UPPER BOUNDS FOR THE VOLUME AND DIAMETER OF $m$-DIMENSIONAL SECTIONS OF CONVEX BODIES

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Abstract. In this paper some upper bounds for the volume and diameter of central sections of symmetric convex bodies are obtained in terms of the isotropy constant of the polar body. The main consequence is that every symmetric convex body $K$ in $\mathbb{R}^n$ of volume one has a proportional section $K \cap F$, $\dim F = \lambda n$ ($0 < \lambda < 1$), of diameter bounded by

$$R(K \cap F) \leq C n^{3/4} \log(n+1) \left(1 - \lambda^2 L_K \right)^{3/4},$$

whenever the polar body $K^\circ$ is in isotropic position ($C > 0$ is some absolute constant).

1. Introduction, notation and results

We consider $\mathbb{R}^n$ endowed with the canonical euclidian structure. The set $K$ will be a convex body in $\mathbb{R}^n$, i.e., a compact convex set in $\mathbb{R}^n$ having a nonempty interior. We shall say that $K$ is in isotropic position if its centroid is the origin

$$\int_K x dx = 0$$

and there is a constant $\alpha > 0$ such that

$$\int_K (x, \theta)^2 dx = \alpha$$

for all $\theta \in S^{n-1}$. Every convex body in $\mathbb{R}^n$ has a unique, up to orthogonal transformation, isotropic position (by a position of a convex body here we mean any affine transformation of the set, with determinant equal to 1). We recall the definition of the isotropy constant $L_K$ of a convex body $K$

$$nL_K^2 |K|^{2/n} = \inf_{S \in SL(n)} \left( \frac{1}{|K|} \int_K |t + Sx|^2 dx \right),$$

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so, if \( K \) is in in isotropic position and \( |K| = 1 \),
\[
L_K^2 = \int_K (x, \theta)^2 \, dx
\]
for all \( \theta \in S^{n-1} \).

In the sequel we will assume that \( |K| = 1 \). It is a major problem in asymptotic convex geometry to prove that the isotropy constant of any convex body \( K \) in \( \mathbb{R}^n \), \( L_K \), is bounded from above by an absolute constant (independent even on the dimension). It is well known that this fact is equivalent to the slicing problem: there exists an absolute constant \( c > 0 \) with the following property: if \( K \) is a convex body in \( \mathbb{R}^n \) there exists a hyperplane \( H \) such that \( |K \cap H|_{n-1} > c \). Very recently, using a new concentration inequality of G. Paouris ([Pa2], [Pa3]), B. Klartag has just improved Bourgain’s estimate proving that \( L_K \leq Cn^{1/4} \) (see [Kl2]).

Let us assume that the convex body \( K \) of volume one has its centroid in the origin. The circumradius \( R(K) \) is the quantity
\[
R(K) = \max\{|x|; x \in K\} = \max\{1/\|\theta\|_K; \theta \in S^{n-1}\}.
\]
A simple computation shows that \( C\sqrt{n} \leq R(K) \). Moreover if \( K \) is isotropic \( R(K) \leq \sqrt{n(n+2)}L_K \) (cf. [KLS]) and if instead of that, \( K^\circ \) is in isotropic position, then
\[
R(K) \leq \frac{Cn}{L_{K^\circ}} \leq Cn,
\]
where \( C \) is an absolute constant (see item 3 in the Remarks below).

In this paper some upper bounds for the volume and circumradius of sections of symmetric convex bodies with volume one are obtained in terms of the isotropy constant of the polar body.

The proof we use in proving the part ii) of Theorem 1.1 is based in the methods appearing in the paper [Bo2] in which Bourgain obtained some sharp estimates for isotropic \( \psi_2 \)-convex bodies. Our result is equivalent to that appearing in [Bo2] (lemma on page 116), at least for high-dimensional sections. However, the approach we use here presents some differences. We use the \( M \)-ellipsoid associated to every convex body and sharp estimates on covering numbers given by Pisier (see [Pi]) instead of using the \( \ell \)-ellipsoid and Talagrand majoration theorem.

The main result we achieve is the following.

**Theorem 1.1.** Let \( K \) be a centrally symmetric convex body in \( \mathbb{R}^n \) with volume \( |K| = 1 \) such that \( K^\circ \) is in isotropic position. Let \( E \) be any \( m \)-dimensional subspace, \( 1 \leq m \leq n \). Then

i) \[
|K \cap E|^{1/m} \leq \frac{Cn}{m^{1/4}L_{K^\circ}},
\]
i
ii) if \( 0 < \delta < 1/2 \) there exists another subspace \( F \subset E \), \((1 - 2\delta)m \leq \dim F \leq (1 - \delta)m \) such that
\[
R(K \cap F) \leq \frac{C \log(m+1)n^{5/4}}{\delta^3 m^{1/2}L_{K^\circ}}
\]
for some absolute constant \( C > 0 \) (independent of \( K \) and of the dimensions \( m, n \)).
we have (\text{true at least for one } m)

K_\ast\text{ for the isotropy constant. In fact, it is clear for } 1.

The bound (1.2) for the high dimensions gives the better known bound

K\ast\text{ whenever that }

L\ast F\text{ any subspace }

m\text{. In our situation, if }

K\ast\text{ is in isotropic position and its volume is }

\langle x, \theta \rangle^2 dx \leq C n^2 \|\theta\|^2_K.

(|| \cdot ||_K \text{ represents the norm in } \mathbb{R}^n \text{ whose unit ball is } K.)

If } K \text{ is a general convex body in } \mathbb{R}^n \text{ with its centroid in the origin, using } K \subset K - K \text{ and Rogers-Shephard’s inequality there exists a position of } K, \text{ say } \tilde{K}, \text{ for which }

R(\tilde{K}) \leq C n.

4. We can improve the above estimate on the circumradius only for high-dimensional sections of convex bodies of volume one, whose polars are in isotropic
position. For instance, given $0 < \lambda < 1$, by taking $m = n$ and $\delta = (1 - \lambda)/2$ in Theorem \([\text{11}]\) we obtain that for any convex body $K$ satisfying this hypothesis there exists some $\lambda n$-dimensional section for a subspace $F$ such that

$$R(K \cap F) \leq \frac{C n^{3/4} \log(n + 1)}{(1 - \lambda)^{3/2} L_K}.$$  

Taking polars and renormalizing volumes we can express this result in terms of the polar body in the following way: If $K$ is an isotropic symmetric convex body in $\mathbb{R}^n$ with volume one, then given $0 < \lambda < 1$ there exists some orthogonal projection on the subspace $F$ of $\mathbb{R}^n$ such that

$$P_F(K) \supseteq C(1 - \lambda)^{3/4} \frac{L_K n^{1/4}}{\log(1 + n)} D_{\lambda n},$$

where $D_{\lambda n}$ denotes the Euclidean unit ball in dimension $\lambda n$.

Let us eventually introduce some notation. We represent $\ell_2^n$ the normed space $\mathbb{R}^n$ with the Euclidean norm. Note that $|\cdot|$ will represent the Euclidian norm and also the volume $|K| = |K|_n$ (if we want to emphasise the dimension). By $C, c$ we will represent absolute constants which can vary from line to line. The expression $\sim$ denotes equivalence of two quantities up to an absolute factor.

2. Proof of the theorem

Proof. i) The case $m = n$ is just the estimate for the isotropy constant. We only consider $1 \leq m \leq n - 1$. By using Bourgain-Milman’s reverse Blaschke-Santaló’s inequality (see \([\text{BM}]\)) we have that

$$|K \cap E|_{1/m}^{1/m} \leq \frac{C}{m|K \cap E|_{1/m}^{1/m}} = \frac{C}{m|P_E(K)|_{1/m}^{1/m}},$$

and also using Fubini’s theorem and Brunn-Minkowski inequality

$$|K|_{1/n}^{1/n} \leq |K \cap E_{1/n-m}^{1/n-m}| P_E(K)_{1/m}^{1/n}.$$  

Then

$$|K \cap E|_{1/m}^{1/m} \leq \frac{C}{m|K|_{1/m}^{1/m}} |K \cap E|_{1/m}^{1/m-n}.$$  

Now we use a fact which can be deduced from a well-known result by Milman-Pajor (cf. \([\text{MP}]\), Proposition 3.11) (this result can be found implicit in K. Ball (\([\text{BI}]\))): if $T$ in $\mathbb{R}^n$ is an isotropic symmetric convex body, then

$$|T \cap E_{1/m}^{1/m-n-m} \leq C \frac{L_m}{L_T} |T|_{(n-m)/nm}^{1/m},$$

where $C$ is an absolute constant and $L_m$ is the supremum of the isotropy constants of all $m$-dimensional convex bodies. Hence, we apply this result to $T = K$, taking into account that $|K|_{1/n}^{1/n} \sim C/n$ by reverse Blaschke-Santaló’s inequality. Eventually the better known estimate for the isotropy constant gives the result.
ii) Since the convex body $K^*$ is in isotropic position and its volume is $|K^*|^{1/n} \sim 1/n$, the isotropic constant $L = L_{K^*}$ satisfies

$$L^2 \sim \frac{n}{|K^*|^n} \int_{K^*} |x|^2 dx,$$

according to (2.4).

Let $E$ be any $m$-dimensional subspace of $\mathbb{R}^n$. We denote by $X$ the $m$-dimensional normed space $X = (E, \| \cdot \|_{K \cap E})$, where $\| \cdot \|_{K \cap E}$ is the norm on $E$ whose unit ball is $K \cap E$. We denote by $B$ the polar set of $K \cap E$ in $E$ so the dual space is $X^* = (E, \| \cdot \|_B)$, and it is well known that $B = P_E(K^\circ)$, where $P_E$ denotes the orthogonal projection on $E$.

We consider the $M$-ellipsoid associated to the symmetric convex body (in $E$), $K \cap E$ (see [Pi]), i.e. given any $0 < p < 2$ there exists an isomorphism $u : \ell^m_2 \to X$ such that

$$\max \{ d_k(u), c_k(u^{-1}), e_k(u), e_k(u^*-), e_k(u^{-1*}) \} \leq \frac{C}{\sqrt{2 - p}} \left( \frac{m}{k} \right)^{1/p}$$

for some absolute constant $C > 0$ and for all $k = 1, \ldots, m$, where the $d_k, c_k, e_k$'s are the corresponding Kolmogorov, Gelfand and entropy numbers. Furthermore

$$\max \{ \log N(u(D_m), tK \cap E), \log N(u^*(B), tD_m), \log N(K \cap E, tu(D_m)), \log N(D_m, tu^*(B)) \}$$

$$\leq \left( \frac{C}{\sqrt{2 - p}} \right)^p \frac{m}{t^p}$$

for all $t \geq C/\sqrt{2 - p}$, where $C > 0$ is an absolute constant (see, for instance, [Pi], Theorem 7.13 and Corollaries 7.15 and 7.16).

We can assume that the isomorphism $u$ has a diagonal expression if we take the canonical basis $\{ e_i \}_{i=1}^m$ in $\ell^m_2$ and some orthogonal basis $\{ e_i \}_{i=1}^m$ (with respect to the euclidian structure of $\mathbb{R}^n$) in $E$ such that $u(e_i) = \lambda_i e_i$, for some $\lambda_i > 0$, $1 \leq i \leq m$ in such a way that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Let $1 \leq k < m$. Since

$$c_k(u^{-1}) \leq \frac{C}{\sqrt{2 - p}} \left( \frac{m}{k} \right)^{1/p}$$

(see (2.9)), by definition of the Gelfand numbers, there exists a subspace $F_1 \subseteq E$ with $\dim F_1 > m - k$, such that

$$|u^{-1}(x)| \leq \frac{C}{\sqrt{2 - p}} \left( \frac{m}{k} \right)^{1/p} \| x \|_K$$

for all $x \in F_1$. Let $F_2 = \text{span} \{ e_1, \ldots, e_{m-k} \}$. It is clear that $F = F_1 \cap F_2$ verifies that $m - 2k < \dim F \leq m - k$ and

$$|u^{-1}(x)| = \sum_{i=1}^{m-k} \lambda_i^{-1} x_i \geq \frac{1}{\lambda_{m-k}} \left( \sum_{i=1}^{m-k} x_i^2 \right)^{1/2} = \frac{|x|}{\lambda_{m-k}}$$
for all $x = \sum_{i=1}^{m-k} x_i e_i \in F$. So,

$$|x| \leq \lambda_{m-k} \frac{C}{\sqrt{2-p}} \left( \frac{m}{k} \right)^{1/p} \|x\|_K$$

$$\leq \frac{1}{k} \left( \sum_{i=m-k+1}^{m} \lambda_i \right) \frac{C}{\sqrt{2-p}} \left( \frac{m}{k} \right)^{1/p} \|x\|_K$$

$$\leq \frac{1}{k} \left( \sum_{i=1}^{m} \lambda_i \right) \frac{C}{\sqrt{2-p}} \left( \frac{m}{k} \right)^{1/p} \|x\|_K$$

for all $x \in F$. We will assume the following.

Claim.

$$\sum_{i=1}^{m} \lambda_i \leq \frac{C n^{(3/2-1/2p)m^{1/p}}}{L \sqrt{2-p}}.$$ 

We first finish the proof. If we assume that $k = \delta m$, for some $0 < \delta < 1/2$, then

$$|x| \leq \frac{C}{(2-p)\delta^{1+1/p}} \frac{m^{3/2-1/2p}}{L m^{1-1/p}}$$

for all $x \in F \cap K$, and we achieve the result, by taking $p = 2 - 1/\log m$. \hfill \square

Proof of the Claim. We consider $\ell^m_n$ canonically embedded in $\ell^2_n$, and we denote by $\{\varepsilon_i\}_{i=1}^n$ its canonical basis; in the same way, we extend the orthogonal basis $\{e_i\}_{i=1}^m$ from $E$ to all $\mathbb{R}^n$. Let $P_E : \mathbb{R}^n \to E$ be the orthogonal projection on $E$.

Let $K_1 \subseteq \ell^2_n$ be the convex set defined by

$$K_1 = \left\{ x = \sum_{i=1}^{n} x_i e_i \in \ell^2_n : \sum_{i=1}^{n} x_i e_i \in K^\circ \right\}.$$ 

$K_1$ is an orthogonal copy of $K^\circ$, therefore they have the same volume and isotropy constant, and hence, by (2.4),

$$\frac{C}{n^2} L^2 \leq \frac{1}{|K_1|^n} \int_{K_1} x_i^2 dx$$

for an absolute constant and for all $1 \leq i \leq n$. Hence

(2.9) \hspace{1cm} \frac{C}{n^2} L^2 \sum_{i=1}^{m} \lambda_i \leq \frac{1}{|K_1|^n} \int_{K_1} \sum_{i=1}^{m} \lambda_i x_i^2 dx $$

(2.10) \hspace{1cm} = \frac{1}{|K_1|^n} \int_{K_1} \left\langle \left( \sum_{i=1}^{n} x_i \varepsilon_i, u^* \left( \sum_{i=1}^{m} x_i e_i \right) \right) \right\rangle dx $$

(2.11) \hspace{1cm} = \frac{1}{|K_1|^n} \int_{K_1} \left\langle x, u^* \circ P_E \left( \sum_{i=1}^{m} x_i e_i \right) \right\rangle dx $$

(2.12) \hspace{1cm} \leq \frac{1}{|K_1|^n} \int_{K_1} \max_{z \in \mathbb{R}^n} \langle x, z \rangle dx $$

since $B$, which is the polar of $K \cap E$ in $E$, can be expressed as $B = P_E(K^\circ)$ and $u^* : (E, \| \cdot \|_B) \to \ell^m_2$. 

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Now we follow the method appearing in [D], [Gi], [Pa1] in order to estimate from above the last integral. If we assume that \( u^*(B) \subset RD_m \), for some \( R > 0 \) large enough, we use the Dudley-Fernique’s technique. Let \( N \in \mathbb{N} \) (to be chosen later). By using (2.8), it is clear that for every \( j = 1, \ldots, N \) there exist points \( y_1^{(j)}, \ldots, y_{N_j}^{(j)} \) such that

\[
u^*(B) \subset \bigcup_{i=1}^{N_j} \left( y_i^{(j)} + \frac{R}{2^j} D_m \right),
\]

where

\[
\log N_j \leq \left( \frac{C}{\sqrt{2-p}} \right)^p \frac{m2^j p}{R^p}.
\]

Therefore, for every \( z \in \nu^*(B) \) and \( 1 \leq j \leq N \) we choose the points \( y^{(j)} \) such that

\[
z = 0 + (y^{(1)} - 0) + (y^{(2)} - y^{(1)}) + \cdots + (y^{(N)} - y^{(N-1)}) + (z - y^{(N)}) = w_1 + \cdots + w_N + w,
\]

where

\[
|w_j| \leq |y^{(j)} - z| + |z - y^{(j-1)}| \leq \frac{3R}{2^j},
\]

\[
|w| = |z - y^{(N)}| \leq \frac{R}{2^j}.
\]

Each vector \( w_j \) belongs to a finite set \( \mathcal{F}_j \) of cardinality \( |\mathcal{F}_j| \leq N_j N_{j-1} \), so

\[
\log |\mathcal{F}_j| \leq 2 \left( \frac{C}{\sqrt{2-p}} \right)^p \frac{m2^j p}{R^p}.
\]

Therefore

\[
\int_{K_1} \max_{z \in \nu^*(B)} \langle x, z \rangle \, dx \leq \sum_{j=1}^{N} \int_{K_1} \max_{w_j \in \mathcal{F}_j} \langle x, w_j \rangle \, dx + \int_{K_1} \max_{w \in \frac{R}{2^j} D_m} \langle x, w \rangle \, dx.
\]

Since \( K_1 \) is in isotropic position, the last summand verifies

\[
\frac{1}{|K_1|} \int_{K_1} \max_{w \in \frac{R}{2^j} D_m} \langle x, w \rangle \, dx \leq \frac{R}{2^N} \frac{1}{|K_1|} \int_{K_1} |x| \, dx
\]

\[
\leq \frac{R}{2^N} \left( \frac{1}{|K_1|} \int_{K_1} |x|^2 \, dx \right)^{1/2}
\]

\[
\leq \frac{CR}{2^N} \frac{L}{\sqrt{n}}.
\]

We use the following well-known fact (which is a consequence of Borell’s lemma; see [MS]): if \( \tilde{K} \) is an isotropic convex body with volume \( |\tilde{K}| = 1 \), then

\[
(2.13) \quad \int_{\tilde{K}} \max_{i=1}^{N} |\langle x, y_i \rangle| \, dx \leq CL_{\tilde{K}} \log N \max_{i=1}^{N} |y_i|
\]
for any finite family of unit vectors \( \{y_1, \ldots, y_N\} \), where \( C > 0 \) is an absolute constant. Then we get that

\[
\frac{n}{|K_1^n|} \int_{K_1^n} \max_{w_j \in \mathcal{F}_j} |\langle x, w_j \rangle| \, dx \leq C \frac{|K_1|^{-1/n}}{|K_1^n|} \int_{K_1^n} \max_{w_j \in \mathcal{F}_j} |\langle x, w_j \rangle| \, dx
\]

\[
\leq C_1 L \log |\mathcal{F}_j| \max_{w_j \in \mathcal{F}_j} |w_j|
\]

\[
\leq C_1 L \left( \frac{2^p m 2^{(p-1)}}{R^p} \right) \frac{3R}{2^p}
\]

\[
\leq \frac{CC_1 L m}{(2 - p)^{p/2}} 2^{(p-1)} R^{p-1}.
\]

Taking into account (2.12) and (2.13) we have

\[
L \sum_{i=1}^m \lambda_i \leq Cn \left( \frac{R}{2^n} \sqrt{n} + \frac{CC_1 m}{(2 - p)^{p/2}} \sum_{j=1}^N 2^{(p-1)} R^{p-1} \right)
\]

\[
\leq Cn \left( \frac{R}{2^n} \sqrt{n} + \frac{CC_1 m}{(2 - p)^{p/2}} \left( \frac{2^N}{R} \right)^{(p-1)} \right).
\]

We optimize by taking

\[
R = \left( \frac{CC_1 m}{\sqrt{n}(2 - p)^{p/2}} \right)^{1/p},
\]

and hence the claim holds.

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