty} \beta_k z^k \tag{2.8}$$

be the Taylor expansion of the function $g_n(z)$ around $z = 0$. Then $\beta_{2j-1} = 0$ for $1 \le j < n$.

Proof. It is well known that the function $f(z) = L^\alpha_n(z)e^{-\frac{z}{2}}$ satisfies the differential equation

$$zf'' + (\alpha+1)f' + \left(n + \frac{\alpha+1}{2}\right) f = \frac{zf}{4};$$

see for example [16, p. 85].

Using the above equation and (2.7) we see that $g_n(z)$ must satisfy the differential equation

$$zg''_n - 2(n-1)g'_n = \frac{zg_n}{4},$$

from which it follows immediately that

$$8(2j-1)(j-n)\beta_{2j-1} = \beta_{2j-3}. \tag{2.9}$$

Combining this with $\beta_1 = 0$ we obtain the assertion of the lemma. □

3. Proof of the theorem

The implication (i)⇒(ii) can be proved using the connection between Hadamard's coefficients on the diagonal and the KdV hierarchy (2.1). Indeed we have

$$\frac{\partial H_k(x,x)}{\partial x} = \frac{2^{k-1}}{(2k-1)!!}\,[(\mathcal{L}^{\frac{2k-1}{2}})_+, \mathcal{L}]; \tag{3.1}$$

see for example [19, Theorem 5.2]. If $H_n(x,y) = 0$, then we have $H_k(x,y) = 0$ for all $k \geq n$. Formula (3.1) combined with (2.1) shows that u is stationary under the flows $\partial/\partial s_{2k-1}$ for $k \geq n$. This implies that $u(x)$ is either a constant (in which case we see immediately that $u(x) = 0$, i.e. $\mathcal{L} = d^2/dx^2$ and (ii) is obvious) or $A_W \supset \mathbb{C}[z^2, z^{2n-1}]$. In the second case, the operator $\mathcal{M} = L_{z^{2n-1}}$ satisfies $\mathcal{M}^2 = \mathcal{L}^{2n-1}$, which gives (ii).

Below we prove the implication (ii) ⇒ (i). If (ii) holds, then the operator $\mathcal{L}$ commutes with the operator $\mathcal{M}$ of order $2n-1$, i.e. it belongs to a rank one commutative ring of differential operators $\mathfrak{A}_W$ for some plane $W \in \mathrm{Gr}^{(2)}$. Moreover we have

$$\mathcal{L}(x, \partial/\partial x)\Psi_W(x,z) = z^2\Psi_W(x,z), \tag{3.2}$$

$$\mathcal{M}(x, \partial/\partial x)\Psi_W(x,z) = z^{2n-1}\Psi_W(x,z), \tag{3.3}$$

or equivalently

$$\mathbb{C}[z^2, z^{2n-1}] \subset A_W. \tag{3.4}$$

This combined with (2.3) means that

$$\oint_{S^1} z^k \Psi_W(x,z)\Psi_W^*(y,z)dz = 0 \text{ if } \begin{cases} k \text{ is even or} \\ k \geq 2n-1. \end{cases}$$

On the other hand, Lemma 2.2 says that the first odd power of z in the Taylor expansion of $g_n(z)$ is z^{2n-1}. Thus formula (2.4) shows that $H_n(x,y) = 0$.

Remark 3.1. The subspaces W parametrizing the operators $\mathcal{L}$ described in Theorem 1.1(ii) form a sub-Grassmannian of $\mathrm{Gr}^{(2)}$ denoted by $\mathrm{Gr}_0^{(2)}$ in [20]. It also parametrizes all rank one commutative rings of differential operators $\mathfrak{A}$ with unicursal spectral curve $\mathrm{Spec}\mathfrak{A}$, containing an operator of order two. It is well known (see [20, p. 46]) that it has a cell decomposition with cells indexed by the sets

$$S_n = \{-n, -n+2, -n+4, \dots, n, n+1, n+2, \dots\}.$$

The corresponding cell C_n in $\mathrm{Gr}_0^{(2)}$ consists of the solutions to the KdV hierarchy (2.1) flowing out of the initial value

$$u(x, 0, 0, \dots) = -\frac{n(n+1)}{x^2}.$$

These solutions exhaust all rational solutions to the KdV hierarchy that vanish at $x = \infty$ [1, 2]. The statement of Theorem 1.1 can be reformulated as follows: $H_{n+1}(x,y) = 0$ if and only if $W \in C_k$ for some $k \leq n$.

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