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ON THE ROLEWICZ THEOREM FOR EVOLUTION OPERATORS

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ABSTRACT. We give a short proof of a generalization of the Rolewicz theorem based on the uniform boundedness principle.

We say that a function $f : (0, \infty) \to (0, \infty)$ is *proper* if f is nondecreasing.

Theorem 1. *Suppose that X is Banach space, and a C_0-semigroup $\{T_t : X \to X\}$ is not exponentially bounded, i.e. $\|T_t\|$ does not decrease exponentially as $t \to \infty$. For each proper function f, there exists $x \in X$ such that $\int_0^\infty f(|T_t(x)|)dt = \infty$.*

R. Datko proved this fact in [2] for $f(z) = z^2$ and X a Hilbert space. (There he established an analog to the Liapounov theorem.) A. Pazy generalized it in [4] for $f(z) = z^p,\ p \in [1, \infty)$. For continuous strictly monotone proper functions, this fact was obtained by Littman [3].

Moreover, Rolewicz [7] considered evolution operators (satisfying $U(t, s)U(s, r) = U(t, r)$, $U(t, t) = \mathrm{Id}$ and $U(t, s)(x)$ is continuous with respect to t for each $x \in X$).

Theorem 2 (Rolewicz)**.** *Suppose that $U(t, s) : X \to X,\ t \geq s \geq 0$, are evolution operators on a Banach space X, which are uniformly bounded but not uniformly exponentially bounded (i.e. $\sup_{s\geq 0} \|U(s + p, s)\|$ does not decrease exponentially in p). Assume that $N(\alpha, u) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and the function $f_\alpha(u) := N(\alpha, u)$ is proper for every α. Then there is $x \in X$ such that for all α*

$$\sup_{s<\infty} \int_0^\infty f_\alpha(|U(s + p, s)(x)|)dp = \infty. \tag{1}$$

Rolewicz-type theorems were proved, for example, in [1, 8].

We establish a Rolewicz-like result in the next theorem.

Theorem 3. *Suppose that $U(t, s)$ are evolution operators on a Banach space X, which are uniformly bounded but not uniformly exponentially bounded. If $f_\alpha(u) = N(\alpha, u)$, $\alpha \in \mathbb{R}^m$, is continuous in α and proper on u, then there is $x \in X$ such that the condition* (1) *is fulfilled for each f_α, $\alpha \in \mathbb{R}^m$.*

Note that in [8] the assumption of continuity on u was also removed.

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Lemma 4. *Let $T_n : X \to X$ be linear operators on a Banach space with $\|T_n\| \not\to 0$. For each positive sequence $a_n \to 0$, there exists x such that $\overline{\lim}\{\frac{|T_n(x)|}{a_n}\} = \infty$. If f is a proper function and $b_n \to \infty$, then there is x such that $\overline{\lim}\, b_n f(|T_n(x)|) = \infty$.*

Proof. Apply the Banach-Steinhaus Theorem for the unbounded set of operators $\{\frac{T_n}{a_n}\}$. To prove the second part of the lemma, note that if f is a proper function, then there is a sequence $a_n \to 0$ such that $f(a_n) \geq (b_n)^{-\frac{1}{2}}$ for large n. Now if $|T_{n_k}(x)| \geq a_{n_k}$, then $b_{n_k} \cdot f(|T_{n_k}(x)|) \geq b_{n_k} \cdot f(a_{n_k}) \geq \sqrt{b_{n_k}} \to \infty$. □

Lemma 5. *If $f_\alpha(u) = N(\alpha, u)$, $\alpha \in \mathbb{R}^m$, is continuous in α and proper on u, then there is a proper f such that $f(u) \leq f_\alpha(u)$ as $u \to 0$ for each $\alpha \in \mathbb{R}^m$.*

Proof. Cover $\mathbb{R}^m$ by compacts $K_1 \subset K_2 \subset \ldots$ and put $f_n = \inf\{f_\alpha \mid \alpha \in K_n\}$. Then f_n are proper functions and $f_1 \geq f_2 \geq \ldots$. Now for $u \in (\frac{1}{n+1}, \frac{1}{n}]$ put $f(u) = f_n(u)$. □

Proof of Theorem 3. Let f be as in Lemma 5. If condition (1) is fulfilled for f, then the same condition is fulfilled for each f_α, $\alpha \in \mathbb{R}^m$.

Note that $\sup_s\{\|U(p+s,s)\|\} = \lambda_p \geq 1$ for every $p \geq 0$. Otherwise the family $\{U\}$ would be exponentially bounded, since

$$\|U(pn+s,s)\| \leq \|U(pn+s, p(n-1)+s) \circ \cdots \circ U(p+s,s)\| \leq \lambda_p^n$$

and $\|U(t+s,s)\| \underset{t\to\infty}{=} O(e^{\frac{\ln \lambda_p}{p}t})$. In particular, there exists a sequence s_n such that $\|U(n+s_n,s_n)\| \not\to 0$ as $n \to \infty$.

Put $c = \sup\{\|U\|\}$. From composition law we see that if $n \geq p$, then for each s $\|U(n+s,s)\| \leq c\|U(p+s,s)\|$. Then $|U(p+s,s)(x)| \geq |U(n+s,s)(\frac{x}{c})|$, and we have

$$\int_0^\infty f(|U(p+s_n,s_n)(x)|)dp \geq \int_0^n f(|U(p+s_n,s_n)(x)|)dp \geq n \cdot f(|U(n+s_n,s_n)\left(\frac{x}{c}\right)|).$$

By Lemma 4, the right-hand side of the inequality is unbounded on some x. □

Remark 1. Lemma 4 uses only the uniform boundedness principle, therefore Theorem 5 is valid for normed barreled spaces and for Frechet spaces (if we replace $|x|$ by $\rho(0,x)$ for $x \in X$). Also we can replace $\mathbb{R}^m$ by any σ-compact Y.

Remark 2. Van Neerven [6] generalized the Datko-Pazy theorem in another direction, by replacing $\int$ with a more general functional. He made use of an assertion in [5] which is much stronger than our Lemma 4 (van Neerven's result implies that the first assertion of Lemma 4 still holds even if we replace "$\overline{\lim}$" by "$\underline{\lim}$").

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